$q$-Analogue of summability of formal solutions of linear $q$-difference-differential equations

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Schedule

1. Motivation
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Let $m \geq 1$ be an integer, let \((t, x) = (t, z_1, \ldots, z_d) \in \mathbb{C}_t \times \mathbb{C}_z^d\) be the complex variables. Let us consider the linear partial differential equation

$$
\sum_{j+|\alpha| \leq m} a_{j,\alpha}(t, z)(t \partial_t)^j \partial_z^\alpha X = F(t, z),
$$

(1.1)

with the unknown function $X = X(t, z)$, where $a_{j,\alpha}(t, x)$ \((j + |\alpha| \leq m)\) and $F(t, z)$ are holomorphic functions in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_z^d$. 

**Motivation**

**Known facts (Equations)**
Motivation

Known facts (Conditions)

We suppose: there is an $m_0 \in \mathbb{N}$ with $0 \leq m_0 \leq m$ such that

$$\begin{cases}
    \text{ord}_t(a_{j,0}) \geq \max\{0, j - m_0\}, & \text{if } |\alpha| = 0, \\
    \text{ord}_t(a_{j,\alpha}) \geq \max\{1, j + |\alpha| - m_0 + 1\}, & \text{if } |\alpha| > 0,
\end{cases}$$

where $\text{ord}_t(a)$ denotes the order of the zeros of the function $a(t, z)$ at $t = 0$.

For $r > 0$ we write $D_r = \{ t \in \mathbb{C} ; |t| < r \}$; for $R > 0$ we write $D_R = \{ z \in \mathbb{C}^d ; |z_i| < R \ (i = 1, \ldots, d) \}$ and we denote by $\mathcal{O}_R[[t]]$ the set of all formal power series in $t$ with coefficients in $\mathcal{O}_R$.
Known result 1

For the equation (1.1) we have

Theorem 1.1
(Baouendi-Goulauic[1]). Suppose the conditions (1.2), $m_0 = m$ and

$$a_{m,0}(0,0) \neq 0.$$ 

If the equation (1.1) has a formal solution $\hat{X}(t, z) = \sum_{n \geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$ (with $R > 0$), then it is convergent in a neighborhood of the origin $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_z$. 
Motivation

Known result 2

Theorem 1.2

(Ōuchi[7]). Suppose the conditions (1.2), $0 < m_0 < m$ and

$$a_{m_0,0}(0,0) \neq 0,$$

and

$$\left. \frac{a_{m,0}(t,0)}{t^{m-m_0}} \right|_{t=0} \neq 0.$$ 

If the equation (1.1) has a formal solution

$$\hat{X}(t, z) = \sum_{n \geq 0} X_n(z) t^n \in \mathcal{O}_R[[t]]$$

(with $R > 0$), then it is Borel summable in $t$ in a suitable direction.
Definition of $q$-difference

Let $q > 1$: for a function $f(t, x)$ we define the $q$—difference operator $D_q$ by

$$(D_qf)(t, z) = \frac{f(qt, z) - f(t, z)}{qt - t}.$$ 

In this talk, we will try to $q$—discrete the equation (1.1) to the following $q$—difference-differential equation

$$\sum_{j+|\alpha|\leq m} a_{j, \alpha}(t, z)(tD_q)^j \partial_z^\alpha X = F(t, z), \tag{2.1}$$

and we will consider the following problem.
Problems

Problem 2.1

(1) (q—Analogue of [1]). Under what condition can we get the result that every formal solution $\hat{X} = \sum_{n \geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$ is convergent in a neighborhood of the origin $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_z$?

(2) (q—Analogue of [7]). Under what condition can we get the result that every formal solution $\hat{X} = \sum_{n \geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$ is $q$—summable in $t$ in a suitable direction $\lambda$ (in the sense of Definition 2.2 given below)?
Definition of $q$-summable

For $\lambda \in \mathbb{C} \setminus \{0\}$ and $\epsilon > 0$ we set

$$
S_{\lambda} = \{-\lambda q^m; \ m \in \mathbb{Z}\},
$$

$$
S_{\lambda,\epsilon} = \bigcup_{m \in \mathbb{Z}} \{t \in \mathbb{C} \setminus \{0\}; \ |1 + \lambda q^m / t| \leq \epsilon\}.
$$

It is easy to see that if $\epsilon > 0$ is sufficiently small the set $S_{\lambda,\epsilon}$ is a disjoint union of closed disks. The following definition is due to Ramis-Zhang [8] (though, they did not use the word "$q$-summable").
Definition of $q$-summable

Let $\hat{X}(t, z) = \sum_{n \geq 0} X_n(z) t^n \in \mathcal{O}_{R_0}[[t]]$: we say that the series $\hat{X}(t, z)$ is $q$-summable in $t$ in the direction $\lambda$ if there are $r > 0$, $R > 0$, $M > 0$, $H > 0$ and a holomorphic function $W(t, z)$ on $(D_r \setminus S_\lambda) \times D_R$ such that

$$
\left| W(t, z) - \sum_{n=0}^{N-1} X_n(z) t^n \right| \leq \frac{MH^N}{\epsilon} q^{N(N-1)/2} |t|^N \tag{2.2}
$$

holds on $(D_r \setminus S_\lambda, \epsilon) \times D_R$ for any sufficiently small $\epsilon > 0$ and any $N = 0, 1, 2, \ldots$. 
Reference of $q$-analogue

To solve Problem 2.1 we will use the framework of $q$–Borel and $q$–Laplace transformations via Jacobi theta function, developed by Ramis-Zhang [8] and Zhang [11].

In the case of $q$-difference equations, $q$–analogs of summability of formal solutions have been studied quite well by Zhang [10], Marotte-Zhang [4] and Ramis-Sauloy-Zhang [9].

In the case of $q$–difference-differential equations, we have some references, Malek [5][6], Lastra-Malek [2] and Lastra-Malek-Sanz [3]: but their problem is different from ours.
Throughout this paper, we let $q > 1$ be a fixed real number, $m \geq 1$ be an integer, and $\delta > 0$ be a real number. As a generalization of (2.2), we will treat the following equation

$$\sum_{j+\delta|\alpha| \leq m} a_{j,\alpha}(t, z)(tD_q)^j \partial_z^\alpha X = F(t, z),$$

with the unknown function $X = X(t, z)$, where $a_{j,\alpha}(t, z)$ $(j + \delta|\alpha| \leq m)$ and $F(t, z)$ are holomorphic functions in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_z^d$. 

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Condition

Instead of (1.2), we suppose: there is an integer $m_0$ with $0 \leq m_0 \leq m$ such that

$$
\begin{cases}
    \text{ord}_t(a_{j,0}) \geq \max\{0, j - m_0\}, & \text{if } |\alpha| = 0, \\
    \text{ord}_t(a_{j,\alpha}) \geq \max\{1, j - m_0 + 1\}, & \text{if } |\alpha| > 0.
\end{cases}
$$

(A)

$$
\begin{cases}
    \text{ord}_t(a_{j,0}) \geq \max\{0, j - m_0\}, & \text{if } |\alpha| = 0, \\
    \text{ord}_t(a_{j,\alpha}) \geq \max\{1, j + |\alpha| - m_0 + 1\}, & \text{if } |\alpha| > 0.
\end{cases}
$$

(1.2)
Main results.

A $q$-analogue version of [1]

Theorem 3.1

Suppose the conditions $(A)$, $m_0 = m$ and

$$a_{m,0}(0, 0) \neq 0.$$  \hspace{1cm} (3.1)

Then, if the equation $(E)$ has a formal solution

$$\hat{X}(t, z) = \sum_{n \geq 0} X_n(z) t^n \in \mathcal{O}_R[[t]] \text{ (with } R > 0),$$

it is convergent in a neighborhood of the origin $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_z$. 

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A $q$-analogue version of [7]

Theorem 3.2 (Formal solution)

Suppose the conditions (A), $0 \leq m_0 < m$ and

$$a_{m_0,0}(0,0) \neq 0. \quad (3.2)$$

Then, \(\hat{X}(t,z) = \sum_{n \geq 0} X_n(z)t^n \in O_R[[t]] \) (with $R > 0$) is a formal solution of $(E)$, there are $A > 0$, $h > 0$ and $0 < R_1 < R$ such that

$$|X_n(z)| \leq Ah^n q^{n(n-1)/2} \quad \text{on } D_{R_1}, \quad n = 0, 1, 2, \ldots. \quad (3.3)$$
A $q$-analogue version of [7]

Theorem 3.3 (Summability)

Suppose that the conditions of Theorem 3.2 hold. In addition, if the conditions

$$\frac{a_{m,0}(t,0)}{t^{m-m_0}}\bigg|_{t=0} \neq 0, \quad (3.4)$$

$$\text{ord}_t(a_{j,\alpha}) \geq j - m_0 + 2, \text{ if } |\alpha| > 0 \text{ and } m_0 \leq j < m \quad (3.5)$$

are satisfied, the formal solution $\hat{X}(t, z)$ is $q$–summable in any direction $\lambda \in \mathbb{C} \setminus (\{0\} \cup S)$, where $S$ is the set of singular directions defined below.
Singular direction $S$

By the assumption (A) we have the expression

$$a_{j,0}(t, z) = t^{j-m_0} b_{j,0}(t, z) \text{ for } m_0 < j \leq m$$

for some holomorphic functions $b_{j,0}(t, z)$ $(m_0 < j \leq m)$ in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_z^d$. By (3.4) we have $b_{m,0}(0, 0) \neq 0$. We set

$$P(\xi, z) = \sum_{m_0 < j \leq m} \frac{b_{j,0}(0, z)}{q^{j(j-1)/2}} \xi^{j-m_0} + \frac{a_{m,0,0}(0, z)}{q^{m_0(m_0-1)/2}}.$$

By the assumptions (3.4) and (3.5) we see that $P(\xi, 0)$ is a polynomial of degree $m - m_0$ and it has $m - m_0$ non-zero roots $\tau_1, \ldots, \tau_{m-m_0}$. Then the set of singular directions is defined by

$$S = \{ \tau \in \mathbb{C}; \tau = t\tau_i \text{ for some } t > 0 \text{ and } 1 \leq i \leq m - m_0 \}.$$
Sketch of proof

Let give a proof for a simple example.
By

\[ tD_q f(t, z) = \frac{f(qt, z) - f(t, z)}{q - 1} \]  \hspace{1cm} (4.1)

we study the equation

\[ \sum_{j + \delta|\alpha| \leq m} b_{j,\alpha}(t, z) \sigma_q^j \partial_z^\alpha X = G(t, z), \]  \hspace{1cm} (4.2)

where \( \sigma_q f(t, z) = f(qt, z) \).

In this talk we study the following simple example:

\[ X(t, z) + t \sigma_q X(t, z) - t^2 \partial_z^\alpha X(t, z) = a(z). \]  \hspace{1cm} (4.3)
At first let us give an important proposition.

**Proposition 4.1**

For a series \( \hat{X}(t, z) = \sum_{k=0}^{\infty} a_k(z) t^k \) the formal q-Borel transform \((\hat{B}_q \hat{X})(\tau, z)\) is defined by

\[
(\hat{B}_q \hat{X})(\tau, z) := \sum_{k=0}^{\infty} a_k(z) \frac{\tau^k}{q^{k(k-1)/2}}. \tag{4.4}
\]

Set \( U(\tau, z) = (\hat{B}_q \hat{X})(\tau, z) \). Suppose that \( U(\tau, z) \) satisfies

\[
|U(\lambda q^l, z)| \leq AB^l q^{l^2/2} \quad \text{for} \ l = 0, 1, \ldots \tag{4.5}
\]

for some \( A, B > 0 \). Then \( \hat{X}(t, z) \) is q-summable in \( t \) in the direction \( \lambda \).
Let us give a proof for the example. By operating the formal $q$-Borel transform to the example (4.3) we get

$$(1 + \tau)U(\tau, z) = a(z) + \frac{\tau^2}{q} \sigma_q^{-2} \partial_z^\alpha U(\tau, z).$$  \hspace{1cm} (4.6)$$

We construct $U(t, z) = \sum_{k=0}^{\infty} U_k(\tau, z)$ with

$$(1 + \tau)U_0(\tau, z) = a(z)$$

$$(1 + \tau)U_k(\tau, z) = \frac{\tau^2}{q} \sigma_q^{-2} \partial_z^\alpha U_{k-1}(\tau, z).$$  \hspace{1cm} (4.7)$$
Then we have

**Proposition 4.2**

\[ |U_k(\tau, z)| \leq \frac{1}{q^k q^{k(k-1)}} \left| \partial_z^k a(z) \right| \]  \hspace{1cm} (4.8)

Then for \( z \in D_R \) we get

\[ |U_k(\tau, z)| \leq AB^k \frac{1}{q^k q^{(1-\epsilon)k(k-1)}} \left| \tau^k \right| \]  \hspace{1cm} (4.9)

Hence we have

\[ |U(\lambda q^l, z)| \leq \sum_{k=0}^{\infty} AB^k \frac{\left| \lambda \right|^k q^{kl}}{q^k q^{(1-\epsilon)k(k-1)}} \]  \hspace{1cm} (4.10)

\[ \leq q^{l^2/2} \sum_{k=0}^{\infty} AB^k \frac{\left| \lambda \right|^k q^{k^2/2}}{q^k q^{(1-\epsilon)k(k-1)}} \]

by \( kl \leq k^2/2 + l^2/2 \). Q.E.D.
References


Thank you!