The Banach Center Conference

Formal and Analytic Solutions of Differential and Difference Equations

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Abstracts of lectures and talks

Nour Eddine Askour (Moulay Slimane University, Morocco)

Formal solutions of exponential type for ordinary differential equations near irregular singular points

For any fixed slope $\alpha > 0$ of the Newton diagram at the origin $z = 0$ of a linear differential operator $P(z, \frac{d}{dz})$ with polynomial coefficients, and for each simple root λ of the α-indicial equation, associated to the differential equation

$$P(z, \frac{d}{dz})U(z) = 0, \quad z \in \mathbb{C},$$

we construct a formal solution of exponential type:

$$U(z) = \exp\left(\sum_{j=0}^{+\infty} \lambda_k \phi_{\alpha_k}\right),$$

where $\phi_\alpha(z) = \frac{z^{-\alpha}}{-\alpha})$ if $\alpha \neq 0$ and $\phi_0(z) = \log(z)$ with $(\alpha_0, \lambda_0) = (\alpha, \lambda)$ and the sequence $(\alpha_k, \lambda_k)_{k\geq 1}$ is given by a simple linear algorithm.

For a second order differential equation with α–indicial equation having two different roots, we give two linearly independent formal solution of exponential type. Also, we present a computer program which enables us to compute these solutions.

Applications to confluent cases of Heun equations, Bessel equations and Airy equations are also discussed.

Werner Balser (University of Ulm, Germany)

Semi-formal Stokes phenomenon for non-linear systems of ordinary differential equations

Fairly general ν-dimensional non-linear systems have solutions that are formal power series in ν free parameters. The coefficients of these series are multisums of certain formal logarithmic-exponential expressions that exhibit a Stokes phenomenon which is discussed in the lecture.

Marco Bertola (Concordia University, Canada)

Fredholm determinants and noncommutative Painlevé II

The connection between Fredholm determinants of integral operators and Painlevé equations was observed in statistical mechanics and its most famous example is the Tracy–Widom distribution, connecting the distribution of the largest eigenvalue of a random matrix and the second Painlevé equation. In this joint work with M. Cafasso I will show how the Fredholm determinant of a matrix-valued integral operator can be related to a Riemann–Hilbert problem following the theory developed by Its-Izergin-Korepin-Slavnov. By appropriate choice of the operator we construct a pole-free solution of the noncommutative second Painlevé equation, recently introduced by Retakh and Rubtsov.

Yuliya Bibilo (HSE, Russia)

On the Malgrange isomonodromic deformations of non-resonant meromorphic connections

Movable singularities of the equations governing the Malgrange isomonodromic deformation of a non-resonant rank 2 meromorphic connection are studied: we describe the theta-divisor of the deformation and estimate orders of movable poles of the equations in the case of irreducible monodromy.

Thomas Bothner (Purdue University, USA)

Asymptotics for the second Painlevé equation

This talk discusses joint work with Alexander Its on the asymptotics of real valued singular solutions of the second Painlevé equation. Based on the Riemann-Hilbert approach we reestablish Kapaev's result, which was obtained in the past using WKB analysis for the direct monodromy problem. Here for the first time the relevant inverse problem is investigated rigorously. The asymptotic analysis is based on the Deift-Zhou nonlinear steepest descent method. We will highlight certain new technical features in the implementation of the method related to the presence of the poles in the solution.

Alexander Bruno (Keldish Institute, Russia)

The next level of Differential Calculus

We develop a new Calculus based on Power Geometry. It allows to compute local and asymptotic expansions of solutions to equations of three classes: (A) algebraic, (B) ordinary differential, (C) partial differential, as well as to systems of such equations. Principal ideas and algorithms are common for all classes of equations. But for each class of equations, there are additional algorithms, since the complexity of solutions increases for equations of successive classes.

Thus, at the singular point $x = y = 0$ all solutions of the algebraic equation $f(x, y) = 0$ are expanded into series of the form

$$y = \sum c_s x^s, \tag{1}$$

where exponents $s > 0$ are rational, and coefficients c_s are constants. Solutions to the ODE $f(x, y, y' \ldots, y^{(n)}) = 0$, where f is a polynomial of its arguments, have the following four types of expansions of the form (1).

Type 1. *Power*, with complex s, $|\text{Im } s/\text{Re } s| <$ const and with constant c_s.

Type 2. *Power-logarithmic*, where s are as before, but c_s are polynomial in $\log x$.

Type 3. *Complicated*, where s are as before, but c_s are series in decreasing powers of $\log x$.

Type 4. *Exotic*, where s are real, but c_s are series in powers of x^i.

In addition, there are *exponential* expansions (type 5)

$$y = \sum_{k=0}^{\infty} b_k(x) C^k e^{k\varphi(x)},$$

where $b_k(x)$ and $\varphi(x)$ are power series, but C is arbitrary constant; and other types.

Applications. *Class A.* 1. Sets of stability of multiparameter problems. *Class B.*
2. Asymptotic forms and expansions of solutions to the Painlevé equations. 3. Periodic motions of satellite around its mass center moving around an elliptical orbit. 4. New properties of motion of a top. 5. Families of periodic solutions of the restricted three-body problem and distribution of asteroids. 6. Integrability of ODE systems. *Class C.* 7. Boundary layer on a needle. 8. Evolution of the turbulent flow.

Peter Clarkson (University of Kent, United Kingdom)

The Painlevé equations – nonlinear special functions

In this talk I shall give an overview of the Painlevé equations, which might be thought of nonlinear special functions, and discuss some of their properties. Further I shall discuss some of the "Painlevé Challenges", i.e. open problems in the field of Painlevé equations.

Ovidiu Costin (Ohio State Univ., USA)

Generalized transseries and global asymptotics of ODEs

I will describe recent developments in understanding "large" solutions of nonlinear ODEs, for which no transseries exist. The theory of Borel summed transseries provides a way to extend the asymptotic theory to the general, full parameter solution of ODEs.

Rodica D. Costin (Ohio State Univ., USA)

Differential systems with Fuchsian linear part: correction and linearization, normal forms and matrix valued orthogonal polynomials

The question of analytic equivalence of equations in regions with more than one singularity arises naturally in the study of integrability, as well as in a class of important problems in scattering theory, in quantum mechanics. Nonlinear perturbations of linear equations with one regular singular point are linearizable (generically) in a disk (or, at least, in an annulus) surrounding the singular point (therefore, they are equivalent). This turns out not to be necessarily true in regions containing two, or more, singularities. However, it is shown that there exists a unique correction of the nonlinear part, after which the corrected system becomes linearizable. The classification of systems and the question of convergence are also discussed.

Galina Filipuk (Warsaw University, Poland)

Semi-classical discrete orthogonal polynomials and the Painlevé equations

In the talk I shall present the latest results concerning the connection of the semi-classical discrete orthogonal polynomials and the Painleve equations, solutions of which are nonlinear special functions.

Pedro Fortuny Ayuso (University of Oviedo, Spain)

Power series solutions of first order and first degree non-linear q-difference equations

Using the Newton-Puiseux Polygon, we show an algorithm for computing generalized power series solutions of q-difference equations of arbitrary order and degree (when they exist). This method lets us bound by 2 the rational rank of those solutions in the case of order and degree 1, and to give estimates for the q-Gevrey order of those solutions in terms of the q-Gevrey order of the original equation.

Irina Goryuchkina (Keldysh Institute of Applied Mathematics, Russia)

Asymptotic forms and asymptotic expansions of solutions to the Painlevé equations

1. We found 117 families of asymptotic expansions of solutions to the sixth Painlevé equation (P6) near its 3 singular points for all values its 4 complex parameters. Among them there are power, power-logarithmic, com-

plicated and exotic expansions.

2. We found 17 families of solutions to the P6 equation near its regular points.
3. We found the elliptic asymptotic forms of solutions with 2 parameters to the P1 – P4 equations (for P1 they coincide with Boutroux asymptotic forms.
4. We shown that P6 has not elliptic asymptotic forms.
5. We proved two theorem on convergence of formal solutions to ODE.

Anton Grigor'ev (Belarus State University, Belarus)

Rational solutions of the fourth and fifth Painlevé hierarchies

We will consider four hierarchies of higher order analogues of the fourth and fifth Painlevé equations. The necessary and sufficient conditions for having rational solutions will be presented. Also the algorithm for obtaining such solutions will be described.

Renat Gontsov (Russian Academy of Science, Russia)

On the Malgrange isomonodromic deformations of non-resonant meromorphic connections (this is a joint work with Yu.Bibilo)

Movable singularities of the equations governing the Malgrange isomonodromic deformation of a non-resonant rank 2 meromorphic connection are studied: we describe the theta-divisor of the deformation and estimate orders of movable poles of the equations in the case of irreducible monodromy.

Rod Halburd (University College London, United Kingdom)

Movable singularities of nonlinear ODEs

A movable singularity of a solution of an ordinary differential equation is a singularity whose location is determined by initial conditions rather than by some kind of singular behaviour of the equation itself. Movable singularities play an important role in integrable systems. In this talk we consider classes of ordinary differential equations for which we can show that the only movable singularities that can be reached by analytic continuation along finite length curves are algebraic. Although the local singularity structure of these solutions are simple, the global structure can be very complicated.

Yoshishige Haraoka (Kumamoto University, Japan)

Regular coordinates and reduction of deformation equations for Fuchsian systems

Let $t_1, t_2, \ldots, t_p$ be distinct points in $\mathbb{C}$, and $A_1, A_2, \ldots, A_p \in \mathrm{Mat}(n \times n; \mathbb{C})$. The deformation equation for the Fuchsian system

$$\frac{dY}{dx} = \left(\sum_{j=1}^{p} \frac{A_j}{x - t_j} \right) Y$$

is given by the Schlesinger system, which is a system of differential equations for the entries of the residue matrices $A_1, \ldots, A_p$. Since the rank of the deformation equation is the number of the accessory parameters of the Fuchsian system, Schlesinger system has too much unknowns.

In this talk, we propose a good coordinate for the accessory parameters of the Fuchsian system. Let α be the number of the accessory parameters. A set of α variables $(z_1, z_2, \ldots, z_\alpha)$ is called a *regular coordinate* if all the entries of the residue matrices $A_1, \ldots, A_p$ are rational functions of $(z_1, \ldots, z_\alpha)$. Then the deformation equation becomes a system of equations for the regular coordinate. Moreover, for two Fuchsian systems with distinct spectral types, we may have a reduction of the deformation equations for these systems when they are written in the regular coordinates. An example of the reduction is to obtain the Painlevé equation from the Garnier system in two variables with special values of parameters.

Stefan Hilger (Catholic University of Eichstaett, Germany)

Jacobi polynomials in SIE representations of quivers

We will explain the notion of quivers and their representations with Scalar Intrinsic Endomorphisms (SIEs). We then present several theorems that ensure the SIE property of a representation.

Among others the "Jacobi grid quiver" will serve as a main example for the presented theory.

Kunio Ichinobe (Aichi University of Education, Japan)

Decomposition of solutions of the Cauchy problem of a quasi homogeneous partial differential equation This is a joint work with M. Miyake.

We give a decomposition formula of formal solution of the Cauchy problem for a quasi homogeneous partial differential equation with constant coefficients in two dimensional complex plane. The decomposition formula is given in a form associated with the factorization of the operator which is similar with decomposition of solution of an ordinary differential equation with constant coefficients.

Hideaki Izumi (Chiba Institute of Technology, Japan)

Analytic solutions of iterative functional equations

Firstly, we will present an algorithm which gives a formal solution to a functional equation of the form $\sum_{j=1}^{n} a_i f^i(x) = bx + \sum_{k=1}^{\infty} c_k \exp(d_k x)$, where f is unknown, $f^i(x)$ is the i-th iterate of $f(x)$, $b > 0$, a_i, c_k are constants and d_k are either all positive or all negative constants. For example, the solutions to $f(f(x)) = x + \exp(x)$ and $f(x) + f(f(x)) + f(f(f(x))) = x - \exp(-x) - \exp(-2x)$ are given. In this algorithm, the linear term bx of positive coefficient plays a crucial role. Moreover, real one-parameter groups $\{f_t(x)\}$ of functions which satisfy the composition law $f_t \circ f_s = f_{t+s}$ for $f_1(x) = bx + \sum_{k=1}^{\infty} c_k \exp(d_k x)$ are given explicitly. Secondly, we will solve implicit functions consisting of exponentials by using infinite-fold exponentials. For example, $y^y = x^{y^x}$ or $x^{y^{2x}} = y^{y^3}$ are solved explicitly with respect to y in the form of infinite-fold exponentials involving only x.

Yuri Kosovtsov (Lviv Radio Engineering Research Institute, Ukraine)

Formal exact operator solutions to nonlinear differential equations

The compact explicit expressions for formal exact operator solutions to Cauchy problem for sufficiently general systems of nonlinear differential equations (ODEs and PDEs) in the form of chronological operator exponents are given. The variant of exact solutions in the form of ordinary (without chronologization) operator exponents are proposed. (See arXiv:0910.3923v1 [math-ph].)

Arno Kuijlaars (Leuven, Belgium)

Orthogonal polynomials in the normal matrix model

Joint work with Pavel Bleher (Indianapolis).

The normal matrix model is a random matrix model defined on complex matrices. The eigenvalues in this model fill a two-dimensional region in the complex plane as the size of the matrices tends to infinity. Orthogonal polynomials with respect to a planar measure are a main tool in the analysis. In many interesting cases, however, the orthogonality is not well-defined, since the integrals that define the orthogonality are divergent. I will present a way to redefine the orthogonality in terms of a well-defined Hermitian form. This reformulation allows for a Riemann-Hilbert characterization. For the special case of a cubic potential it is possible to do a complete steepest descent analysis on the Riemann-Hilbert problem, which leads to strong asymptotics of the orthogonal polynomials, and in particular to the two-dimensional domain where the eigenvalues are supposed to accumulate. At a critical time, the boundary of the domain develops a cusp. The local behavior at the cusp point is described by a special solution of the Painlevé I equation.

Oleksandra Kukharenko (National University of Kyiv, Ukraine)

Representation of the solution for systems of linear partial differential equations with constant delay

Systems of the linear stationary partial differential equations with one constant delay are considered in this report:

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} = a_{11}\frac{\partial^2 u(x,t-\tau)}{\partial x^2} + a_{12}\frac{\partial^2 v(x,t-\tau)}{\partial x^2} + b_{11}u\left(x,t\right) + b_{12}v\left(x,t\right), \\ \frac{\partial^2 v(x,t)}{\partial t^2} = a_{21}\frac{\partial^2 u(x,t-\tau)}{\partial x^2} + a_{22}\frac{\partial^2 v(x,t-\tau)}{\partial x^2} + b_{21}u\left(x,t\right) + b_{22}v\left(x,t\right). \end{cases}$$

A solution of the first boundary value problem is obtained for a case, when eigenvalues of the matrixes of coefficients are real and different. The method of separation variables is used for solving. A solution of the corresponding Cauchy problem for delay equations is obtained in an analytical form.

Radoslaw Antoni Kycia (Jagiellonian University, Poland)

Self-similar solutions of semilinear wave equations

Self-similar solutions play very important role in singularity formation of many nonlinear PDEs. After a short introduction to this phenomenon existence and analytic properties of self-similar solutions of semilinear wave equations will be presented.

Alberto Lastra (University of Lille, France)

On $q-$asymptotics for $q-$difference-differential equations with Fuchsian and irregular singularities

We consider a Cauchy problem for some family of $q-$difference-differential equations with Fuchsian and irregular singularities, that admit a unique formal power series solution in two variables $\hat{X}(t,z)$ for given formal power series initial conditions. Under suitable conditions and by the application of certain $q-$Borel and Laplace transforms (introduced by J.-P. Ramis and C. Zhang), we are able to deal with the small divisors phenomenon caused by the Fuchsian singularity, and to construct actual holomorphic solutions of the Cauchy problem whose $q-$asymptotic expansion in t, uniformly for z in the compact sets of $\mathbb{C}$, is $\hat{X}(t,z)$. The small divisors's effect is an increase in the order of $q-$exponential growth and the appearance of a power of the factorial in the corresponding $q-$Gevrey bounds in the asymptotics.

Donald Lutz (San Diego State University, USA)

Liouville/Green/Olver asymptotics for solutions of some second order linear difference equations. Joint work with S. Bodine.

L/G/O theory for second order differential equations is well established and has many applications for special functions. Second order linear difference equations correspond to 3-term linear recurrence relations, which arise in the study of orthogonal polynomials. The goal of this talk is to analyze several types of asymptotic results which have appeared in the literature and compare those ad hoc results with a more general approach based on asymptotic factorization of fundamental matrices.

Alexander P. Lyapin (Siberian Federal University, Russia)

A multidimensional analogue of the Moivre theorem

Moivre considered the recursive series as the power series $F(z) = a_0 + a_1 z + \ldots + a_k z^k + \ldots$ with the constant coefficients $a_0, a_1, \ldots$ that make recursive sequence (a_n) satisfying the difference equation $c_0 a_{m+p} + c_1 a_{m+p-1} + \ldots + c_m a_p = 0$ with some constants $c_j \in \mathbb{C}$. In 1722 he proved that the series $F(z)$ are rational functions.

We consider the multidimensional difference equations $\sum\limits_{\alpha \in A} c_\alpha f(x+\alpha) = 0$ with constant coefficients $c_\alpha \in \mathbb{C}$, where A is the finite subset of the integer lattice $\mathbb{Z}^n_+$ and function $f : \mathbb{Z}^n_+ \to \mathbb{C}$. For each such equation we can formulate the Cauchy problem assuming that the function f is defined on special infinite subset along the axes (the ¡¡initial¿¿ data).

The aim of this research is to find the generating function (GF) of the solution of the Cauchy problem for a multidimensional difference equation with constant coefficients. Namely, under certain restrictions on the difference equation we establish the dependence between the GF of the initial data and the GF of the solutions to the Cauchy problem of the difference equation under study. As a consequence, we prove that the GF of the solution to the difference equation is rational if and only if the GF of the initial data is rational.

These results are used to solve certain problems in enumerative combinatorial analysis.

Grzegorz Łysik (Polish Academy of Sciences, Poland)

Mean-values and analytic solutions of the heat equation

We give an extension of the mean-value property and its converse to the case of real analytic functions and to functions of Laplacian growth. As an application we give a characterization of analytic solutions in time variable of the initial value problem to the heat equation $\partial_t u = \Delta u$ in terms of holomorphic properties of the solid and/or spherical means of the initial data.

Stephane Malek (University of Lille, France)

On singularly perturbed q-difference-differential equations with irreg-

ular singularity

We study a q-analog of a singularly perturbed Cauchy problem with irregular singularity in the complex domain. We construct solutions of this problem which are holomorphic on open half q-spirals. Using a version of a q-analog of the Malgrange-Sibuya theorem obtained by J-P. Ramis, J. Sauloy and C. Zhang, we show the existence of a formal power series solution in the perturbation parameter which is the q-asymptotic expansion of these holomorphic solutions.

SLAWOMIR MICHALIK (UKSW, Poland)

Analytic solutions of moment-PDEs

In the talk we study the Cauchy problem for linear moment-PDEs (introduced recently by W. Balser and M. Yoshino) with constant coefficients. We construct the integral representation of the solution to the problem and we show when the solution is analytic. As a consequence we also obtain a characterization of summable formal solutions of the Cauchy problem.

MASATAKE MIYAKE (Nagoya University, Japan)

Reduction into Hukuhara-Turrittin's canonical form of a singular system of ordinary differential equations

We introduce a notion of T-expansion for matrix functions which is different from the usual Taylor expansion. By using this idea we can show an algorithm of reduction procedure into an irreducible decomposition and also into Hukuhara-Turrittin's canonical form of a system in explicit form. This enable us easy to know the structure of fundamental matrix solution and its multi-summability property.

YASUNORI OKADA (Chiba University, Japan)

A notion of boundedness for hyperfunctions and Massera type theorems

For some classes of periodic linear ordinary differential equations and functional equations, it is known that the existence of a bounded solution in the future implies the existence of a periodic solution. In this talk, we study hyperfunction solutions to some classes of periodic linear functional equations, and we are interested to see if there is a counterpart in our setting.

For this purpose, we introduce a notion of boundedness at infinity for univariate hyperfunctions, and translate our problem into the existence of periodic analytic solutions to equations in complex domains. We also mention vector valued cases.

Satyendra Nath Pandey (National Institute of Technology, Allahabad, India)

Integrability aspects of the general damped nonlinear oscillators and systems

Identifying integrable nonlinear differential equations and exploring their under- lying solutions is one of the challenging problems in nonlinear dynamical systems. Different methods have been proposed in order to identify new integrable cases and to understand the underlying dynamics associated with the finite dimensional nonlinear dynamical systems. The most widely used methods include Painlevé analysis, Lie symmetry analysis, Noether's theorem and direct linearization etc. In this paper, we consider a general damped second-order nonlinear ordinary differential equation of the form $\ddot{x} + k_1 \frac{\dot{x}^2}{x} + (k_2 x^2 + k_3)\dot{x} + k_4 x^4 + k_5 x^3 + k_6 x^2 = 0$, where over dot denotes differentiation with respect to t, and $k_i^{'} s, i = 1, 2, 3, 4, 5$ and 6 are arbitrary parameters. It is interesting to see that the above equation includes a large number of physically important nonlinear oscillators and systems. We investigate the integrability properties of this equation with the help of recently introduced semialgorithm to find elementary first integrals of a class of rational second order ordinary differential equations. Further, we prove the integrability of all the equations obtained in the present paper either by obtaining the integrating factors, integrals of motion or by constructing time independent Hamiltonians (Liouville integrability). Several of these equations are being identified for the first time.

Anastasia Parusnikova (Lomonosov State University, Russia)

Asymptotic expansions of solutions to the fifth Painlevé equation

The aim of the present work is to find all asymptotic expansions of solutions to the the fifth Painlevé equation (P5) near its nonsingular points ($z = 0$ and $z = \infty$) and near its nonsingular points. By means of Power Geometry we are looking for the expansions of the following form near the singular points of the equation:

$$w = c_r(z)z^r + \sum_{s\in\mathbf{K}} c_s(z)z^s, \qquad (2)$$

where $c_r(z), c_s(z), r, s \in \mathbb{C}$, $\mathbf{K} \subset \{\ s \mid \text{Re } s > \text{Re } r\}$ for the expansions in the neighbourhood of $z = 0$ and $\mathbf{K} \subset \{\ s \mid \text{Re } s < \text{Re } r\}$ for the expansions in the neibourhood of $z = \infty$; the set $\mathbf{K}$ is countable. We obtain the expansions (2) of the following five types:

1. $c_r(z)$ and $c_s(z)$ are constant.
2. $c_r(z)$ is constant, $c_s(z)$ are polynomials in $\log z$.
3. $c_r(z)$ and $c_s(z)$ are power series in $\log z$.
4. $r,\ s \in \mathbb{R}$, $c_r(z)$ and $c_s(z)$ are series in z^i, and c_r is a sum of countable number of terms, the set of power exponents of z^i in c_r is bounded either from above, or from below.

5. $r,\ s \in \mathbb{R}$, $c_r(z)$ is a finite sum of powers of z^i with complex coefficients and $c_s(z)$ are power series over z^i.

The expansions in the neigbourhood of the nonsingular points $z = \infty$ and $z = 0$ form 16 and 30 families correspondingly.

In the neighbourhood of the nonsingular point $z = z_0$ of the P5 equation there exist 10 families of asymptotic expansions of its solutions (they are Laurent and Taylor series). The expansions of all 10 families converge in the neighbourhood (or deleted neighbourhood for Laurent series) of $z = z_0$.

Over 20 families obtained are new.

Akira Shirai (Jogakuen University, Japan)

Convergence of the formal solutions of 1st order singular partial differential equations of nilpotent type

We shall study the solvability of singular first order partial differential equations of nilpotent type. A typical example is the following:

$$P(X, \partial_X)u(X) := (y\partial_x - z\partial_y)u(x, y, z) = f(x, y, z) \in \mathcal{O}_X$$

where $X = (x, y, z) \in \mathbb{C}^3$, $\partial_X = (\partial_x, \partial_y, \partial_z)$.

In this talk, we shall study the conditions that the formal solutions of PDEs of nilpotent type converge.

Christophe Smet (Katholieke Universiteit Leuven, Belgium)

Discrete Painlevé equations, satisfied by the recurrence coefficients of orthogonal polynomials on a bi-lattice

We investigate generalizations of the Charlier and the Meixner polynomials on the lattice $\mathbb{N}$ and on the shifted lattice $\mathbb{N}+1-\beta$. We combine both lattices to obtain the bi-lattice $\mathbb{N}\cup(\mathbb{N}+1-\beta)$ and show that the orthogonal polynomials on this bi-lattice have recurrence coefficients which satisfy a non-linear system of recurrence equations, which we can identify as a limiting case of an (asymmetric) discrete Painlevé equation.

Catherine Stenger (University of La Rochelle, France)

On complex singularity analysis of holomorphic solutions of linear partial differential equations

We construct formal series solutions of linear partial differential equations as linear combinations of powers of solutions of first order nonlinear differential equations, following the classical tanh method. We give sufficient conditions under which the constructed formal series define holomorphic functions on some punctured polydiscs of $\mathbb{C}^2$. Moreover, we study the rate of growth of these solutions near their singular points.

Tetyana Stulova (University of Kharkiv, Ukraine)

Gevrey class operator series and an integral representation of entire solutions of some differential equations in a Banach space

Joint work with Sergey Gefter.

Let T be a bounded linear operator in a Banach space E and $g : E \to \mathbb{C}$ be an entire function. We study entire solutions of the following inhomogeneous implicit linear differential equation

$$Tw' + g(z) = w, \tag{3}$$

Theorem 1 *Let $\rho(T)\,\sigma(g) < 1$, where $\rho(T)$ is a spectral radius T and $\sigma(g)$ is exponential type of g. Then Equation (3) has the unique entire solution $w(z) = \sum\limits_{n=0}^{\infty} T^n g^{(n)}(z)$ of exponential type $\sigma(g)$.*

Theorem 2. *Let the condition of Theorem 1 be fulfilled, $w(z)$ be a solution of Equation (3),*
$\mathcal{E}_T(\zeta) = \sum\limits_{n=0}^{\infty} \frac{n! T^n}{\zeta^{n+1}}$ be the Borel-Laplace formal transformation of the Fredholm resolvent of the operator T and $\mathcal{E}_T(\zeta - z) \overset{def}{=} \sum\limits_{n=0}^{\infty} \frac{n! T^n}{\zeta^{n+1}} \left(\sum\limits_{j=0}^{\infty} \left(\frac{z}{\zeta}\right)^j \right)^{n+1}$. Then the product $\mathcal{E}_T(\zeta - z)\ g(\zeta)$ is well defined as an element of the space $E[[z]]\left[\left[\zeta, \frac{1}{\zeta}\right]\right]$, that is the product is a formal Laurent series with respect to ζ, that the coefficients are vector formal power series to powers of z and

$$w(z) = \frac{1}{2\pi i} \oint \mathcal{E}_T(\zeta - z)\, g(\zeta)\, d\zeta,$$

where the integral in $E[[z]]\left[\left[\zeta, \frac{1}{\zeta}\right]\right]$ is defined in the following way

$$\oint \left(\sum_{m=-\infty}^{+\infty} c_m(z)\, \zeta^m \right) d\zeta \overset{def}{=} 2\pi i c_{-1}(z)$$

Hidetoshi Tahara (Sophia University, Japan)

Maillet type theorem and Gevrey regularity in time of solutions to nonlinear partial differential equations

I will consider the nonlinear partial differential equation

$$(E) \qquad t^\gamma (\partial/\partial t)^m u = F(t, x, \{(\partial/\partial t)^j (\partial/\partial x)^\alpha u\}_{j<m, |\alpha| \le L})$$

(with $\gamma \ge 0$ and $1 \le m \le L$) and show the following two results: (1)(Maillet type theorem) if (E) has a formal solution it is in some formal Gevrey class, and

(2)(Gevrey regularity in time) if (E) has a solution $u(t,x) \in C^\infty([0,T], G_\sigma(V))$ it is in some Gevrey class also with respect to the time variable t. It will be explained that the mechanism of these two results are quite similar, and so the argument in (1) can be applied to (2) by a small modification.

Kouichi Takemura (Chuo University, Japan)

Introduction to middle convolution for differential equations with irregular singularities

Middle convolution was originally introduced by Katz, and Dettweiler and Reiter reformulated it for systems of Fuchsian differential equations in terms of linear algebras. In the talk, we introduce middle convolution for systems of linear differential equations with irregular singular points and we presend a tentative definition of the index of rigidity for them. (arXiv:1002.2535)

Alexander Tovbis (University of Central Florida, USA)

Universality of transitions at the point of gradient catastrophe for some integrable systems and some orthogonal polynomials

By a point of gradient catastrophe we mean a point where the leading order asymptotic behavior loses smoothness (for example, derivatives are not square integrable). In the case of the small dispersion (semiclassical) focusing NLS, this is a point where a slowly modulated high frequency plane wave wave suddenly burst into rapid amplitudial oscillations (spikes). Adjusting the nonlinear steepest descent (Deift-Zhou) method for Riemann-Hilbert problems, we give complete description of the leading order term near the point of gradient catastrophe in terms the *tritronquée* solution to the Painlevé I and rational breathers for the NLS. In fact, each spike corresponds to a pole of the *tritronquée* and has the universal shape of a scaled rational breather. Similar phenomenon was recently studied (work in progress) for the asymptotic of orthogonal polynomials with complex varying weight $e^{-N(\frac{1}{2}z^2+\frac{1}{4}tz^4)}$ near the critical value of the parameter t. In this case the spikes in the asymptotics for the recurrence coefficients turn out to be bounded for $t \in \mathbb{R}$ but unbounded for complex t. This is a joint work with Marco Bertola.

Kseniia Verbinina (V. Karazin Kharkiv National University, Ukraine)

The differential operators of infinite order based on the Gevrey formal power series

We are interested in differential operators of infinite order $\varphi\left(\frac{d}{dz}\right) g = \sum_{n=0}^{\infty} a_n g^{(n)}$ in the spaces of entire functions, where $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to some Gevrey class.

Definition 1. *Let* $f(z)=\sum\limits_{n=0}^{\infty} f_n z^n \in \mathbb{C}[[z]]$. *We say that* f *is of a Gevrey order* $\beta \geq 0$ *and a type at most* M, *if* $|f_n| \leq CM_1^n (n!)^\beta$ *for some* $C>0$ *and every* $M_1 > M$, $n \geq 0$. *The space of all such Gevrey series we denote by* $\mathbb{C}[[z]]_{\beta,M}$.

Then the following theorems take place:

Theorem 1. *Suppose that* $\varphi(z)=\sum\limits_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]]_{\beta,M}$ *and* $g(z)=\sum\limits_{n=0}^{\infty} c_n z^n$ *is an entire function of an order* ρ *and a type* σ. *Let one of the following conditions holds true: i)* $\rho(\beta+1)<1$ *; ii)* $\rho(\beta+1)=1$ *and* $(\sigma\rho)^{\frac{1}{\rho}} M < 1$. *Then the series* $\sum\limits_{n=0}^{\infty} a_n g^{(n)}(z)$ *is convergent in* $\mathbb{C}$, *and its sum* $w(z)$ *is an entire function of an order at most* ρ.

In addition, we obtain a few integral representations for $\sum\limits_{n=0}^{\infty} a_n g^{(n)}(z)$.

Definition 2. *Let* V *be a vector space. We suppose to take for* $V[[\zeta,\frac{1}{\zeta}]]$ *the space of all formal Laurent series with coefficients from* V. *For* $g(\zeta)=\sum\limits_{n=-\infty}^{+\infty} b_n \zeta^n \in V[[\zeta,\frac{1}{\zeta}]]$ *set*

$$\oint g(\zeta)d\zeta \stackrel{def}{=} 2\pi i b_{-1}. \tag{4}$$

We call the last linear map from $V[[\zeta,\frac{1}{\zeta}]]$ *into* V *the Cauchy-Laurent integral.*

Definition 3. *To a formal power series* $f(z)=\sum\limits_{n=0}^{\infty} c_n z^n$ *we associate its Laplace-Borel transform, which is the formal Laurent power series*

$$F(s)=\sum_{n=1}^{\infty} \frac{(n-1)!c_{n-1}}{s^n}.$$

Theorem 2. *Under the conditions and notations from Theorem 1 one can represent the function* $w(z)$ *in the following forms:*

$$w(z)=\frac{1}{2\pi i}\oint \Phi(s-z)g(s)ds=\frac{1}{2\pi i}\oint e^{zs}\varphi(s)G(s)ds,$$

where Φ, G *are respectively the Laplace-Borel transforms of* φ *and* g, *all integrands are well-defined as elements of* $\mathbb{C}[[z]][[s,\frac{1}{s}]]$, *and the integrals are Cauchy-Laurent ones (4).*

Corollary 1 (Parseval's identity). *Under the conditions from Theorem 1 and the notations from Theorem 2 the following equality holds true:*

$$\oint \Phi(s)g(s)ds=\oint \varphi(s)G(s)ds.$$

FREEK VERSTRINGE (University of Hasselt, Belgium)

Every Gevrey-α vector field with nilpotent linear part admits a Gevrey-$(1+\alpha)$ normal form

We explore the convergence/divergence of the normal form for a singularity of a vector field on $\mathbb{C}^n$ with nilpotent linear part. We prove that a Gevrey-α vector field X with a nilpotent linear part can be reduced to a normal form of Gevrey-$(1+\alpha)$ type with the use of a Gevrey-$(1+\alpha)$ transformation

HIDESHI YAMANE (Kwansei Gakuin University, Japan)

Nonlinear Cauchy problems with small analytic data

We study the lifespan of solutions to fully nonlinear second-order Cauchy problems with small real- or complex-analytic data. The nonlinear term is an analytic function not only in ∇u and $\nabla^2 u$ but also in u, $\partial_t u$ and $\partial_t \nabla u$. Moreover, we can deal with equations involving the modulus of the unknown function like $(\partial_t^2 - \partial_x^2)u = |\partial_x u|^2 = \partial_x u\, \overline{\partial_x u}$.

HIROSHI YAMAZAWA (Caritas Junior College, Japan)

Borel summability of a formal solution for $\frac{\partial}{\partial t}u(t,x) = (\frac{\partial}{\partial x})^2 u(t,x) + t(t\frac{\partial}{\partial t})^3 u(t,x)$

We study the following initial value problem:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) &= (\frac{\partial}{\partial x})^2 u(t,x) + At(t\frac{\partial}{\partial t})^3 u(t,x) \\ u(0,x) &= \phi(x) \end{cases} \tag{5}$$

where $(t,x) \in \mathbf{C} \times \mathbf{C}$ and $A \in \mathbf{C}$.

We will consider a formal power series solution $\tilde{u}(t,x) = \sum_{i=0}^{\infty} u_i(x) t^i$ of the equation (5).

The formal solution $\tilde{u}(x,t)$ diverges for general initial value. In the case of $A = 0$, the equation (5) is the heat equation, and we have a result of Lutz-Miyake-Schäfke for Borel summability of the formal solution $\tilde{u}(t,x)$. For a general equation, we have a result of Balser by using an idea of a normalized formal solution.

Here we consider Borel summability of the formal solution $\tilde{u}(t,x)$ in the case of $A \neq 0$. We get a result of Borel summability under the condition that the initial value $\phi(x)$ is an entire function of exponential order at most 2, that is, the following inequality holds for some positive constants C and c,

$$|\phi(x)| \le C \exp c|x|^2, \ x \in \mathbf{C}.$$

In the case without a term $(\frac{\partial}{\partial x})^2 u(t,x)$, we have a result of Ōuchi for some inhomogeneous linear and nonlinear partial differential equations. Ōuchi showed (Multi) summability of the formal solution $\tilde{u}(t,x)$ by using the Borel transform and the Laplace transform.Our proof method adopts some ideas of Balser and Ōuchi.

Masafumi Yoshino (Hiroshima University, Japan)

Summability of first integrals of a resonant Hamiltonian system

In this talk we consider a resonant Hamiltonian system

$$\dot{q} = \nabla_p H, \quad \dot{p} = -\nabla_q H, \tag{6}$$

where $H = H(q,p)$ is a Hamiltonian function and $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$ are the variables in $\mathbb{R}^n$ or in $\mathbb{C}^n$ $(n \geq 2)$. We denote the Hamiltonian vector field χ_H by

$$\chi_H := \{H, \cdot\} = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \right), \tag{7}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. We say that ϕ is a first integral of χ_H if $\chi_H \phi = 0$. Eq. (1) is said to be C^ω-Liouville integrable if there exist first integrals $\exists \phi_j \in C^\omega$ $(j = 1, \ldots, n)$ which are functionally independent on an open dense set and Poisson commuting, i.e., $\{\phi_j, \phi_k\} = 0$, $\{H, \phi_k\} = 0$. If $\phi_j \in C^\infty$ $(j = 1, \ldots, n)$, then we say C^∞- Liouville integrable.

Bolsinov and Taimanov (Invent. Math. 2000) showed that there exists a Hamiltonian system related with geodesic flow on a Riemannian manifold which is not C^ω-integrable, and showed that non C^ω- integrability is closely related with the non Abelian property of a monodromy group. Because the (formal) integrability of a given Hamiltonian vector field is closely related with the existence of a (formal) solution with n parameters satisfying a certain non-degeneracy condition (complete solution), the monodromy condition is closely related with a monodromy property of formal n first integrals. We discuss the construction and summability of formal first integrals and non-integrability.

In view of a fundamental solution of a linear ordinary differential equation with irregular singularity we therefore construct first integrals in a formal exponential- log series, which is called, in the terminology of Ecalle's, a transseries. We discuss the summability of such series in each level as well as the global behavior the resummed first integrals.

Michał Zakrzewski (Warsaw University, Poland)

Asymptotic analysis and special values of multiple L-functions

Multiple Zeta Function is defined by the series

$$\zeta(s_1, s_2, ..., s_p) := \sum_{n_1 > n_2 > ... > n_p} n_1^{-s_1} n_2^{-s_2} ... n_p^{-s_p},$$

whenever it converges and it generalizes Riemann zeta function. If $(s_1, s_2, ..., s_p) \in \mathbb{Z}^p$, then $\zeta(s_1, s_2, ..., s_p)$ is called Multiple Zeta Value (MZV). There are many interesting relations between MZVs. For example, we have

$$\zeta(s_1)\zeta(s_2) \quad = \quad \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_1).$$

With the multiplication defined as above, the space spanned by MZVs L-Values (MLV), modeled on Dirichlet L-functions.
The generating function for MZVs, is a special value, $f(\lambda) = F(\lambda, 1)$, of a function $F(\lambda, t)$, which satisfies differential equation

$$(T+\lambda^{s_1+\ldots+s_p})F \; = \; 0, \qquad \text{where} \qquad T \; := \; (1-t)\partial_t(t\partial_t)^{s_1-1}...(1-t)\partial_t(t\partial_t)^{s_p-1}. \tag{8}$$

The generating function for MLVs can be also constructed is this way. I'm going to speak about connections between MZVs and MLVs and asymptotic properties of the eigenequation (8).

Henryk Żołądek (Warsaw University, Poland)

Linear meromorphic differential equations and multiple zeta values

Some generating functions of so-called multiple zeta values are closely related with solutions of some hypergeometric equations with a parameter. For example, the function $f_2(x) = 1 - \zeta(2)x^2 + \zeta(2,2)x^4 - ...$ is the hypergeometric function $F(x, -x; 1; t)$ evaluated at $t = 1$ and the function $f_3(x) = 1-\zeta(3)x^3+...$ is related with a hypergeometric function which satisfies a third order differential equation with a parameter x. I together with my student MichałZakrzewski began study of solutions of these differential equation when the parameter x is large. The WKB method and Stokes phenomena are used in our approach. In this way we obtain the well known formula $f_2(x) = \sin(\pi x)/(\pi x)$ and we prove that the function $f_3(x)$ satisfies a sixth order linear differential equation with an irregular singularity at $x = \infty$.