On complex singularity analysis of holomorphic solutions of linear partial differential equations.

Catherine STENGER

Joint work with Stéphane MALEK

University of La Rochelle - France.

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We construct formal series solutions of linear PDE as linear combinations of powers of solutions of a first order nonlinear ODE: the \textit{tanh method}.

Initiated by W. Malfliet: an effective algebraic method for exact solutions for nonlinear PDEs

\[
H \left( u, \partial_t u, \partial_x u, \partial_x^2 u, \ldots \right) = 0
\]

using finite expansions

\[
u(t, x) = \sum_{j \in J} u_j \left( \varphi(\kappa(x - wt)) \right)^j
\]

where \( \varphi \) is a solution of a Ricatti equation

\[
\varphi' = a + b\varphi + c\varphi^2.
\]
We will consider special solutions of a linear PDE

\[ \partial_z^S X(t, z) = \sum_{k=(k_0, k_1) \in J} a_k(t, z) \partial_t^{k_0} \partial_z^{k_1} X(t, z) \]

in the form

\[ X(t, z) = \sum_{j \geq 0} X_j(t, z)(\phi(t))^j \]

where \( \phi(t) \) is a solution of some nonlinear first order ODE

\[ \phi'(t) = P(t, \phi(t)) \]

with

- \( P(t, X) \in \mathbb{C} \{t\} [X] \)
- \( a_k(t, z) \in \mathcal{O} [\mathcal{D}(t_0) \times \mathcal{D}(0)] \)
- a finite set \( J \subset \{ k = (k_0, k_1) \in \mathbb{N}^2 | k_1 \leq S - 1 \} \).
Motivations

- *existence* of such a formal solutions
- *sufficient conditions* for which this formal solutions are holomorphic in some punctured polydiscs of $\mathbb{C}^2$
- *rate of growth* of this solutions near the singularities
  (example: $P(t, X) = X^2, \phi' = \phi^2, \phi(t) = -1/(t + t_0)$)

Schedule

- Formal solutions.
- Majorant series method.
  - An auxiliary linear Cauchy problem.
  - Banach space of entire functions with exponential growth.
  - A Cauchy-Kowalevskii theorem.
  - Classification of singularities for first order ODEs.
  - Main result.
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  - Main result.
We look for \textit{transseries} solutions \( \hat{X}(t, z) \)

\[
\hat{X}(t, z) = \sum_{\beta \geq 0} \sum_{\ell \geq 0} X_{\ell, \beta}(t) \frac{\phi(t)^\ell}{\ell!} \frac{z^\beta}{\beta!}.
\]

An induction relation for the coefficients \( X_{\ell, \beta}(t) \)

\[
X_{\ell, \beta + \lambda} = R \left( (X_{\ell_1, \beta_1})_{\ell_1, \beta_1} \right)
\]

is obtained by the Faa di Bruno formula

\[
\partial_t^m \left( \frac{\phi(t)^\ell}{\ell!} \right) = \sum_{(p_1, \ldots, p_m) \in \mathbb{N}^m} \frac{m!}{p_1! \ldots p_m! (\ell - |p|)!} \prod_{k=1}^{m} \left( \frac{\phi(k)(t)}{k!} \right)^{p_k}
\]

\[
\text{Proposition}
\]

\textit{Let} \( X_{\ell, \beta}(t) \in \mathcal{O}(\mathcal{D}(t_0)), \ell \geq 0, 0 \leq \beta \leq S - 1. \)

\textit{Then, there exists a formal solution} \( \hat{X}(t, z) \) \textit{of the PDE for the given initial conditions}

\[
(\partial_z^j \hat{X})(t, 0) = \sum_{\ell \geq 0} X_{\ell, j}(t) \frac{\phi(t)^\ell}{\ell!}, \quad 0 \leq j \leq S - 1.
\]
Let
\[ v_{n_0, \ell, \beta} := \sup_{|t-t_0| \leq r} |\partial_t^{n_0} X_{\ell, \beta}(t)|. \]

As \( X_{\ell, \beta}(t) \) satisfies \( X_{\ell, \beta+s} = R((X_{\ell_1, \beta_1})_{\ell_1, \beta_1}) \), \( v_{n_0, \ell, \beta} \) satisfies
\[ v_{n_0, \ell, \beta+s} \leq \tilde{R}((v_{n_1, \ell_1, \beta_1})_{n_1, \ell_1, \beta_1}). \]

We define the formal series
\[
U(t, z, T) = \sum_{\beta \geq 0} \sum_{\ell \geq 0} \sum_{n_0 \geq 0} u_{n_0, \ell, \beta} t^{n_0} \frac{T^\ell}{\ell!} z^\beta \frac{1}{n_0! \beta!}
\]
where \( u_{n_0, \ell, \beta} \) is the unique solution of \( u_{n_0, \ell, \beta+s} = \tilde{R}((u_{n_1, \ell_1, \beta_1})_{n_1, \ell_1, \beta_1}). \)

**Proposition**

The formal series \( U(t, z, T) \) is the unique solution of the Cauchy problem
\[
\partial_z^S U(t, z, T) = \sum_{q=(q_0,q_1,q_2) \in \mathcal{Q}} B_q(t, z, T) \partial_t^{q_0} \partial_T^{q_1} \partial_z^{q_2} U(t, z, T)
\]
for the given initial conditions \( (\partial_z^j U)(t, 0, T) = \sum_{\ell \geq 0} \sum_{n_0 \geq 0} v_{n_0, \ell, j} t^{n_0} \frac{T^\ell}{\ell!} \)
\[ 0 \leq j \leq S - 1. \]
\( \mathcal{G}_q (\delta_1, \delta_2; \sigma) \): subspace of the vector space of entire functions / to \( T \) and holomorphic / to \((t, z)\):

\[
V(t, z, T) = \sum_{n, \beta \geq 0} v_{n, \beta}(T) \frac{t^n}{n!} \frac{z^\beta}{\beta!} \in \mathcal{G}_q (\delta_1, \delta_2; \sigma),
\]

such that

\[
\sum_{n, \beta \geq 0} ||v_{n, \beta}(T)||_{\beta; \sigma} \frac{\delta_1^n \delta_2^\beta}{(n + \beta)!} < +\infty, \text{ where}
\]

\[
||v_{n, \beta}(T)||_{\beta; \sigma} = \sup_{T \in \mathbb{C}} |v_{n, \beta}(T)| (1 + |T|)^{-m} \exp (-\sigma r_b(\beta)|T|^q).
\]

\( \|\cdot\|_{\delta_1, \delta_2; \sigma} :\)

\[
\|V(t, z, T)\|_{\delta_1, \delta_2; \sigma} = \sum_{n, \beta \geq 0} ||v_{n, \beta}(T)||_{\beta; \sigma} \frac{\delta_1^n \delta_2^\beta}{(n + \beta)!}.
\]

Banach space of entire functions with exponential growth.
Theorem

If

\[ \partial_z^S U(t, z, T) = \sum_{q \in \mathcal{Q}} B_q(t, z, T) \partial_t^{q_0} \partial_T^{q_1} \partial_z^{q_2} U(t, z, T) \]

satisfies some conditions and if for all \(0 \leq j \leq S - 1\),

\[ (\partial_z^j U)(t, 0, T) = \psi_j(t, T) \in \mathcal{G}_q(\delta_1, 0, \delta_2, 0; \sigma_0). \]

Then, there exists \(\delta_1, \delta_2, \sigma > 0\), such that the Cauchy problem has a unique solution \(U(t, z, T) \in \mathcal{G}_q(\delta_1, \delta_2; \sigma)\).

For the proof, we need

\[ \| (T^s \partial_t^\nu \partial_T^\kappa \partial_z^{-S} V)(t, z, T) \|_{\delta_1, \delta_2; \sigma} \leq C \delta_1^{-\nu} \delta_2^S \| V(t, z, T) \|_{\delta_1, \delta_2; \sigma}, \]

we have to estimate \( \| (T^s \partial_T^\kappa v_{n+\nu, \beta-S})(T) \|_{\beta, \sigma} \).
we use the Cauchy-integral formula

\[
(T^s \partial_T^{\kappa} \nu_{n+\nu,\beta-S})(T) = \frac{\kappa!}{2i\pi} \int_{|\xi-T|=a} \frac{T^s \nu_{n+\nu,\beta-S}}{(\xi - T)^{\kappa+1}} d\xi
\]

to obtain our result, we use a good choice for the radius \(a\) (introduced by Y. Dubinskï)

\[
a = (|T|^q + 1)^{1/q} - |T|
\]

and we yield the conditions

\[
\frac{b(s + \kappa(q - 1))}{q} + \nu < S.
\]
For
\[ \phi'(t) = P(t, \phi(t)), \]
by P. Painlevé, the only movable singularities in \( \mathbb{C} \) of \( \phi(t) \) are poles and/or algebraic branch points.

We define \( D_{\theta}(t_0, r) = D(t_0, r) \setminus [t_0, re^{i\theta}) \).
Let \( \phi(t) \) solution on \( D_{\theta}(t_0, r_0) \), there can be represented by a Puiseux series
\[ \phi(t) = \sum_{n \geq -n_0} f_n(t - t_0)^{n/\mu} \]
where \( \mu, n_0 \in \mathbb{N}^* \) et \( f_{-n_0} \neq 0 \).
Theorem

Under some conditions, the formal series

\[ X(t, z) = \sum_{\beta \geq 0} \sum_{\ell \geq 0} X_{\ell, \beta}(t) \frac{\phi(t)^{\ell}}{\ell!} \frac{z^{\beta}}{\beta!} \]

defines a holomorphic function on \( D_{\theta}(t_0, r_0) \times D(0, \delta) \).

Moreover, there exists \( C_1, C_2 \) such that

\[
\sup_{|z| \leq \delta} |X(t, z)| \leq C_1 |t - t_0|^{-n_0 m / \mu} \exp \left( C_2 |t - t_0|^{-n_0 q / \mu} \right)
\]

for all \( t \in D_{\theta}(t_0, r_0) \).

Sketch of proof:

- \( U(t, z, T) \) solution of the auxiliary problem belongs to \( G_q(\delta_1, \delta_2; \sigma) \)
  (Cauchy-Kowalevskii)
\[ W(t, z, T) = \sum_{\beta \leq 0} \sum_{\ell \leq 0} X_{\ell, \beta}(t) \frac{T^\ell z^\beta}{\ell! \beta!} \]

\[
\sup_{|z| < \delta} \sup_{t \in D} |W(t, z, T)| \leq |U(0, \delta, |T|)| \leq C(1 + |T|)^m \exp \left( \sigma \zeta(b)|T|^q \right)
\]

\( W(t, z, T) \) is holomorphic / \( t, z \) and at most of exponential growth / \( T \). Therefore \( X(t, z) = W(t, z, \phi(z)) \) is holomorphic solution.
\[
\left\{
\begin{array}{ll}
\phi' = \phi^2 & \text{Ricatti equation} \\
\partial_z^2 X = \partial_t X & \text{heat equation}
\end{array}
\right.
\]

Auxiliary equation: \( \partial_z^2 U = \partial_t U + T^2 \partial_T U \).

The condition from Cauchy-Kowalevskii theorem for \( T^2 \partial_T \):

\[
\frac{b(2 + 1(q - 1))}{q} < 2 \iff \frac{b}{q} + b < 2 \quad (b > 1)
\]

holds for \( q \) large enough.

\begin{equation*}
\begin{cases}
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\end{cases}
\end{equation*}

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