Integrability of certain homogeneous Hamiltonian systems

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Introduction

- Let $H : \mathbb{C}^{2n} \to \mathbb{C}$ be a holomorphic Hamiltonian, and

$$\frac{d}{dt}\mathbf{x} = \boldsymbol{\nu}_H(\mathbf{x}), \quad \boldsymbol{\nu}_H(\mathbf{x}) = \mathbf{I}_{2n}\nabla_{\mathbf{x}}H, \qquad \mathbf{x} \in \mathbb{C}^{2n}, \quad t \in \mathbb{C}, \tag{1}$$

  the associated Hamilton equations.
- Let $t \to \boldsymbol{\varphi}(t) \in \mathbb{C}^{2n}$ be a non-equilibrium solution of (1).
- The maximal analytic continuation of $\boldsymbol{\varphi}(t)$ defines a Riemann surface $\boldsymbol{\Gamma}$ with t as a local coordinate.

$$\boldsymbol{\Gamma} := \{\mathbf{x} \in \mathbb{C}^{2n} | \mathbf{x} = \boldsymbol{\varphi}(t),\ t \in U \in \mathbb{C}\}.$$

- Variational equations along $\boldsymbol{\varphi}(t)$ have the form

$$\frac{d}{dt}\boldsymbol{\xi} = \mathbf{A}(t)\boldsymbol{\xi}, \qquad \mathbf{A}(t) = \frac{\partial \boldsymbol{\nu}_H}{\partial \mathbf{x}}(\boldsymbol{\varphi}(t)). \tag{2}$$

- We can attach to Eq. (2) the differential Galois group $\mathcal{G}$.

Morales-Ramis theorem

Theorem

Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of the analytic phase curve Γ. Then the identity component of the differential Galois group of the variational equations along Γ is Abelian.

- Morales Ruiz, J. J., *Differential Galois theory and non-integrability of Hamiltonian systems*, volume 179 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 1999.
- Audin, M., *Les systèmes hamiltoniens et leur intégrabilité*, Cours Spécialisés 8, Collection SMF, SMF et EDP Sciences, Paris, 2001.

Aplications of Morales–Ramis theory

- to prove non-integrability of Hamiltonian systems,

A. J. Maciejewski and M. Przybylska, Non-integrability of *ABC* flow, *Phys. Lett. A*, 303(4):265–272, 2002.

T. Stachowiak and W. Szumiński, Non-integrability of constrained double pendula, *Phys. Lett. A*, under review.

Maria Przybylska, Wojciech Szumiński, Non-integrability of flail triple pendulum, *Chaos, Solitons & Fractals*, Vol. 53, August 2013.

- to detection possible integrable cases for Hamiltonian systems depending on parameters.

A. J. Maciejewski, M. Przybylska and H. Yoshida, Necessary conditions for the existence of additional first integrals for Hamiltonian systems with homogeneous potential, *Nonlinearity*, Vol. 25, no 2, s. 255–277, 2012.

W. Szumiński, A. J. Maciejewski and M. Przybylska, Note on integrability of certain homogeneous Hamiltonian systems, *Phys. Lett. A*, In press.

Main steps during applications

- Find a particular solution different from equilibrium points,
- calculate VE and NVE,
- check if G^0 is Abelian (most difficult step): we try to transform NVE into the equation with known differential Galois group:
  - Riemann equation,
  - Lammé equation,
  - an equation of the second order with rational coefficients.

Kovacic, J. An algorithm for solving second order linear homogeneous differential equations. *J. Symbolic Comput.*, 2(1):3–43,

Integrability of homogeneous Hamiltonian equations

Integrability of Hamiltonian systems given by

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\mathbf{q}), \qquad (\mathbf{q},\mathbf{p}) \in \mathbb{C}^{2n},$$

V — homogeneous of degree $k \in \mathbb{Z}$

$$V(\lambda q_1, \ldots, \lambda q_n) = \lambda^k V(q_1, \ldots, q_n)$$

Definition (standard)

Darboux point $\mathbf{d} \in \mathbb{C}^n$ is a non-zero solution of

$$V'(\mathbf{d}) = \mathbf{d}$$

Particular solution

$$\mathbf{q}(t) = \varphi(t)\mathbf{d},\ \ \mathbf{p}(t) = \dot{\varphi}(t)\mathbf{d} \quad \text{provided} \quad \ddot{\varphi} = -\varphi^{k-1}.$$

Integrability of homogeneous Hamiltonian equations

On the energy level:

$$H(\varphi(t)\mathbf{d}, \dot{\varphi}(t)\mathbf{d}) = e \in \mathbb{C}^{\star},$$

hyperelliptic curve

$$\dot{\varphi}^2 = \frac{2}{k}\left(\varepsilon - \varphi^k\right), \quad \varepsilon = ke \in \mathbb{C}^{*}.$$

The variational equations

$$\ddot{x} = -\lambda\varphi(t)^{k-2}x, \tag{3}$$

where λ is an eigenvalue of $V''(\mathbf{d})$.

Morales Ruiz, J. J., Differential Galois theory and non-integrability of Hamiltonian systems, volume 179 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 1999.

What is analog of homogeneous systems in curved spaces?

No obvious answer

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\mathbf{q}), \qquad (\mathbf{q}, \mathbf{p}) \in \mathbb{C}^{2n},$$

Our proposition

$$H = T + V, \qquad T = \frac{1}{2} r^{m-k}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right), \qquad V = r^m U(\varphi),$$

where m and k are integers, and $k \neq 0$.

Main integrability theorem. Auxiliary sets

$$\mathcal{I}_0(k,m) := \left\{ \frac{1}{k}\left(mp+1\right)\left(2mp+k\right) \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_1(k,m) := \left\{ \frac{1}{2k}\left(mp-2\right)\left(mp-k\right) \;\middle|\; p = 2r+1, r \in \mathbb{Z} \right\},$$

$$\mathcal{I}_2(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{2}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_3(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{3}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_4(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{4}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_5(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{5}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_6(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{2}{5}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_{\mathrm{a}}(k,m) := \mathcal{I}_0(k,m) \cup \mathcal{I}_1(k,m) \cup \mathcal{I}_2(k,m).$$

Main integrability theorem

Theorem

Assume that $U(\varphi)$ is a complex meromorphic function and there exists $\varphi_0 \in \mathbb{C}$ such that $U'(\varphi_0) = 0$ and $U(\varphi_0) \neq 0$. If the Hamiltonian system defined by Hamiltonian

$$H = T + V, \qquad T = \frac{1}{2} r^{m-k} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right), \qquad V = r^m U(\varphi),$$

is integrable in the Liouville sense, then number

$$\lambda := 1 + \frac{U''(\varphi_0)}{kU(\varphi_0)},$$

belongs to set $\mathcal{I}(k, m)$ which is defined by the following table

Main integrability theorem. Integrability table

No.	k	m	$\mathcal{I}(k,m)$
1	$k=-2(mp+1)$	m	$\mathbb{C}$
2	$k\in\mathbb{Z}\setminus\{0\}$	m	$\mathcal{I}_{\mathrm{a}}(k,m)$
3	$k=2(mp-1)\pm\frac{1}{3}m$	$3q$	$\bigcup_{i=0}^{6}\mathcal{I}_i(k,m)$
4	$k=2(mp-1)\pm\frac{1}{2}m$	$2q$	$\mathcal{I}_{\mathrm{a}}(k,m)\cup\mathcal{I}_4(k,m)$
5	$k=2(mp-1)\pm\frac{3}{5}m$	$5q$	$\mathcal{I}_{\mathrm{a}}(k,m)\cup\mathcal{I}_3(k,m)\cup\mathcal{I}_6(k,m)$
6	$k=2(mp-1)\pm\frac{1}{5}m$	$5q$	$\mathcal{I}_{\mathrm{a}}(k,m)\cup\mathcal{I}_3(k,m)\cup\mathcal{I}_5(k,m)$

Table : Integrability table. Here $k,m,p,q\in\mathbb{Z}$ and $k\neq 0$.

Proof. Particular solution

$$\dot{r} = \frac{\partial H}{\partial p_r} = r^{m-k} p_r,$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = r^{m-k-3} p_\varphi^2 - \frac{1}{2}(m-k) r^{m-k-1} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) - m r^{m-1} U(\varphi),$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = r^{m-k-2} p_\varphi, \quad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = -r^m U'(\varphi).$$

If $U'(\varphi_0) = 0$ for a certain $\varphi_0 \in \mathbb{C}$, then system has invariant manifold

$$\mathcal{N} = \left\{ (r, p_r, \varphi, p_\varphi) \in \mathbb{C}^4 | \varphi = \varphi_0, \ p_\varphi = 0 \right\}.$$

particular solution lying on $\mathcal{N}$

$$\dot{r} = r^{m-k} p_r, \qquad \dot{p}_r = -\frac{1}{2}(m-k) r^{m-k-1} p_r^2 - m r^{m-1} U(\varphi_0)$$

$$E = \frac{1}{2} r^{m-k} p_r^2 + r^m U(\varphi_0)$$

Proof. Variational equations

Let $[R, P_R, \Phi, P_\Phi]^T$ are variations of $[r, p_r, \varphi, p_\varphi]^T$, then variational equations along the particular solution

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} R \\ P_R \\ \Phi \\ P_\Phi \end{bmatrix} = \mathbf{C}\begin{bmatrix} R \\ P_R \\ \Phi \\ P_\Phi \end{bmatrix},$$

with

$$\mathbf{C} = \begin{bmatrix} lr^{l-1}p_r & r^l & 0 & 0 \\ -\frac{1}{2}(l-1)lr^{l-2}p_r^2 - (m-1)mr^{m-2}U(\varphi_0) & -lr^{l-1}p_r & 0 & \\ 0 & 0 & 0 & r^{l-2} \\ 0 & 0 & -r^m U''(\varphi_0) & 0 \end{bmatrix}$$

where auxiliary parameter $l = m - k$.

Proof. Normal variational equations (NVEs)

Equations for Φ and P_Φ form a closed subsystem NVEs. that give one second-order differential equation

$$\ddot{\Phi} + P\dot{\Phi} + Q\Phi = 0, \quad P = (k-m+2)r^{m-k-1}p_r, \quad Q = r^{2m-k-2}U''(\phi_0).$$

Rationalization

$$t \longrightarrow z = \frac{U(\varphi_0)}{E} r^m(t),$$

for $E \neq 0$, that gives immediately

$$\dot{z}^2 = -2Em^2 r^{m-k-2} z^2 (z-1), \ \ddot{z} = Emr^{m-k-2} z \left[(k-4m+2)z + 3m - k - 2\right].$$

NVEs after such a change of independent variable takes the form

$$z(z-1)\Phi''(z) + \left[\frac{2m+k+2}{2m} z - \frac{k+m+2}{2m}\right]\Phi'(z) + \frac{k(1-\lambda)}{2m^2}\Phi(z) = 0,$$

where prime denotes derivative with respect to z and

$$\lambda = 1 + \frac{U''(\varphi_0)}{kU(\varphi_0)}.$$

Proof. Gauss hypergeometric differential equation

Form of Gauss hypergeometric differential equation

$$z(z-1)\Phi''(z) + \left[(\alpha+\beta+1)z - \gamma\right]\Phi'(z) + \alpha\beta\Phi(z) = 0,$$

with parameters

$$\alpha = \frac{k+2-\Delta}{4m}, \qquad \beta = \frac{k+2+\Delta}{4m}, \qquad \gamma = \frac{k+2+m}{2m},$$

where

$$\Delta = \sqrt{(k-2)^2 + 8k\lambda}.$$

The differences of exponents at singularities $z=0$, $z=1$ and at $z=\infty$

$$\rho = 1-\gamma, \quad \sigma = \gamma - \alpha - \beta = \frac{1}{2}, \quad \tau = \beta - \alpha$$

and for our equation

$$\rho = \frac{m-k-2}{2m}, \qquad \sigma = \frac{1}{2}, \qquad \tau = \frac{\Delta}{2m}.$$

Solvability of Riemann P equation. Kimura theorem

Theorem

The identity component of the differential Galois group of the Riemann P equation is solvable iff

A. *at least one of the four numbers $\rho+\sigma+\tau$, $-\rho+\sigma+\tau$, $\rho-\sigma+\tau$, $\rho+\sigma-\tau$ is an odd integer, or*

B. *the numbers ρ or $-\rho$ and σ or σ and τ or $-\tau$ belong (in an arbitrary order) to some of appropriate fifteen families forming the so-called Schwarz's table fifteen families*

1	$1/2+l$	$1/2+s$	arbitrary complex number	
2	$1/2+l$	$1/3+s$	$1/3+q$	
3	$2/3+l$	$1/3+s$	$1/3+q$	$l+s+q$ even
4	$1/2+l$	$1/3+s$	$1/4+q$	
5	$2/3+l$	$1/4+s$	$1/4+q$	$l+s+q$ even
6	$1/2+l$	$1/3+s$	$1/5+q$	
7	$2/5+l$	$1/3+s$	$1/3+q$	$l+s+q$ even
8	$2/3+l$	$1/5+s$	$1/5+q$	$l+s+q$ even
9	$1/2+l$	$2/5+s$	$1/5+q$	
10	$3/5+l$	$1/3+s$	$1/5+q$	$l+s+q$ even
11	$2/5+l$	$2/5+s$	$2/5+q$	$l+s+q$ even
12	$2/3+l$	$1/3+s$	$1/5+q$	$l+s+q$ even
13	$4/5+l$	$1/5+s$	$1/5+q$	$l+s+q$ even
14	$1/2+l$	$2/5+s$	$1/3+q$	
15	$3/5+l$	$2/5+s$	$1/3+q$	$l+s+q$ even

where $l, s, q \in \mathbb{Z}$.

Kimura theorem. Condition A

The condition A of Kimura theorem is fulfilled if at least one of the following numbers

$$\begin{aligned}
\rho + \sigma + \tau &= \frac{2m - k - 2 + \Delta}{2m}, \\
-\rho + \sigma + \tau &= \frac{k + 2 + \Delta}{2m}, \\
\rho - \sigma + \tau &= \frac{-k - 2 + \Delta}{2m}, \\
\rho + \sigma - \tau &= \frac{2m - k - 2 - \Delta}{2m}
\end{aligned}$$

is an odd integer.

- If it is the first one, then $\lambda \in \mathcal{I}_0(k, m)$,
- if it is the second one, then $\lambda \in \mathcal{I}_1(k, m)$,
- if the third or fourth of the above numbers is an odd integer, then $\lambda \in \mathcal{I}_0(k, m) \cup \mathcal{I}_1(k, m)$.

Kimura theorem. Condition B

In this case the quantities ρ or $-\rho$, σ or $-\sigma$ and τ or $-\tau$ must belong to Schwarz's table. As $\sigma = \frac{1}{2}$ only items 1, 2, 4, 6, 9, or 14 of the Schwarz table are allowed.

Case 1.

- $\pm\rho = 1/2 + s$, for a certain $s \in \mathbb{Z}$, then $k = -2(mp + 1)$ for a certain $p \in \mathbb{Z}$. In this case τ is an arbitrary number, so λ is arbitrary.
- $\pm\tau = 1/2 + p$, for a certain $p \in \mathbb{Z}$, then $\lambda \in \mathcal{I}_2(k, m)$. In this case ρ-arbitrary, and thus k can be arbitrary.

Case 2. In this case $\pm\tau = 1/3 + p$, for a certain $p \in \mathbb{Z}$, and $\pm\rho = 1/3 + s$, for a certain $s \in \mathbb{Z}$. The first condition implies that $\lambda \in \mathcal{I}_3(k, m)$. If the second condition is fulfilled, then

$$k = 2(mp - 1) \pm \frac{1}{3}m. \tag{4}$$

Similar analysis for items 4, 6, 9, or 14 of the Schwarz table

Example 1. Separable cases

Hamilton-Jacobi equation for our Hamiltonian

$$\frac{1}{2}r^{m-k}\left[\left(\frac{\partial S}{\partial r}\right)^2 + r^{-2}\left(\frac{\partial S}{\partial \varphi}\right)^2\right] + r^m U(\varphi) = E,$$

where $S = S(r,\varphi)$ is Hamilton's characteristic function.
We look for S postulating its additive form

$$S = S_r(r) + S_\varphi(\varphi).$$

Substitution into Hamilton-Jacobi equation gives

$$r^{-k}\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 + r^{-(k+2)}\left(\frac{\mathrm{d}S_\varphi}{\mathrm{d}\varphi}\right)^2 + 2U(\varphi) = 2r^{-m}E.$$

It separates when $k = -2$

$$r^2\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 - 2r^{-m}E = \alpha, \qquad \left(\frac{\mathrm{d}S_\varphi}{\mathrm{d}\varphi}\right)^2 + 2U(\varphi) = -\alpha,$$

and then where α is a separation constant.

Example 1. Separable cases

For $k = -2$ the Hamiltonian of the system takes the form

$$H = \frac{1}{2} r^{m+2} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) + r^m U(\varphi).$$

and it is integrable with the following additional first integral

$$F := \frac{p_\varphi^2}{2} + U(\varphi).$$

Case $k = -2$ is contained in the first item of the Integrability Table.

Example 2.

$$H = \frac{1}{2}r^{m-k}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^m\cos\varphi. \tag{5}$$

It corresponds to $U(\varphi) = -\cos\varphi$. As $U'(\varphi) = \sin\varphi$, we take $\varphi_0 = 0$. Since $U''(0)/U(0) = -1$, we have $\lambda = (k-1)/k$.
Comparing this value with forms of λ in sets $\mathcal{I}_j(k,m)$ for $j = 0,\ldots,6$:

- if $\lambda \in \mathcal{I}_0(k,m)$, then $2m^2p^2 + (k+2)mp + 1 = 0$, and this implies

$$[4mp + k + 2]^2 = k^2 + 4k - 4,$$

- if $\lambda \in \mathcal{I}_1(k,m)$, then $m^2p^2 - (k+2)mp + 2 = 0$, and this implies

$$[2(mp-1) - k]^2 = k^2 + 4k - 4,$$

- if $\lambda \in \mathcal{I}_2(k,m)$, then

$$[m(2p+1)]^2 = k^2 + 4k - 4,$$

Example 2.

- if $\lambda \in \mathcal{I}_3(k,m)$, then

$$[2m(3p+1)]^2 = 9(k^2+4k-4),$$

- if $\lambda \in \mathcal{I}_4(k,m)$, then

$$[m(4p+1)]^2 = 4(k^2+4k-4),$$

- if $\lambda \in \mathcal{I}_5(k,m)$, then

$$[2m(5p+1)]^2 = 25(k^2+4k-4),$$

- if $\lambda \in \mathcal{I}_6(k,m)$, then

$$[2m(5p+2)]^2 = 25(k^2+4k-4).$$

If one of the above conditions is fulfilled, then we have equality

$$k^2+4k-4 = q^2, \quad \text{for a certain } q \in \mathbb{Z}$$

that can be rewritten as

$$(k+2+q)(k+2-q) = 8$$

Example 2.

$$(k+2+q)(k+2-q)=8$$

- Considering all decompositions of $8=(\pm1)\cdot(\pm8)=(\pm2)\cdot(\pm4)=(\pm4)\cdot(\pm2)=(\pm8)\cdot(\pm1)$, we obtain that $k\in\{-5,1\}$.
- With these values of k one can easily find that $\lambda=(k-1)/k\in\mathcal{I}_0(k,m)$ iff $m\in\{-1,1\}$.
- Hence, we have the following four cases with m, k and $l=m-k$:

$$\begin{array}{llll} 1. & m=1, & k=-5, & l=6,\\ 2. & m=-1, & k=1, & l=-2,\\ 3. & m=1, & k=1, & l=0,\\ 4. & m=-1, & k=-5, & l=4, \end{array} \tag{6}$$

Example 2.

- Similarly, if $\lambda \in (k-1)/k \in \mathcal{I}_1$ with $k \in \{-5, 1\}$, then $m \in \{-2, -1, 1, 2\}$.
- Besides the above cases we have additionally

$$\begin{array}{llll} 5. & m=2, & k=1, & l=1, \\ 6. & m=-2, & k=1, & l=-3, \\ 7. & m=2, & k=-5, & l=7, \\ 8. & m=-2, & k=-5, & l=3. \end{array} \tag{7}$$

- No other cases when the necessary conditions for the integrability given by our Theorem.
- Surprisingly all cases (**??**) are integrable and in fact superintegrable.

Example 2. Superintegrable cases

Case 1: $m = 1,\ k = -5$.

$$H = \frac{1}{2} r^6 \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r \cos\varphi,$$

$$F_1 := r^2 p_\varphi^2 \cos(2\varphi) - r^3 p_r p_\varphi \sin(2\varphi) + r^{-1} \sin\varphi \sin(2\varphi),$$

$$F_2 := r^2 p_\varphi^2 \sin(2\varphi) + r^3 p_r p_\varphi \cos(2\varphi) - r^{-1} \sin\varphi \cos(2\varphi).$$

Case 2: $m = -1,\ k = 1$.

$$H = \frac{1}{2} r^{-2} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r^{-1} \cos\varphi,$$

$$F_1 := r^{-2} p_\varphi^2 \cos(2\varphi) + r^{-1} p_r p_\varphi \sin(2\varphi) + r \sin\varphi \sin(2\varphi),$$

$$F_2 := -r^{-2} p_\varphi^2 \sin(2\varphi) + r^{-1} p_r p_\varphi \cos(2\varphi) + r \sin\varphi \cos(2\varphi).$$

Example 2. Superintegrable cases

Case 3: $m = 1, \ k = 1.$

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r\cos\varphi,$$

$$F_1 := r^{-1}p_\varphi^2\cos\varphi + p_r p_\varphi \sin\varphi + \frac{1}{2}r^2\sin^2\varphi,$$

$$F_2 := \left(p_r^2 - r^{-2}p_\varphi^2\right)\cos\varphi\sin\varphi + r^{-1}p_r p_\varphi\cos(2\varphi) - r\sin\varphi.$$

Case 4: $m = -1, \ k = -5.$

$$H = \frac{1}{2}r^4\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^{-1}\cos\varphi,$$

$$F_1 := rp_\varphi^2\cos\varphi - r^2 p_r p_\varphi\sin\varphi + \frac{1}{2}r^{-2}\sin^2\varphi,$$

$$F_2 := r^4\left(p_r^2 - r^{-2}p_\varphi^2\right)\cos\varphi\sin\varphi - r^3 p_r p_\varphi\cos(2\varphi) - r^{-1}\sin\varphi.$$

Example 2. Integrable cases

- In cases with parameters given in (**??**) we have integrable as well as non-integrable systems.
- Namely cases 5 and 8 are integrable.

Case 5: $m = 2,\ k = 1$**.**

$$H = \frac{1}{2}r\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^2\cos\varphi,$$

$$F := r^{-1}(p_\varphi^2 - r^2p_r^2)\cos\varphi + r^2(1+\cos^2\varphi) + 2p_rp_\varphi\sin\varphi.$$

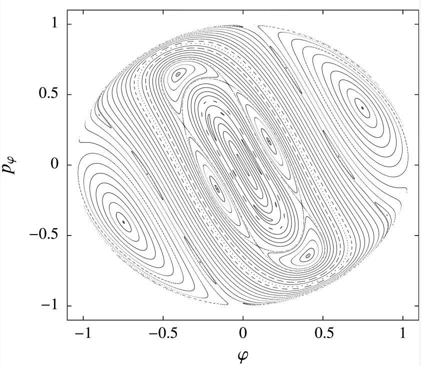
Case 8: $m = -2,\ k = -5$**.**

$$H = \frac{1}{2}r^3\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^{-2}\cos\varphi,$$

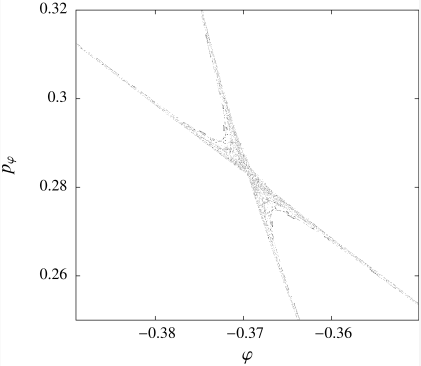
$$F := r(p_\varphi^2 - r^2p_r^2)\cos\varphi + r^{-2}(1+\cos^2\varphi) - 2r^2p_rp_\varphi\sin\varphi.$$

- Poincaré sections for Hamiltonian systems with parameters given in cases 6 and 7 in (**??**) show chaotic area.

Example 2. Non-integrable case 6



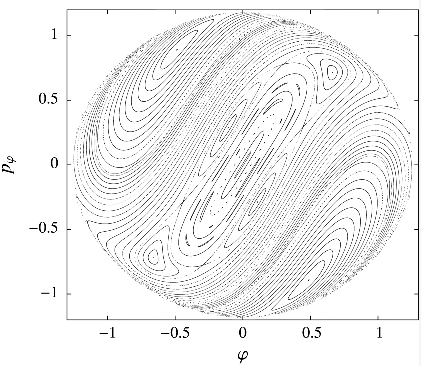
(a) section plane $r = 1$ with coordinates (φ, p_φ)



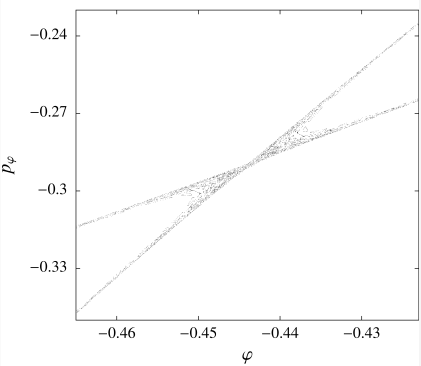
(b) magnification of region around unstable periodic solution

Figure : Poincaré cross sections on energy level $E = -0.5$ for Hamiltonian system given by (**??**) with $m = -2, k = 1$ corresponding to case 6

Example 2. Non-integrable case 7



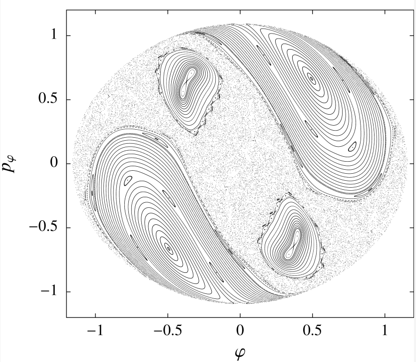
(a) section plane $r = 1$ with coordinates (φ, p_φ)



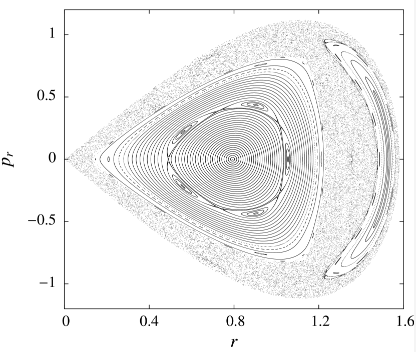
(b) magnification of region around unstable periodic solution

Figure : Poincaré cross sections on energy level $E = -0.3$ for Hamiltonian system given by (??) with $m = 2, k = -5$ corresponding to case 7

Example 2. Non-integrable cases for family $k = -2(mp + 1)$



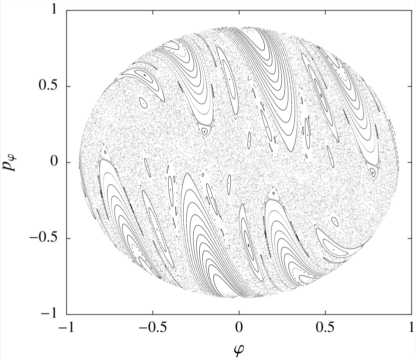
(a) section plane $r = 1$ with coordinates (φ, p_φ)



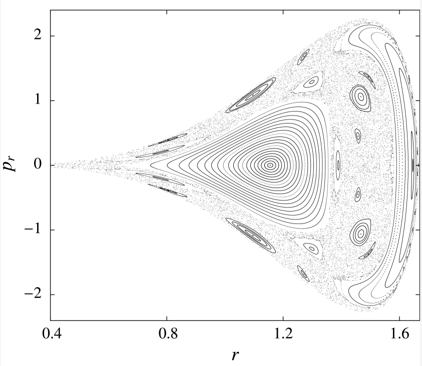
(b) section plane $\varphi = 0$ with coordinates (r, p_r)

Figure : Poincaré cross sections on energy level $E = -0.5$ for Hamiltonian system given by (**??**) with $m = -2, k = 2$

Example 2. Non-integrable cases for family $k = -2(mp+1)$



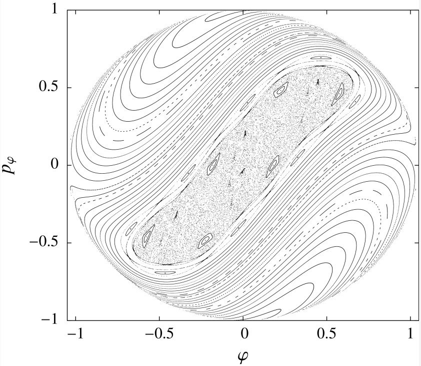
(a) section plane $r = 1$ with coordinates (φ, p_φ)



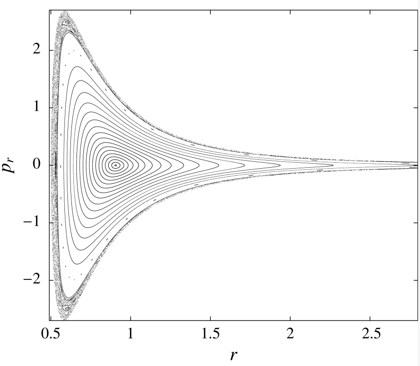
(b) section plane $\varphi = 0$ with coordinates (r, p_r)

Figure : Poincaré cross sections on energy level $E = -0.6$ for Hamiltonian system given by (??) with $m = -1, k = 8$

Example 2. Non-integrable cases for family $k = -2(mp + 1)$



(a) section plane $r = 1$ with coordinates (φ, p_φ)



(b) section plane $\varphi = 0$ with coordinates (r, p_r)

Figure : Poincaré cross sections on energy level $E = -0.5$ for Hamiltonian system given by (??) with $m = 1, k = -6$