Regular coordinates and reduction of deformation equations for Fuchsian systems

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Problem

“Construct Fuchsian systems for prescribed Riemann schemes”

Fuchsian system:

\[ \frac{dY}{dx} = \left( \sum_{j=1}^{p} \frac{A_j}{x - t_j} \right) Y \]

\( A_j : n \times n \)-constant matrix \((1 \leq j \leq p)\)

\[ A_0 := - \sum_{j=1}^{p} A_j, \quad t_0 = \infty \]

Assume: for each \( j \),

\( A_j \) is \( \left\{ \begin{array}{l} \text{diagonalizable} \\ \lambda, \mu : \text{eigenvalues of } A_j, \lambda \neq \mu \Rightarrow \lambda - \mu \notin \mathbb{Z} \end{array} \right\} \)
Riemann scheme: the table which describes the characteristic exponents at each singular point

\begin{align*}
\begin{cases}
  x = t_0 : 
  \begin{array}{c}
    \lambda_{01}, \ldots, \lambda_{01}, \ldots, \\
    \lambda_{0n_0}, \ldots, \lambda_{0n_0}
  \end{array} \\
  \vdots \\
  x = t_j : 
  \begin{array}{c}
    \lambda_{j1}, \ldots, \lambda_{j1}, \ldots, \\
    \lambda_{jn_j}, \ldots, \lambda_{jn_j}
  \end{array} \\
  \vdots \\
  x = t_p : 
  \begin{array}{c}
    \lambda_{p1}, \ldots, \lambda_{p1}, \ldots, \\
    \lambda_{pn_p}, \ldots, \lambda_{pn_p}
  \end{array}
\end{cases}
\end{align*}

\( m_j := (m_{j1}, \ldots, m_{jn_j}) \): the spectral type of \( A_j \)
Problem: Construct tuples \((A_0, A_1, \ldots, A_p)\) with sum zero and with prescribed eigenvalues \(\{\lambda_{01}(m_{01}), \ldots, \lambda_{pn_p}(m_{pn_p})\}\)

The Problem

- seems fundamental
- is open (far from the perfect solution)
- is deeply related to the deformation theory
Precise formulation of the problem

\[
A_j \sim \begin{pmatrix}
\lambda_{j1} I_{m_{j1}} \\
\vdots \\
\lambda_{jn_j} I_{m_{jn_j}}
\end{pmatrix} =: C_j
\]

\[
O_j := \{ A \in M(n \times n, \mathbb{C}) \mid A \sim C_j \}
\]

We set

\[
\mathcal{M} = \mathcal{M}_{O_0, \ldots, O_p}
\]

\[
:= \{ (A_0, \ldots, A_p) \in O_0 \times \cdots \times O_p \mid \sum_{j=0}^{p} A_j = O \} / \sim,
\]

where

\[
(A_0, \ldots, A_p) \sim (B_0, \ldots, B_p)
\]

\[
def \iff \exists P \in \text{GL}(n, \mathbb{C}), A_j = PB_jP^{-1} \quad (\forall j)
\]
We have a map

\[ [(A_0, \ldots, A_p)] \leftrightarrow (O_0, \ldots, O_p) \]

Our problem is to describe

\[ \varphi^{-1}((O_0, \ldots, O_p)) = M_{O_0, \ldots, O_p} \]
Related results

1. \( \varphi \) is not surjective.
   We have an obvious necessary condition \( \sum_{j=0}^{p} \text{tr} \mathcal{O}_j = 0 \), which is not sufficient.

\[
\tilde{m} := (m_0, m_1, \ldots, m_p) : \text{the spectral type of } (F)
\]

For which \( \tilde{m} \), does an irreducible \( [(A_0, \ldots, A_p)] \) exist?
   (for generic values of \( \{\lambda_{jk}\} \))

Deligne-Simpson Problem

- V.P. Kostov
- W. Crawley-Boevey — in terms of Kac-Moody root systems

2. For an irreducibly realizable \( \tilde{m} \),

\[
\dim \mathcal{M} = (p - 1) n^2 - \sum_{j=0}^{p} \dim \mathcal{Z}(\mathcal{O}_j) + 2 =: \alpha
\]

A coordinate system of \( \mathcal{M} \) is called \textit{accessory parameters}. 
3. Scalar equation case. Toshio Oshima solved the Problem for scalar equations

\[ y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0. \]

The moduli space is a smooth manifold.

However,

\[ \# \text{ of a.p. for scalar equation} = \frac{\alpha}{2}. \]

\(\alpha\) parameters are necessary for the deformation, because \(\alpha\) is equal to the dimension of the conjugacy classes of the monodromy representations.

4. Case \(\vec{m} = (11, 11, 11, 11)\). \(\overline{\mathcal{M}}\) is constructed by Saito-Inaba-Iwasaki. \(\Rightarrow\) Painlevé property for Painlevé VI.
Our Approach

- Do not go into the compactification (too serious)
- Consider only generic points of $\mathcal{M}$
- Find *good* representatives $(A_0, A_1, \ldots, A_p)$

It would be good if there is a set of a.p. $z = (z_1, z_2, \ldots, z_\alpha)$ s.t. \( \forall \) entries of \( \forall A_j \) are rational functions in $z$.

We call such set of a.p. a *regular coordinate*.

A regular coordinate may be different from the canonical coordinate.
How to find regular coordinates

**Lemma 1.** For a generic pair $A, B$ of $n \times n$-matrices, there exists $P \in \text{GL}(n, \mathbb{C})$ such that

$$P^{-1}AP = \text{lower triangular}$$

$$P^{-1}BP = \text{upper triangular}$$

**Lemma 2.** Let $C$ be a diagonalizable $n \times n$-matrix with spectral type $(n_1, n_2, \ldots, n_q)$.

(i) $C$ can be parametrized by $n^2 - \sum_{i=1}^{q} n_i^2$ parameters besides the eigenvalues.
(ii) Let $\gamma_i$ be the eigenvalue of multiplicity $n_i$. Then $C$ can be (generically) parametrized as follows.

$$C = \gamma_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_{n-n_1} & P_1 \end{pmatrix} \quad C_1 : (n - n_1) \times (n - n_1)$$

$$C_1 + P_1 U_1 = \gamma_2 - \gamma_1 + \begin{pmatrix} C_2 \\ U_2 \end{pmatrix} \begin{pmatrix} I_{n-n_1-n_2} & P_2 \end{pmatrix}$$

$$C_2 + P_2 U_2 = \gamma_3 - \gamma_2 + \begin{pmatrix} C_3 \\ U_3 \end{pmatrix} \begin{pmatrix} I_{n-n_1-n_2-n_3} & P_3 \end{pmatrix}$$

$$\vdots$$

$$C_{q-1} + P_{q-1} U_{q-1} = \gamma_q - \gamma_{q-1}$$

parameters: $P_i, U_i$ (1 $\leq$ i $\leq$ q - 1)

Note that $\sum_{i=1}^{q} n_i^2 = \dim Z(C)$
\( \vec{m} = (m_0, m_1, \ldots, m_p) \): given
First we assume two \( m_i \) are \( 1^n \).

\[
m_0 = m_p = 1^n
\]

By Lemma 1, we can take a representative \((A_0, A_1, \ldots, A_p)\) s.t.

\[
A_0 = \begin{pmatrix} a_{01} & \cdots & O \\ \vdots & \ddots & \vdots \\ * & \cdots & a_{0n} \end{pmatrix}, \quad A_p = \begin{pmatrix} a_{p1} & \cdots & * \\ \vdots & \ddots & \vdots \\ O & \cdots & a_{pn} \end{pmatrix}
\]

Parametrize \( A_1, \ldots, A_{p-1} \) by Lemma 2.

The number of parameters we use is

\[
\sum_{j=1}^{p-1} \left( n^2 - \dim Z(A_j) \right).
\]
We can normalize the tuple \((A_0, \ldots, A_p)\) by \(\text{GL}(1)^n\) (with center \(\mathbb{C}^\times\)).

Since \(\sum_{j=0}^p A_j = O\), we have

\[
(*) \quad a_{oi} + \sum_{j=1}^{p-1} ((i, i)\text{-entry of } A_j) + a_{pi} = 0
\]

for \(i = 1, \ldots, n-1\), which are \(n-1\) relations for the parameters.

Thus

\[
\sum_{j=1}^{p-1} \left( n^2 - \dim Z(A_j) \right) - (n - 1) - (n - 1)
\]

\[
= (p - 1) n^2 - \sum_{j=1}^{p-1} \dim Z(A_j) - n - n + 2
\]

\[= \alpha.\]
If we can take $\alpha$ parameters $(z_1, z_2, \ldots, z_\alpha)$ s.t. the solution of (*) can be written as rational functions of $(z_1, z_2, \ldots, z_\alpha)$, this set of the parameters is a regular coordinate.

Note that the off-diagonal entries of $A_0$ and $A_\rho$ are determined by $\sum_{j=1}^{\rho} A_j = O$:

\[
\begin{pmatrix}
a_{01} & \cdots & O \\
\vdots & \ddots & \vdots \\
a_{0\rho} & & a_{0n}
\end{pmatrix}
+ A_1 + \cdots + A_{\rho-1} +
\begin{pmatrix}
a_{\rho1} & \cdots & * \\
\vdots & \ddots & \vdots \\
O & & a_{\rho n}
\end{pmatrix} = O
\]
Next we relax the assumption by a coalescence of eigenvalues.

\[ m_0 = 1^n \rightarrow 2, 1^{n-2} \]

\[ A_0 = \begin{pmatrix} a_{01} & \ast & a_{01} & O \\ \ast & a_{03} & \ddots \\ & \ast & \ddots & a_{0n} \end{pmatrix} = \begin{pmatrix} a_{01} & 0 & a_{01} & O \\ 0 & a_{03} & \ddots \\ & \ast & \ddots & a_{0n} \end{pmatrix} \]

Then by $GL(2) \times GL(1)^{n-2}$ action, we have

\[ A_p = \begin{pmatrix} a_{p1} & 0 & \ast \\ a_{p2} & a_{p3} & \ast \\ 0 & \ddots & \ddots \end{pmatrix} \]
\[ m_0 = 1^n \rightarrow 3, 1^{n-3}; \quad m_p = 1^n \rightarrow 2, 1^{n-2} \]

\[ A_0 = \begin{pmatrix} a_{01} & 0 & 0 \\ 0 & a_{01} & 0 \\ 0 & 0 & a_{01} \end{pmatrix} \]

\[ A_p = \begin{pmatrix} a_{p1} & 0 & 0 \\ 0 & a_{p1} & 0 \\ 0 & 0 & a_{p3} \end{pmatrix} \]

GL(2) × GL(1)^{n-2} action keeps these forms of \( A_0 \) and \( A_p \).
Reductions

1. Katz operations

addition: \( Y(x) \mapsto \prod_{j=1}^{p} (x - t_j)^{a_j} \cdot Y(x) \)

middle convolution: \( Y(x) \mapsto \int_{\Delta} (u - x)^{\lambda} Y(u) \, du \)

These operations are realized as operations on \((A_0, A_1, \ldots, A_p)\).

Katz operations keep the number of accessory parameters, irreducibility and the deformation equation invariant.

**Theorem.** If \((A_0, A_1, \ldots, A_p)\) has a regular coordinate, the result of a Katz operation also has a regular coordinate.

Thus it is enough to find regular coordinates for basic \(\tilde{m}\).
Basic spectral types.

\( \alpha = 2: \)

\((11, 11, 11, 11)\); \((111, 111, 111)\), \((22, 1^4, 1^4)\), \((33, 2^3, 1^6)\)

\( \alpha = 4: \)

\((11, 11, 11, 11, 11)\);
\((21, 21, 1^3, 1^3)\), \((31, 22, 22, 1^4)\), \((22, 22, 22, 211)\);
\((211, 1^4, 1^4)\), \((221, 221, 1^5)\), \((32, 1^5, 1^5)\), \((2^3, 2^3, 2211)\),
\((33, 2211, 1^6)\), \((44, 2^4, 22211)\), \((44, 332, 1^8)\), \((55, 3331, 2^5)\),
\((66, 4^3, 2^511)\)
Example. \((33, 222, 1^6) \ (\alpha = 2)\)

\[
A = \begin{pmatrix}
  a_1 & 0 \\
  0 & a_1 \\
  a_2 & 0 \\
  0 & a_2 \\
  * & a_3 \\
  0 & a_3 
\end{pmatrix},
B = \begin{pmatrix}
  b_1 & 0 \\
  0 & b_2 \\
  b_3 & 0 \\
  0 & b_4 \\
  O & * \\
  b_5 & 0 \\
  0 & b_6 
\end{pmatrix}
\]

\[C = c_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_3 & P_1 \end{pmatrix}, \quad C_1 + P_1 U_1 = c_2 - c_1\]

Normalization by GL(1)^6 gives

\[U_1 = \begin{pmatrix}
  1 & * & * \\
  1 & * & * \\
  1 & 1 & 1 
\end{pmatrix}\]
\[ P_1 = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \]

Parameters we use: \( 4 + 9 = 13 \)
Relations: \( 4 + 4 + 4 - 1 = 11 \)
Thus we have
\[ 13 - 11 = 2 = \alpha. \]
We find we can take a regular coordinate \((p_{11}, p_{21})\).
2. Good reductions

We consider a coalescence of eigenvalues which sends $\vec{m}$ to $\vec{m}'$. For example, for $\vec{m} = (m_0, m_1, \ldots, m_p)$ with $m_0 = m_p = 1^n$, we consider the coalescence

$$m_0 = 1^n \mapsto 21^{n-2} =: m'_0.$$
Assume that the tuple \((A_0, A_1, \ldots, A_p)\), in particular \(f\) and \(g\), are written rationally by a regular coordinate \(z = (z_1, \ldots, z_\alpha)\).

The coalescence \(a_{02} \rightarrow a_{01}\) yields two equations

\[
f = 0, \quad g = 0.
\]

If this system is linear in two entries \(z_i, z_j\) of the regular coordinate \(z\), we can solve the system to get a regular coordinate \(z' := (z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_\alpha)\) for \(\vec{m}'\).

We call such reduction \(\vec{m} \rightarrow \vec{m}'\) a **good reduction**.
Example.

\((11, 11, 11, 11, 11) \rightarrow (11, 11, 11, 11, 2) = (11, 11, 11, 11)\)

\[
A_0 = \begin{pmatrix} a_{01} & 0 \\ f & a_{02} \end{pmatrix}, \quad A_4 = \begin{pmatrix} a_{41} & g \\ 0 & a_{42} \end{pmatrix}
\]

\[
A_j = \begin{pmatrix} a_{j2} - u_j p_j & (a_{j2} - a_{j1} - u_j p_j) p_j \\ u_j & a_{j1} + u_j p_j \end{pmatrix} \quad (j = 1, 2, 3)
\]

Normalization: \(p_1 = 1\)
Relation: \(a_{01} + \sum_{j=1}^{3} (a_{j2} - u_j p_j) + a_{41} = 0\)

We have a regular coordinate \((u_2, p_2, u_3, p_3)\).

\[
\begin{cases}
  f = -(u_1 + u_2 + u_3) \\
  g = - \sum_{j=1}^{3} (a_{j2} - a_{j1} - u_j p_j) p_j
\end{cases}
\]
Coalescence: $a_{41}, a_{42} \rightarrow (a_{41} + a_{42})/2$

$$
\begin{align*}
\Rightarrow \quad & \begin{cases} 
    u_1 + u_2 + u_3 = 0 \\
    \sum_{j=1}^{3} (a_{j2} - a_{j1} - u_j p_j) p_j = 0
\end{cases}
\end{align*}
$$

This system is linear in $u_2, u_3$, and then they can be written rationally in $p_2, p_3$. Thus we obtain a regular coordinate $(p_2, p_3)$ after the coalescence.

This is a good reduction, and gives a reduction from Garnier system to Painlevé VI.
The isomonodromic deformation of the Fuchsian system

\[ \frac{dY}{dx} = \left( \sum_{j=1}^{p} \frac{A_j}{x - t_j} \right) Y \]

is described by the Schlesinger system

\[ \frac{\partial A_i}{\partial t_i} = - \sum_{k \neq i} \frac{[A_i, A_k]}{t_i - t_k} \quad \frac{\partial A_j}{\partial A_i} = \frac{[A_i, A_j]}{t_i - t_j} \quad (i \neq j) \]

under the condition

\[ A_j \sim C_j \quad (0 \leq j \leq p). \]
The unknowns of (S) are the entries of $A_1, \ldots, A_p$: $pn^2$ unknowns, while the rank of (S)+(J) is $\alpha$.
Thus we must reduce the unknowns of (S) to get a slim deformation equation.

If we have a regular coordinate for $(A_0, A_1, \ldots, A_p)$, we obtain, as isomonodromic deformation equations, algebraic differential equations for the regular coordinate.

If, moreover, we have a good reduction, we get an explicit reduction formula for the deformation equations such as Garnier to Painlevé.
Questions

Q1. Does a regular coordinate exist for any basic spectral type \( \tilde{m} \)? If it does not so, describe the condition.

Q2. Are there any general procedures to find a regular coordinate?

Q3. Can we obtain a regular coordinate for any basic spectral type from a regular coordinate for \((1^n, 1^n, \ldots, 1^n)\) by a finite iteration of good reductions?

Q4. For which pair of spectral types does a good reduction exist? Give the condition in terms of Kac-Moody root systems.

Q5. Irregular singular case?