THERE ARE NOT MANY PERIODIC ORBITS IN BUNCHES FOR ITERATION OF COMPLEX QUADRATIC POLYNOMIALS OF ONE VARIABLE

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Abstract. It is proved that for every complex quadratic polynomial f with Cremer's fixed point z_0 (or periodic orbit) for every $\delta > 0$, there is at most one periodic orbit of minimal period n for all n large enough, entirely in the disc (ball) $B(z_0, \exp -\delta n)$ (at most p for a Cremer orbit of period p). Next, it is proved that the number of periodic orbits of period n in a bunch $\mathcal{P}_n$, that is for all $x, y \in \mathcal{P}_n$, $|f^j(x) - f^j(y)| \leq \exp -\delta n$ for all $j = 0, ..., n-1$, does not exceed $\exp \delta n$. We conclude that the geometric pressure defined with the use of periodic points coincides with the one defined with the use of preimages of an arbitrary typical point. I. Binder, K. Makarov and S. Smirnov (Duke Math. J. 2003) proved this for all polynomials but assuming all periodic orbits were hyperbolic, and asked about general situations. We prove here a positive answer for all quadratic polynomials.

1. Introduction

In this paper we consider holomorphic maps $f : U \to \overline{\mathbb{C}}$, where $U \subset \overline{\mathbb{C}}$ is an open domain in the Riemann sphere $\overline{\mathbb{C}}$, where $\mathbb{C}$ is the complex plane. Usually f will be a rational function on $U = \overline{\mathbb{C}}$ or just a quadratic polynomial $f(z) = f_c(z) = z^2 + c$ for a complex number c.

The following notions of geometric pressure for a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are of interest, see e.g. [PRS2] and [P-ICM18]. Let us start with *variational geometric pressure* with $\phi = -t \log |f'|$ considered on $X = J(f)$ being Julia set for f, and for real t.

Definition 1.1 (variational pressure)**.**

$$P_{\mathrm{var}}(f, \phi) = \sup_{\mu \in \mathcal{M}(f)} \left(h_\mu(f) + \int_X \phi \, d\mu \right), \tag{1.1}$$

where $\mathcal{M}(f)$ is the set of all f-invariant Borel probability measures on X and $h_\mu(f)$ is measure theoretical entropy.

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Definition 1.2 (Tree pressure)**.** For every $z \in K$ and $t \in \mathbb{R}$ define

$$P_{\text{tree}}(z,t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{f^n(x)=z,\, x\in K} |(f^n)'(x)|^{-t}. \tag{1.2}$$

These notions follow adequate notions in equilibrium statistical physics, e.g. free energy for say generalizations of Ising model with a hamiltonian potential on the space of configurations over $\mathbb{Z}^d$, see e.g. [Ruelle] and [Bowen].

Another definition is also natural:

Definition 1.3 (Periodic pressure)**.** For every $t \in \mathbb{R}$ define

$$P_{\text{per}}(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{f^n(z)=z,\, z\in J(f)} |(f^n)'(x)|^{-t}. \tag{1.3}$$

It was proved e.g. in [PRS2], see also [P-conical], that for every $z \in \overline{\mathbb{C}}$ except in a zero Hausdorff dimension subset of $\overline{\mathbb{C}}$ both pressures coincide.

It was proved in [PRS2, Theorem C], that for $t \geq 0$ they coincide with the *periodic pressure* for every rational function f if the following Hypothesis H holds, see [PRS2] and [P-ICM18].

Denote the standard spherical metric on the Riemann sphere $\overline{\mathbb{C}}$ by ρ. For rational f denote by Per_n the set of all periodic points of minimal period n in the Julia set $J(f)$.

Hypothesis H. For every rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, for every $\delta > 0$ and all n large enough, if for a set $\mathcal{P} \subset \mathrm{Per}_n$, for all $x, y \in \mathcal{P}$ and all $i : 0 \leq i < n$ $\rho(f^i(x), f^i(y)) < \exp -\delta n$, then $\#\mathcal{P} \leq \exp \delta n$.

We prove here, in Section 5, that Hypothesis H holds for all quadratic polynomials. Therefore we conclude with

Theorem 1.4. *For every quadratic polynomial f and real all $t \geq 0$ geometric pressures for f acting in $\mathbb{C}$, variational and tree ones, are equal to the periodic pressure.*

A condition stronger than Hypothesis H has been formulated in [BMS]. Namely

Hypothesis BMS. For every $\epsilon > 0$ there exists $r = r(f, \epsilon)$ such that if $n \geq n(f, \epsilon)$ and $x \in \mathrm{Per}_n$, then

$$\#\{y \in \mathrm{Per}_n : \rho(f^i(x), f^i(y)) < r \forall 0 \leq i < n\} \leq \exp \epsilon n.$$

The authors of [BMS] proved it for all polynomials, but under the assumption REP saying that all periodic orbits of f in $J(f)$ are hyperbolic repelling, that is $|(f^n)'(x)| > 1$ where n is a period of x. They used external rays, as we do here, and Yoccoz's puzzle structure (which we do not use). They asked whether the assumption REP can be skipped. See [BMS, Lemma 7 and the comment preceding it].

As we already mentioned, we succeed here, proving the equality of pressures in Theorem 1.4 for all quadratic polynomials, without assuming REP, via proving a weaker than Hypothesis BMS, but sufficient for us, Hypothesis H.

Hypothesis BMS, hence Theorem 1.4, hold also for all generalized multimodal maps of interval, with all periodic orbits hyperbolic repelling, see [PR, Section 5].

Let now $f : U \to \overline{\mathbb{C}}$ be a holomorphic mapping as above. Let $z_0 \in U$ be a fixed point for f. We call it indifferent if $|f'(z_0)| = 1$. Then the derivative $f'(z_0)$ is either a root of unity in which case we call z_0 parabolic, or not a root of unity when we call z_0 irrationally indifferent. In the latter case if f is not holomorphically linearizable in a neighbourhood of z_0 we call z_0 Cremer. See e.g. [Milnor-book] for an introduction to this theory. We use the same language for any periodic orbit of period n, replacing f by f^n.

If z_0 is a Cremer fixed point and $f'(z_0) = \exp \alpha 2\pi i$, where α is irrational then its convergents p_n/q_n in the continued fraction algorithm satisfy

$$\sum^{n\geq 1} \frac{\log q_{n+1}}{q_n} = \infty. \tag{1.4}$$

(So convergence of the series, Bryuno condition, implies the existence of a so-called Siegel disc around z_0, that is a rotation in a disc by the angle $2\pi\alpha$ in respective holomorphic coordinates.)

If additionally

$$\sum^{n\geq 1} \frac{\log \log q_{n+1}}{q_n} < \infty \tag{1.5}$$

then f has a sequence of periodic orbits converging to z_0 of periods being a subsequence of (q_n). see [Perez-Marco1, Perez-Marco2].

A question arises how many at most such periodic orbits converging to z_0 in full, may happen. In this paper we prove the following

Theorem 1.5. *Set $f = f_c(z) = z^2 + c$. Let $z = z_0 \in \mathbb{C}$ be a fixed point for f. Then for every $\delta > 0$ there exists $r_0 > 0$ such that for every integer $n > 1$ and $r_n = r_0 \exp -n\delta$ there is at most one orbit of minimal period n, entirely contained in $B(z_0, r_n)$.*

Theorem 1.6. *Set $f(z) = z^2 + c$ as above. Let $z = z_0 \in \mathbb{C}$ be a Cremer periodic point for f, of minimal period p. Then for every $\delta > 0$ there exists $r_0 > 0$ such that for every n and $r_n = r_0 \exp -n\delta$ there are at most* *p orbits of minimal period np, entirely contained in the union of the open discs* $\bigcup_{i=0,\dots,p-1} B(f^i(z_0), r_n)$.

By estimates in Section 5.2 the assumption "entirely" can be replaced by formally weaker (in fact equivalent) assumption "at least one point of each orbit" in a disc (ball) or union of the discs in Theorems 1.5 or 1.6 respectively.

Note that for every quadratic polynomial there is at most one Cremer periodic orbit. See [Shishikura].

A combination of these two theorems, a version not mentioning Cremer orbits, allows us to prove Hypothesis H, hence Theorem 1.4.

An advantage when one deals with polynomials is a possibility to use external rays and angles (arguments), [Milnor], which makes the problem partially real, considering the angle doubling map $F(\theta) := 2\theta \mod 2\pi\mathbb{Z}$, in place of complex. This was pioneered in [BMS]. Here we apply Milnor's point of view of *orbit portrait* for a Cremer periodic orbit $O(z_0)$ of period p, by rays starting at hyperbolic orbits of period n, close to $O(z_0)$, rather than at $O(z_0)$ itself. For the reader's convenience we provide Milnor's approach in Appendix. We prove preservation of cyclic order, but we are not able to prove the unlinking property, which we fortunately can omit by a direct argument.

We call sometimes the union of these rays, or various versions of truncated ones, *quasi-spiders*, following a terminology by J. Hubbard and D. Schleicher in [Hub-Sch], see Section 4.

Note that all our theorems proved here for quadratic polynomials hold for all uni-critical polynomials, that is $z^d + c$ for $d \geq 2$, with the same proofs. A challenge is to verify them for all polynomials of degree at least 2; our approach might work. For rational functions Theorem 1.4 holds if Hypothesis H holds. The latter holds easily under some weak hyperbolicity assumptions, e.g. uniform exponential decay to 0 of diameters of pullbacks under f^{-n} of small discs, see e.g. [PRS1]. For a rational map the existence of a geometric coding tree, see [P-ICM18], without self-intersections, could make proving Hypothesis H doable, similarly to polynomials. But for general rational maps new ideas seem to be needed.

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2. Distortion for iterates

We shall need the following auxiliary

Lemma 2.1. *Let* z_0 *be a fixed point for a holomorphic function* $g : U \to \mathbb{C}$ *for a domain* $U \subset \mathbb{C}$*, such that* $|g'(z_0)| = 1$*. Then, for every* $\alpha > 0$ *(in particular close to 0) there exists* $n_0 > 0$ *such that for all* $n \geq n_0$ *and* $i = 0, ..., n$ *and every*

$$r \leq r_n = r_0 \exp -n\delta$$

as above, it holds

$$g^i(B(z_0), r) \subset B(z_0, r \exp n\delta\alpha). \tag{2.1}$$

Proof. Notice that except for a small neighbourhood of Crit(g) so r_0 small enough, for every n and $\eta : 0 < \eta \leq r_0$

$$\sup_{x \in B(z_0, r)} |\log |g'(x)|| \leq Lr \tag{2.2}$$

(or $|g'(x)| \leq 1 + Lr$), where L is $\sup_{x \in B(z_0, \eta)} \frac{|\log |g'(x)||}{|x - z_0|}$.

Let n_α be the smallest integer which satisfies

$$2Lr_0 \exp(-n_\alpha \delta\alpha) \leq \exp \delta\alpha. \tag{2.3}$$

Then, for each $j : 0 \leq j \leq n - n_\alpha$, due to (2.2),

$$g(B(z_0, r \exp(j\delta\alpha))) \subset B(z_0, r \exp((j+1)\delta\alpha)). \tag{2.4}$$

Now compose (2.4), starting from $j = 0$ up to $j = n - n_\alpha$. We obtain for all $0 \leq i \leq n - n_\alpha$

$$g^i(B(z_0), r) \subset B(z_0, r \exp(i\delta\alpha)). \tag{2.5}$$

Setting here $i = n - n_\alpha$ and adding A_α arising from the last n_α iterates we can replace $(n - n_\alpha)\delta\alpha + A_\alpha$ by the smaller $n\delta(1 - 2\alpha)$. Indeed, the last n_α iterates, thus the addition of at most $A_\alpha := n_\alpha L$, changes the upper estimate to the bigger $\exp -n\delta(1 - 2\alpha)$ for n large enough.

□

The inclusion (2.1) is self-reinforcing as follows:

Corollary 2.2.

$$g^i(B(z_0, r_n)) \subset B(z_0, r_n(1 + \lambda^n))$$

for a constant $\lambda : 0 < \lambda < 1$*, for all* n *large enough and* $i = 1, ..., n$*. Moreover, the complex distortion of each* g^i *on* $B = B(z_0, r_n)$ *is at most of order* λ^n*, i.e.*

$$\sup_{x,y \in B} \Big| \frac{(g^i)'(x)}{(g^i)'(y)} - 1 \Big| \leq \lambda^n. \tag{2.6}$$

The same for every $r \le r_n$.

Proof. For all $x \in B(z_0, r_n)$ we have, for $i \le n - n_\alpha$, due to (2.4) and (2.2)

$$g^i(B(z_0, r_n) \subset B(z_0, r_n \sup_{x \in B(z_0,r_n)} |(g^i)'(x)|)$$

with

$$(2.7) \qquad |(g^i)'(x)| = \exp \log \prod_{j=0}^{i-1} |g'(g^j(x))| \le \exp \sum_{j=0}^{i-1} Lr_0 \exp(-n\delta + j\delta\alpha).$$

Hence for a constant $C_{\delta,\alpha,L} > 0$

$$g^i(B(z_0), r_n) \subset B(z_0, C_{\delta,\alpha,L} r_n \exp(-n\delta(1-2\alpha))$$

This includes all $i \le n$ as at the end of Proof of Lemma 2.1 Thus, we prove Corollary 2.2 with $\lambda := (\exp -\delta(1-2\alpha)) - 1$. The inequality (2.6) follows from the calculation as in (2.7):

□

3. Cremer fixed points

Proof of Theorem 1.5. Step 1. Preliminaries.

If z_0 is repelling or parabolic, the theorem follows from the topological behaviour at its neighbourhood, see [Milnor-book, Camacho's theorem]. There are no periodic orbits there except z_0. This holds for any holomorphic f. So we need consider only Cremer points.

In presence of a Cremer fixed point, Julia set of f_c is connected because it contains the only critical point 0 n it. Hence $A_c(\infty)$, the basin of attraction to ∞ by the action of f_c, is simply connected. Let $\Phi_c : \{z : |z| > 1\} \to A_c(\infty)$ be Riemann (holomorphic injective) map from the outer disc in the Riemann sphere onto $A_c(\infty)$, where $f(\infty) = \infty$ and $f'(\infty) = 1$ (in the chart $z \mapsto 1/z$). Note that the function $\log |\Phi_c^{-1}|$ is Green's function $G = G_c$ on $A_c(\infty)$.

For every $\theta \in [0, 2\pi \mod 2\pi\mathbb{Z}]$ we call $\Phi(\tau \cdot (\exp i\theta))$ for $1 < \tau < \infty$ the external ray with the external argument θ. We denote this ray by $\mathcal{R}_\theta$ and if there exists a limit $\mathcal{R}_\theta(\tau)$ as $\tau \searrow 1$ we say that the ray starts at this limit point. We denote $\mathbb{C}$ compactified by the circle at infinity $S(\infty)$ of external arguments, by $\widetilde{\mathbb{C}}$. We denote f extended continuously to $\widetilde{\mathbb{C}}$ by $\widetilde{f}$.

Fix an arbitrary positive integer n. Let $O_1, ..., O_K$ be periodic orbits for f_c of minimal period n, all in the disc $B(z_0, r)$ with $r \le r_n$. Each point x in this family of orbits is a starting point of a nonempty finite number of external rays, $\mathcal{R}(x, s)$. The existence is known as Douady-Eremenko-Levin-Petersen theorem, see [P-accessibility] for generalizations. Finiteness is a classical Douady-Hubbard easy observation. Namely f permutes the family of the rays, hence external arguments, and the permutation restricted to external arguments for each $x \in O_k, k = 1, ..., K$ (leading to $f(x) \in O_k$) is the doubling map F preserving cyclic order along $S(\infty)$, hence preserving order and finite [Milnor, Definition preceding Theorem 1.1 and Lemma 2.3].

We shall prove that the preservation of this cyclic order holds for the set of all external arguments of the family $\bigcup_x \mathcal{R}(x,s)$ (as if all $\mathcal{R}(x,s)$ started at z_0).

Step 2. Sub-rays.

Enumerate the external arguments according to the cyclic order, by $j = 0,...,m-1$, and denote the permutation induced by f (more precisely by F) by σ. For an arbitrary $r : 0 < r \leq 2r_n$, consider the part $\mathcal{R}'_j = \mathcal{R}'_j[r]$ (sometimes written just $\mathcal{R}_j[r]$) of each $\mathcal{R}_j, j = 0,...,m-1$ starting at the last intersection with the circle $S(r) := \partial B(z_0, r)$ and going to ∞, numbered along the circle order. We shall consider domains **sectors** $Q_j[r] \subset \mathbb{C}, j = 0,...,m-1$ bounded respectively by $\partial Q_j[r]$ consisting of $\mathcal{R}'_j$, $\mathcal{R}'_{j+1 \mod m}$ and the arc $\Gamma_j[r]$ in the circle joining their starting points. We can consider $\overline{Q}_j[r]$ and $\widetilde{Q}_j[r]$, the closures of $Q_j[r]$ in $\overline{\mathbb{C}}$ or in $\widetilde{\mathbb{C}}$ respectively.

Note that the cyclic order of the arguments of $\mathcal{R}'_j$ at ∞ coincides with the order of the rays' beginnings along $S(r)$ (use Jordan theorem). (The same holds if $S(r)$ is replaced by any small simple loop around z_0.)

For all $\mathcal{R}'_j$ we can find sub-rays $\widehat{\mathcal{R}}_j \subset \mathcal{R}'_j$, ending at ∞ and beginning $b\widehat{\mathcal{R}}_j$ in the disc $B(z_0, 2r)$, that is

$$r \leq |b\widehat{\mathcal{R}}_j - z_0| < 2r \tag{3.1}$$

such that

$$f(\widehat{\mathcal{R}}_j) \subset \widehat{\mathcal{R}}_{\sigma(j)} \tag{3.2}$$

for each j.

Indeed, for each $\mathcal{R}'_j$ we find $j_{\max}$ such that in its f-orbit the beginning of $\mathcal{R}'_{j_{\max}}$ has maximal value of Green's function G_c. Define for $j = 0,...,m-1$

$$\widehat{\mathcal{R}}_{\sigma^j(j_{\max})} = f^{j-j_{\max}}(\mathcal{R}'_{j_{\max}}).$$

Since under the action by f the value of G grows by $\log 2$, the inclusion (3.2) holds (it is equality for $j = 0,...,m-2$). Moreover $\widehat{\mathcal{R}}_j \subset \mathcal{R}'_j$ holds. Finally beginnings of all $\widehat{\mathcal{R}}_j$ are in $B(z_0, 2r)$ by Corollary 2.2.

It will be convenient to replace each $\widehat{\mathcal{R}}_j$ by its sub-ray (by (3.1)) $\mathcal{R}'_j[2r]$. Summarizing

$$\mathcal{R}'_j[r] \supset \widehat{\mathcal{R}}_j \supset \mathcal{R}'_j[2r]. \tag{3.3}$$

Note that the union $\bigcup_j \partial Q_j[2r]$ has no self-intersections (as for any other radius, not just $2r$).

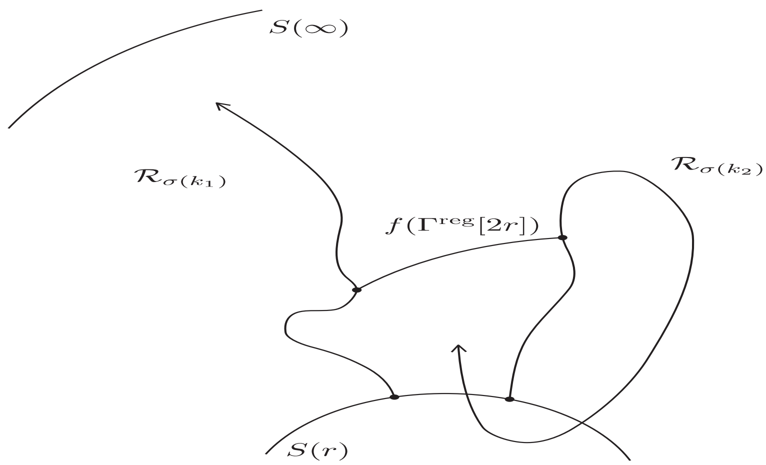

FIGURE 1. No bottom returns

We can assume in our notation that $Q_0[r]$ is the only sector, called the *critical sector*, containing the critical point 0. Denote the union of all other sectors (together with the rays separating them), thus called the *regular sector*, by $Q^{\rm reg}[r]$. Denote $\Gamma^{\rm reg}[r] := \bigcup_{j=1,\dots,m-1} \Gamma_j[r]$.

Step 3. f-images of the sectors.

Consider now $f(\partial Q^{\rm reg}[2r]) \subset \mathbb{C}$. The only self-intersections points which this curve could have, are the points of $f(\Gamma^{\rm reg}[2r]) \cap f(\mathcal{R}_k[2r])$, for $k = k_1, k_2$ and $\mathcal{R}_k[2r]$ being one of two boundary rays of $Q^{\rm reg}[2r]$. We shall analyze two cases:

If the curve $f(\mathcal{R}_k[2r])$ intersects $f(\Gamma^{\rm reg}[2r])$ and the first intersection point K comes from the side of $f(B(z_0, 2r))$ then it must intersect $S(r)$ before, to be able to leave the sector Q between $\mathcal{R}_{\sigma(k_1)}$ and $\mathcal{R}_{\sigma(k_2)}$ when going back to its beginning (the beginning of $f(\mathcal{R}_k[2r])$). See Figure 1. Such intersection is impossible by (3.2).

On the other hand each point K of intersection of $f(\mathcal{R}_k[2r])$ with $f(\Gamma^{\rm reg}[2r])$ from $\mathbb{C} \setminus f(B(z_0, 2r))$ must be followed by an intersection K' back, since otherwise $f(\mathcal{R}_k[2r])$ must leave Q through $S(r)$, again impossible. Consider Jordan domain J_K between this curve from K to K' and the curve Γ_K from K to K' in $f(\mathcal{R}_k[2r])$. Each two such Jordan domains are either disjoint or one is contained in the other, yielding the order by inclusion. Consider any maximal J_K. We can lift it by the branch g of f^{-1} mapping z_0 to itself. Indeed $\mathrm{dist}(J_K, z_0) > \frac{3}{2}r$ if g is sufficiently close to id, i.e. r small, use Corollary 2.2. So lifting starting from Γ_K cannot leave $Q^{\rm reg}[r]$. It cannot hit the critical point 0 which is in the critical sector $Q_0[r]$.

More precisely, J_K stays in a Jordan domain in $f(Q^{\rm reg}[r])$ bounded by $f(\mathcal{R}_k[r]), k = k_1, k_2,\ f(\Gamma^{\rm reg}[r])$ and $S(R)$ for R close to ∞ (or just $R = \infty$ in $\tilde{\mathbb{C}}$), by Janiszewski Theorem B, [Janiszewski, Bing], saying

THEOREM B. The sum of two compact continua cuts the plane provided their common part is not connected.

Here one continuum in Janiszewski's Theorem is the union of the rays above truncated at $S(R)$ and of the adequate arc in $S(R)$, the other is $f(\Gamma^{\rm reg}[r])$. Similarly te existence of K' follows, where the second continuum is $f(\Gamma^{\rm reg}[2r])$.

So the lifting is possible.

Lift a domain $J'_K \subset \mathbb{C}$ slightly larger than J_K. Do this for each K (the family of K's is finite by the analyticity). Let Q' be $Q^{\rm reg}[2r]$ slightly extended in $\mathbb{C}$. We obtain f acting as local homeomorphism on an open topological disc $U' = Q' \cup \bigcup_K g(J'_K)$ in the complex plane $\mathbb{C}$. Consider $\widetilde{\partial} U'$ the boundary of U' in $\widetilde{C}$. It is a Jordan curve being union of $\partial_{\mathbb{C}} \in \mathbb{C}$ and ∂_∞ the first being the boundary in $\mathbb{C}$ and an arc ∂_∞ in $S(\infty)$. Assume it is oriented counterclockwise

Claim. F is injective on ∂_∞, that is the length $|F(\partial_\infty)| < 2\pi$. Indeed, otherwise the index with respect to the critical value $\mathrm{ind}_c(\widetilde{f}(\partial U'))$ is 1 or 2 (depending on which side of $f(\partial_{\mathbb{C}})$ the critical value c is). But since the only critical point 0 is not in U', hence $\widetilde{\partial} U'$ is contractible outside 0, and there is no other pre-image of the critical value c in U' because 0 is its only f-pre-image, the index above is 0. A contradiction. See Figure 3, right.

In conclusion $\widetilde{f}$ is a homeomorphism between the Jordan curves $(\partial U')$ and its $\widetilde{f}$-image. It extends continuously a local homeomorphism f between the Jordan domains U' and $f(U')$. The latter is the Jordan domain bounded by $\widetilde{f}(\partial U')$ by maximum principle, hence f is a homeomorphism (biholomorphic).

We also conclude (a posteriori) that the restriction of f to the closure of $Q^{\rm reg}[2r]$ is injective so $f(\partial Q^{\rm reg}[2r])$ cannot have self-intersections. See also Figure 2. Hence in particular the family of K's is empty. See Figure 3, left.

By the injectivity in **Claim**, $F = f|_{S(\infty)}$ preserves the cyclic order of the boundary (in $S(\infty)$) arcs of the non-critical sectors. We can complete it so that σ maps ends of the critical arc to the ends of the arc missing in the σ-images of the non-critical arcs, called also *the critical value arc*.

Note, there is no cycle of regular sectors under the induced permutation. Otherwise the longest boundary arc in the cycle would be injectively mapped to a twice longer arc, a contradiction. So the permutation σ has no cycles of order less than m. So σ is just cyclic permutation of order m.

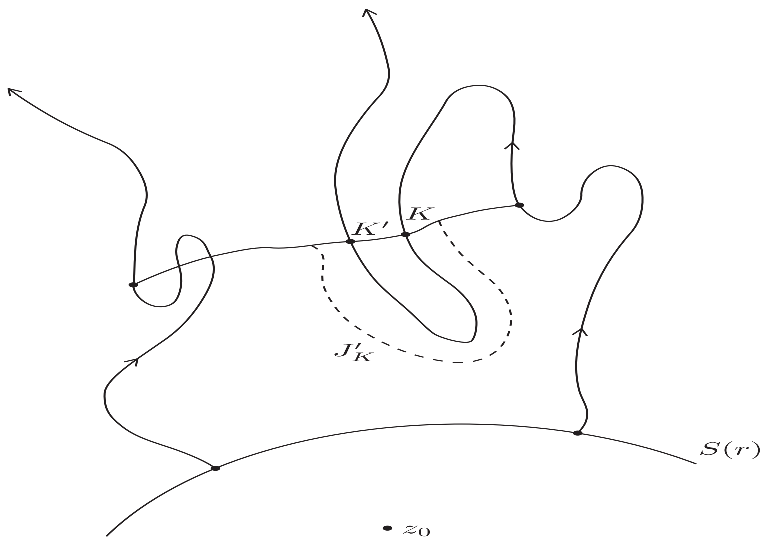

FIGURE 2. Avoiding self-intersections

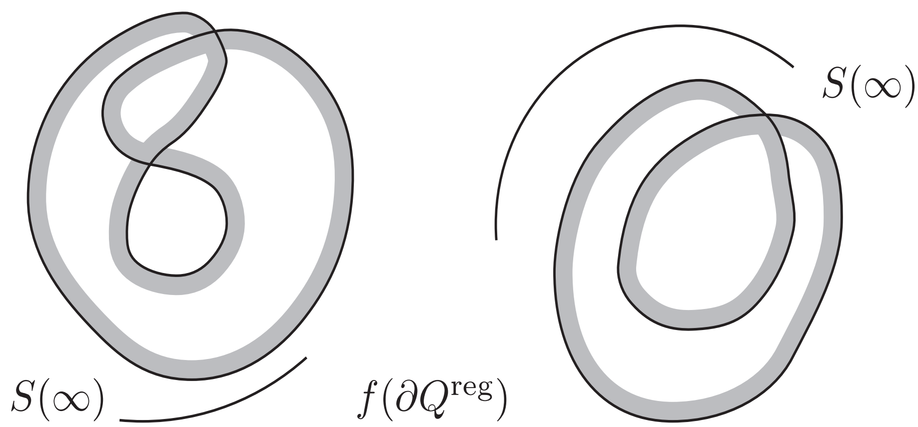

FIGURE 3. Impossible features

Since σ maps neighbour sectors (arcs at ∞) to neighbour sectors it must be a "rotation" (preserving cyclic order) permutation of our sectors and rays.

In other words F induces an order preserving map on the family of the lifts to $\mathbb{R}$ of the external angles of the rays (sides of the regular sectors, that is all the rays $\mathcal{R}_j$, and sectors bounded by them, since F is orientation preserving on $Q^{\mathrm{reg}}(\infty)$ and since $|Q^{\mathrm{reg}}(\infty)| < \pi$ "the last sector" cannot jump over "the first one".

The conclusion is that there can be at most 2 orbits of rays in $\bigcup \mathcal{R}(x,s)$, see [Milnor, Lemma 2.7], hence at most $K = 2$ periodic orbits of minimal period n in $B(z_0, r)$ In fact since the period of z_0 is 1 in this Section, there is just one orbit of rays for $n > 1$, so $K = 1$.

Writing the argument directly, we have proved that σ on the sectors, hence on the rays, is transitive, hence has only one orbit. Hence the number K of our periodic orbits of minimal period n is 1. □

Remark 3.1. Note that f on each regular $Q_j[r]$ is only a local homeomorphism onto its image; it is a priori not clear whether it is injective on $Q_j[r]$ or $Q^{\text{reg}}[r]$. Though the rays cannot intersect one another, the sub-rays $f(\mathcal{R}_j[r])$ can intersect $f(S[r])$ (if $\mathcal{R}_j[r]$ pass close to $\{f^{-1}(z_0)\}$. So we do not know a priori that the correspondence given by the rays $f(\mathcal{R}_j[r])$ between their ends P_j in $f(S(r))$ and $P_j(\infty) \in S(\infty)$ preserves their order. Namely it is not clear that such a ray does not meet $(S(r))$ again (between P_j and $P_j(\infty)$), i.e. that P_j is its last exit from $f(B(z_0, r))$.

In other words, though the cyclic order of the beginnings P_j of $\mathcal{R}'_j$ in $S(r)$ coincide with the cyclic order of $f(P_j)$ in $f(S(r)$, it is a priori not clear that $f(P_j)$ are the last exit points from $f(B(z_0, r))$, so it is not clear that their order coincides with the order at ∞, as $S(r)$ is not precisely invariant under f (this difficulty is not present if all the rays start at one repelling fixed point, in place of $S(r)$). We resolve it using $Q^{\text{reg}}[2r]$. The rays $\mathcal{R}'_j$ considered separately could sneak through gaps between $f(S(r))$ and $S(r)$.

Here is an easy complementing

Proposition 3.2. *For a Cremer fixed point z_0 for a quadratic polynomial f_c and $r > 0$ small enough every orbit of period $n \geq 2$ has the valence ν equal at most 2, that is each its point is a limit of at most two external rays.*

Proof. Let $O = (x_1, ..., x_n)$ be a repelling periodic orbit in $B(z_0, r)$, with period at least 2, Set $g = f^n$. Let $\mathcal{R}_1(x_1), ..., \mathcal{R}_\nu(x_1)$ be all the external rays starting at x_1. Their set will be denoted by A_1. Similarly we define $A_j, j = 1, ..., n$ Let $Q_s(x_j)$ be sectors between them, that is domains bounded by $\mathcal{R}(x_j, s), \mathcal{R}(x_{j+1 \bmod \nu}, s)$ and ∞. Then g yields σ a cyclic (preserving order in external arguments) permutation (combinatorial rotation) of the rays and of the sectors close to z_0.

More precisely for every simple loop γ around $B(z_0, r)$ close to it, we can consider the truncated sectors $Q_s(x_j)(\gamma)$ with the boundary consisting of the parts of the ray $\mathcal{R}(x_j, s)$ joining x_j to the point $P_{s,j}$ being the first point of the exit of this ray from $\operatorname{cl} B(z_0, r)$, the analogous part of the ray $\mathcal{R}_{s+1}(x_j))$ and the arc in γ joining $P_{s,j}$ to $P_{s+1,j}$. For each j and γ the order of $P_{s,j}$ is the same in γ compatible with the order in $S(\infty)$ (the circle of external arguments), f, hence g, preserve this order. To see this use γ and $f(\gamma)$. Compare [Milnor, Lemma 2.1].

Since $x = x_1$ is fixed for g then $x_2 = f(x_1)$ is fixed for g as well. But then, since the sectors $Q_{s,1}$ are pairwise disjoint, and g is a homeomorphism near $B(z_0, r)$, the g images of the truncates sectors are contained in the respective sectors $Q_{\sigma(s),j}$ So σ is the identity permutation.

So for the diagram (phase portrait) $A_1, ..., A_j$, the valence ν is 1, or 2 [Milnor, Lemma 2.7] – a primitive case. See also Appendix.

□

4. Cremer periodic orbits

For a Cremer periodic orbit Theorem 1.6 holds, generalizing Theorem 1.5. This section is devoted to this. It will use arguments from Section 3.

We assume that r_0, hence all r_n, are small enough that all the closed discs $\mathrm{cl}\, B(f^i(z_0), r_n), i = 0, ..., p-1$ are pairwise disjoint (far from each other) and far from the critical point 0. We write z_i for $f^i(z_0)$, for z_0 Cremer's of period p and $i = 0, ..., p-1$.

To simplify notation assume that $|f'(f^i(z_0))| = 1$ for all $i = 0, ..., p-1$. In general it is appropriate e.g. to consider the radii $r_n|(f^i)'(z_0)|$, so that the the differential of f maps each i-th disc to the $i+1$-th one, mod p. One chooses z_0 in the Cremer orbit so that $|(f^i)'(z_0)| \leq 1$ for all $i = 0, 1, ...$ Compare Remark 5.4.

Lemma 4.1. *For each orbit of minimal period np as in Theorem 1.6 the valence of all its points (the number of external rays starting at it) is 1 or 2.*

Proof. See Proof of Proposition 3.2. □

Below we shall prove Theorem 1.6. It will follow the proof of Theorem 1.5, with mild complications. Originally we attempted proving Milnor's "pairwise unlinked" condition (4), see Appendix. Unexpectedly proving that condition occurred difficult (a "proof" in the first version arXiv:2503.03738v1 was incomplete). Here we proceed directly, relying only on the preservation of the cyclic order, Milnor's condition (2), and the proof occurs easy.

Proof of Theorem 1.6. Fix an arbitrary n. Let $O_1, ..., O_K$ be periodic orbits, each of minimal period np, entirely contained in the union of the open discs $B(z_i, r)$, for an arbitrary $r \leq r_n$. Here $r_n = r_0 \exp -\delta np$. We shall find an upper bound for K.

Consider, as in Section 3, all external rays $\mathcal{R}(x, s)$ with their starting points $x \in \bigcup_k O_k$ with arguments $\theta_s(x), s = 0, ...m-1$, where $Knp \leq m \leq 2Knp$. The constant 2 due to Lemma 4.1.

Now, for each ray $\mathcal{R}(x, s)$ consider $\tau(x, s)$ being the largest τ so that $\mathcal{R}(x, s)(\tau) \in \mathrm{cl}\, B(z_i, r)$where $x \in B(z_i, r)$. For each i and $x \in B(z_i, r)$ define the truncated ray $\mathcal{R}'(x, s)) = \mathcal{R}'(x, s)[r]$, running from $\mathcal{R}(x, s)(\tau(x, s))$ to ∞, as in Section 3.

For each i consider the cyclic order in the family of all the rays $\mathcal{R}'(x)$ over $x \in B(z_i, r)$ along the circle $\partial B(z_i, r)$ coinciding with the cyclic order of the set A_i of their arguments (angles) in $S(\infty)$, compare Section 3. Then the doubling map F restricted to A_i maps it 1-to-1 to A_{i+1}.

It preserves the cyclic order, by the arguments the same as in Proof of Theorem 3.

As we already mentioned we would be happy to apply [Milnor, Lemma 2.7]. The sets A_i would play the role of the respective sets in the formal orbit portrait in [Milnor, Lemma 2.3]. For the comfort of the reader we provide the portrait definition in Appendix. This would give the estimate of the number of periodic orbits of minimal period np being $\exp -\delta n$ close to our Cremer's orbit, at most 2, that is $K \leq 2$. However we cannot prove Milnor's property (4), even for a large f-invariant part of the family of rays, namely that all A_i's are *pairwise unlinked*. Miraculously we do not need it and the proof of Theorem 1.6 using the proof of Theorem 1.5 happens to be easy.

We use notation from Section 3. To distinguish objects related to z_i such as e.g. $\mathcal{R}_j[r]$ starting in $\partial B(z_i, r)$ we write $\mathcal{R}_j[r]i$, where i is mod p. Here j enumerates in cyclic order of our external rays (arguments) starting in $B(z_0, r)]$. As mentioned above f preserves this order so it is the same cyclic for each i. Write $\Gamma_j[r]i$ and $Q_j[r]i$ for the corresponding arcs in $B(z_i, r)$, defined as in Section 3, and sectors bounded by $\mathcal{R}_j[r]i, \mathcal{R}_{j+1}[r]i$ and $\Gamma_j[r]i$.

Denote the mapping induced by the doubling map F from A_i to A_{i+1} by σ_i. For each i denote the arcs in $S(\infty)$ between cyclic consecutive points of A_i by $S_j i(\infty), j = 0, ..., m-1$. Call them *angle arcs*. Each is the $S(\infty$ part of the boundary of the sector $Q_j[r]i$. We call the arc (the sector) *regular* if $Q_j[r]i$ does not contain the critical point 0. For each i the arc (sector) containing 0 is called *critical*. Compare Section 3.

We shall consider trajectories of the angle arcs, remembering that F is a homeomorphism between $S_j i(\infty)$ and $S_{\sigma_i j}(i+1)(\infty)$ for $S_j i(\infty)$ regular, and makes an additional round along $S(\infty)$ otherwise. For $S_j i(\infty)$ critical there is no choice of the next arc. It must be the arc (gap) $S(\infty) \setminus \bigcup_{j=0}^{m-1} F(S_j i(\infty)$ by the preservation of the cyclic order. We conclude that the trajectories are disjoint.

When iterating f, for each $i : 0 \leq i < p$ the point 0 serves one sector. So for the period of our rays np or $2np$ (common, as for rotations), at most p trajectories of the angle arcs can be served.

But trajectories of only regular arcs cannot exist because they are periodic so the length of the arcs doubles at each step tending to ∞, in particular exceeding 2π, which is not possible. So the number of the trajectories of the angles or rays must be upper bounded by p. So, after projecting onto the set of the beginnings of the rays, $\bigcup_{k=1}^{K} O_k$, we obtain $K \leq p$. Compare the end of the proof of Theorem 1.5.

□

.

Example 4.2. The period 3 *primitive* renormalization example $f(z) = z^2 - 7/4$ has orbit portrait at period 3 parabolic orbit $A_1 = \{3/7, 4/8\}, A_2 =$

$\{6/7, 1/7\}, A_3 = \{5/7, 2/7\}$ with two trajectories of arcs defined above, one meeting a critical arc (sector) once, the other twice, see [Milnor, Figure 6]. This is the primitive case in Lemma 6.1.

.

Remark 4.3. The proof above holds for any polynomial, provided $F : A_i \to A_{i+1}$ preserves cyclic order for each i. Then $K \leq pd$ where d is the number of critical points in $\mathbb{C}$.

Remark 4.4. Similarly to Theorem 1.6 one proves that for a Siegel S disc fixed for quadratic f with the critical point $0 \notin \partial S$ and ∂S being a Jordan curve, there are few periodic orbits exponentially close to S. The same for any periodic Jordan Siegel disc.

5. Hypothesis H

Here we prove

Theorem 5.1. *Hypothesis H holds for all quadratic polynomials, with $\sharp\mathcal{P} \leq 2n$.*

As announced in Introduction, Section 1, Theorem 5.1 implies Theorem 1.4 (equality of pressures) due to [PRS2].

The proof of Theorem 5.1 is similar to the one in Section 3 and in a version of a Cremer periodic orbit in Section 4, though some additional observations are needed, caused by a possible interference of the critical points. Let us start with two lemmas similar to Lemma 2.1 and Corollary 2.2. They repeat in fact [PRS2, Proposition 3.10] with slightly different proofs; we provide them for completeness.

Denote by $\mathrm{Crit}(f)$ the set of all critical points of f in $J(f)$. For each n consider the metric $\rho_n(x, y) := \max_{i=0,\dots,n-1} \rho(f^i(x), f^i(y))$, where ρ is the spherical metric, see Introduction.

Lemma 5.2. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function and $\mathcal{P} \subset \mathrm{Per}_n$ the set of all periodic points of period n in $J(f)$ as in Hypothesis H with $\rho_n(x, y) < \exp -\delta n$, for n large enough and all $x, y \in \mathcal{P}$. Then there exists $\epsilon > 0$ (not depending on n) and there exists $\mathcal{P}' \subset \mathcal{P}$ such that*

$$\#\mathcal{P}' \geq \exp(n\delta/2), \tag{5.1}$$

and for an arbitrary $x \in \mathcal{P}'$

$$\rho(f^i(x), \mathrm{Crit}(f)) \geq \exp(-n\delta/2) \tag{5.2}$$

for all i except at most $a = a(\delta)$ of "bad" ones for a constant a, and for "bad" i and every $y \in \mathcal{P}'$

$$\rho(f^i(x), f^i(y)) \leq (\exp -n\epsilon)\rho(f^i(x), \mathrm{Crit}(f)). \tag{5.3}$$

Proof. There exists $C > 0$ depending only on f, such that $B(\mathrm{Crit}(f), \exp -Cn) \cap \mathrm{Per}_n = \emptyset$. This follows from so called Rule I in [P-Lyapunov, Lemma 1], saying that for any critical point $\mathrm{cr} \in J(f)$ and $x, f^N(x) \in B(\mathrm{cr}, r)$ it holds $N \geq C^{-1} \log 1/r$, applied for $n = N$ and $r = \exp -n$.

Suppose for simplification $f(z) = z^2 + c$ so there is only one critical point 0 in $J(f)$. Consider the annulus $A = B(0, \exp(-n\delta/2)) \setminus B(0, \exp(-Cn))$. In the logarithmic coordinates A (more precisely $\log A$, with the branch of logarithm with range in $\{x + iy : -\pi < y \leq \pi\}$), can be covered by

$$\frac{2\pi(Cn - n\delta/2)}{\exp -2n\epsilon} \leq \exp 3\epsilon n$$

squares of sides of length $\exp -n\epsilon$ for n large enough. Enumerate them by Q_s.

If i is bad, then $f^i(x)$ belongs to one of these squares. We select $s = s(i)$ with the largest number of points in $f^i(\mathcal{P})$ belonging to it, so

$$s(i) \geq \frac{\exp \delta n}{\exp 3\epsilon n} = \exp n(\delta - 3\epsilon).$$

This (first) selection defines

$$\mathcal{P}'_1 := \{y \in \mathcal{P} : f^{s(i)}(y) \in Q_s\}.$$

Next, for another bad i we make a selection in $\mathcal{P}'_1$ and obtain $\mathcal{P}'_2$, etc. until $\mathcal{P}'_a$ having at least

$$\exp(n(\delta - 3a\epsilon))$$

elements for n large enough.

Here a is the number of returns of the trajectory $f^i(x)_{i=1,\dots,n-1}$ to $B(0, \exp(n\delta/2))$, which is inverse of smallest gap between consecutive times of bad returns, which by Rule I is bounded below by $C^{-1}n\delta/2$. Thus $a \leq 2C/\delta$.

We conclude (5.1) for $\epsilon < \delta^2/6C$. The inequalities (5.2) and (5.3) follows immediately from the definitions.

□

Lemma 5.3 (small distortion)**.** *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function. Then for every $\delta > 0$ there exist $\epsilon > 0$ and $\beta > 0$ such that for all n large enough the following holds. Let $f^i(x), f^i(y), i = 0, ..., n-1$ be two f-periodic orbits of minimal period n which are $r_n = \exp -n\delta$ close, that is such that $\rho_n(x, y) \leq r_n$. Write $\rho(x, y) := r \leq r_n$.*

Assume that the conditions either (5.2) *or* (5.3) *are satisfied for each i, as in Lemma 5.2.*

Then, all $f^i(B(x, (1+\beta)^n r)), i = 0, ..., n-1$ are uniformly boundedly distorted, with distortion of order at most $1 + \exp(-n\epsilon)$ and

$$\mathrm{dist}(f^i(B(x, (1+\beta)^n r)), \mathrm{Crit}(f)) \geq \exp(n\epsilon) \,\mathrm{diam}\, f^i(B(x, (1+\beta)^n r)).$$

In particular f^i is injective on $B(x, (1+\beta)^n r)$ for each $i = 0, ..., n$.

Geometrically, f^i is almost a similarity on $B(x,(1+\beta)^n r)$ for each $i = 0,...,n$.

Proof. Notice that due to (5.2) and (5.3)

$$\Big|\frac{|\rho(f^{i+1}(x_0), f^{i+1}(y_0))/\rho(f^i(x_0,y_0))}{|f'(f^i(x_0))|} - 1\Big| \leq \exp -n\epsilon$$

.

Repeat the proofs of Lemma 2.1 and Corollary 2.2 estimating complex $f'(x_i)$ consecutively. The orbit $(x = x_0,...,x_{n-1}, x_n = x_0)$ plays the role of Cremer's periodic $(z_0,...,z_n = z_0)$.

□

Remark 5.4. In the periodic orbit of x we can choose a point, say x_0, such that

$$\rho(f^i(x_0),(f^i(y_0)) \leq \rho(x_0,y_0) \tag{5.4}$$

Indeed, choose k for which $\rho(f^k(x),(f^k(y))$ attains maximum.

Proof of Theorem 5.1. Fixed n consider for $x = x_0 \in \mathcal{P}'$ the disc $B_n := B(x,r)$ for $r \leq r_n = r_0 \exp -\delta n$ as in Lemma 5.3 (we can omit r_0 for n large enough), that is all $f^i, i = 0,...,n$ are almost similarities on $B(x,(1+\beta)^n r)$, compare notation in Theorem 1.6. Here $\mathcal{P}'$ is as in Lemma 5.2, in particular satisfying (5.1).
Write $B_{n,i} := f^i(B_n)$.

We take $C > 1$ large enough that the proof below makes sense. Let $i = k_1$ be the first time of return of x under the action by f^i to $CB_n = B(x,Cr)$. Since $f^n(x) = x \in CB_n$, $k_1 \leq n$. There are two cases now

I. $k_1 = n$. Then we have te situation similar to that in Theorem 1.6. The quasi-discs $f^i(B(x,\frac{1}{4}Cr))$ for all $0 < i < n$ are disjoint from $B(x,\frac{1}{4}Cr)$. In particular $B_{n,i}$ are well disjoint from B_n for C large enough.

Hence, for $0 \leq i < n$, $B_{n,i}$ are pairwise disjoint, with annuli separating each of the from the others of modulus at least $\frac{1}{2\pi}\log C/3$.

The pairwise disjointness follows, since otherwise, for some $0 < i < k < n$, we would have $f^{n-i}(B_{n,i}) \cap f^{n-i}(\widehat{B}_{n,k}) \neq \emptyset$. The latter quasi-discs are contained in $B(x,2r)$ and $f^{k-i}(B(x,r)$ which therefore intersect. Contradiction. The same consideration holds for moduli

Finally we apply Theorem 1.6 with $p = n$. So the number of periodic orbits of minimal period n in $\mathcal{C}'$ we have discussed, does not exceed n. This contradicts (5.1).

Note that no Cremer periodic orbit is involved; it is not needed. See Theorem 5.5

II. $k_1 < n$.

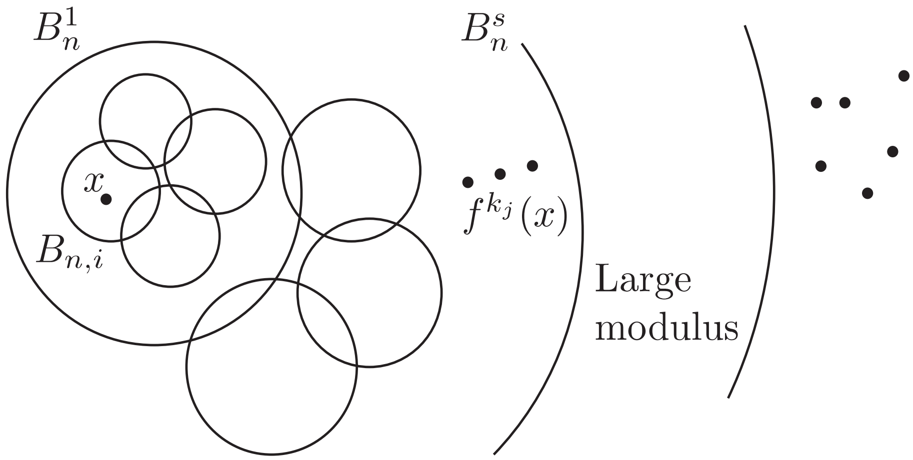

FIGURE 4. Moduli

See Figure 4 below.

Consider $\bigcup_{i=0}^{n/k_1} f^{ik_1}(B(x,2Cr))$ and the disc $B_n^1 := B(x, n8Cr)$ containing it. This containing follows from Remark 5.4, for a right choice of x in its orbit. Let $i = k_2 \leq k_1$ be the first time i of return of x under the action by f^i to CB_n^1. Then either

I_2. $k_2 = k_1$. Then analogously to the case I. we consider the pairwise disjoint quasi-discs $B_{n,i}^1 := f^i(B_n^1)$ for $i = 0, ..., k_1 - 1$ and we consider $p = k_1$

or

II_2. $k_2 < k_1$. Then we define B_n^2 and $B_{n,i}^2$ for $i = 0, ..., k_2 - 1$.

Etc. until $k_s = k_{s-1}$ or $k_s = 1$ for certain s.

Notice that $s \leq \log_2 n$, since for each time $i < s$ we have $k_{i+1} \leq k_i/2$.

Notice that due to our construction $\operatorname{diam} B_{n,i}^s \leq r(4C)^s \prod_{i=0}^{s-1} k_i/k_{i+1}$, where $k_0 := n$. So, for each $i : 0 \leq i < n$,

$$\operatorname{diam} B_{n,i}^s \leq r(4C)^{\log_2 n} n \leq r n^{\log_2 4C+1}, \tag{5.5}$$

where $n^{\log_2 4C+1}$ grows slower than exponentially with n, in particular slower than $(1+\beta)^n$.

So, if $k_s = 1$ then we have the situation as in Section 3 and $B(z_0, r)$ and r exponentially small. If $k_s = k_{s-1}$ we have the situation as in Section 4 with $p = k_s$. Again we receive the number of periodic orbits we have discussed does not exceeding $p \leq n$. This contradicts (5.1).

□

Actually we do not refer above to Theorems 1.5 and Theorem 1.6 directly, because those theorems treat the situations in presence of Cremer fixed point (or periodic orbit).

Instead we apply the following fact having the same proof:

Theorem 5.5. *For every quadratic polynomial $f = f_c$, for every $\delta > 0$ there exist $C > 0$ and $r_0 > 0$ such that for every integers p and n and every sequence of (quasi)discs $B_i, i = 0, ..., p-1$ such that*

- $\operatorname{diam} B_i \leq r_0 \exp(-\delta np) \operatorname{dist}(CB_i, \operatorname{Crit}(f))$,
- *in particular* $\operatorname{diam} B_i \leq r_0 \exp(-\delta np)$,
- *and CB_i are pairwise disjoint,*

there are at most p points $x \in B_0$ of minimal period np, such that $f^{kp+i}(x) \in B_i$ for each $k : 0 \leq k < n$ and $i : 0 \leq i < p$.

6. Appendix. Orbit portraits, following [Milnor]

Let $A_1, ..., A_p$ be subsets of the circle of arguments (angles) at ∞. This collection is called *the formal orbit portrait* if the following conditions are satisfied:

(1) Each A_i is a finite subset of $\mathbb{R}/2\pi\mathbb{Z}$.
(2) For each $i \mod p$, the doubling map $F : \theta \to 2\theta (\mod 2\pi\mathbb{Z})$ carries A_i bijectively onto A_{i+1} preserving cyclic order around the circle,
(3) All of the angles in $A_1 \cup ... \cup A_p$ are periodic under doubling, with a common period rp, and
(4) the sets $A_1, ..., A_p$ are pairwise unlinked; that is, for each $i \neq k$ the sets A_i and A_k are contained in disjoint sub-intervals of $\mathbb{R}/2\pi\mathbb{Z}$.

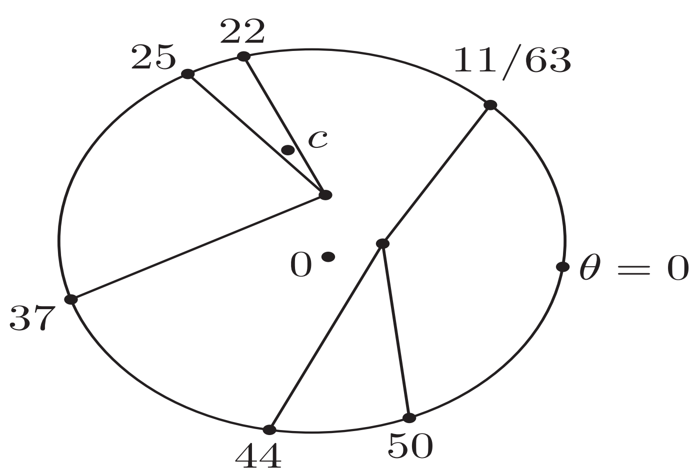

Figure 5. Formal orbit portrait, [Milnor, Fig. 2]

Here at Figure 5, is Milnor's schematic diagram illustrating the formal orbit portrait associated to $z \mapsto z^2 + c$ for $c = \frac{1}{4}e^{2\pi i/3} - 1$ for a period 2 parabolic orbit, with its external ray orbit of period 6.
$A_1/2\pi = \{22/63, 25/63, 37/63\}$ and $A_2/2\pi = \{11/63, 44/63, 50/63\}$. The critical value c lies in the smallest sector. Compare [Milnor, Figures 1 and 2].

Lemma 6.1 (Milnor, Lemma 2.7, primitive vs satellite)**.** *Any formal orbit portrait of valence $\nu > r$ must have $\nu = 2$ and $r = 1$. It follows then that there are just two possibilities:*

Primitive Case. If $r = 1$ so that every every ray which lands on the period p orbit is mapped by to itself by f^p, then at most two rays land on each orbit point.

Satellite Case. If $r > 1$, then $\nu = r$, so that exactly r rays land on each orbit point, and all of these rays belong to a single cyclic orbit.

The number r is defined as the number of points in each A_i in one F-orbit (by (3) it does not depend on the orbit nor i).

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