THERMODYNAMIC FORMALISM METHODS IN ONE-DIMENSIONAL REAL AND COMPLEX DYNAMICS

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Abstract. We survey some results on non-uniform hyperbolicity, geometric pressure and equilibrium states in one-dimensional real and complex dynamics. We present some relations with Hausdorff dimension and measures with refined gauge functions of limit sets for geometric coding trees for rational functions on the Riemann sphere. We discuss fluctuations of iterated sums of the potential $-t\log|f'|$ and of radial growth of derivative of univalent functions on the unit disc and the boundaries of range domains preserved by a holomorphic map f repelling towards the domains.

Contents

1. Thermodynamic formalism, introductory notions

Among founders of this theory are [71], [3] and David Ruelle, who wrote in [69]: "thermodynamic formalism has been developed since G. W. Gibbs

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to describe [...] physical systems consisting of a large number of subunits". In particular one considers a *configuration space* Ω of functions $\mathbb{Z}^n \to \mathbb{A}$ on the lattice $\mathbb{Z}^n$ with interacting values in $\mathbb{A}$ over its sites, e.g. "spin" values in the Ising model of ferromagnetism. One considers probability distributions on Ω, invariant under translation, called *equilibrium states* for potential functions on Ω.

Given a mapping $f : X \to X$ one considers as a configuration space the set of trajectories $n \mapsto (f^n(x))_{n\in\mathbb{Z}_+}$ or $n \mapsto \Phi(f^n(x))_{n\in\mathbb{Z}_+}$ for a test function $\Phi : X \to Y$.

The following simple fact [3, Lemma 1.1] and [69, Introduction], [63, Introduction], resulting from Jensen's inequality applied to the function logarithm, stands at the heart of thermodynamic formalism.

Lemma 1.1 (Finite Variational Principle)**.** *For given real numbers $\phi_1, \dots, \phi_d$, the function $F(p_1, \dots p_d) := \sum_{i=1}^n -p_i \log p_i + \sum_{i=1}^d p_i\phi_i$ defined on the simplex $\{(p_1, \dots, p_d) : p_i \geq 0, \sum_{i=1}^d p_i = 1\}$ attains its maximum value $P(\phi_1, \dots, \phi_d) = \log \sum_{i=1}^d e^{\phi_i}$ at and only at $\widehat{p}_j = e^{\phi_j} \left(\sum_{i=1}^d e^{\phi_i}\right)^{-1}$.*

We can read $i \mapsto \phi_i, i = 1, \dots, d$ as a *potential* function and $\widehat{p}_i$ as the equilibrium probability distribution on the finite space $\{1, \dots, d\}$. $P(\phi_1, \dots, \phi_d)$ is called the *pressure* or *free energy*, see [69].

Let $f : X \to X$ be a continuous mapping of a compact metric space X and $\phi : X \to \mathbb{R}$ be a continuous function (the potential). We define the *topological pressure* or free energy by

Definition 1.2.

$$P_{\mathrm{var}}(f, \phi) = \sup_{\mu\in\mathcal{M}(f)} \left(h_\mu(f) + \int_X \phi \, d\mu \right), \tag{1.1}$$

where $\mathcal{M}(f)$ is the set of all f-invariant Borel probability measures on X and $h_\mu(f)$ is measure theoretical entropy. Sometimes we write $\mathcal{M}(f, X)$.

Recall that $h_\mu(f) = \sup_{\mathbb{A}} \lim_{n\to\infty} \frac{1}{n+1} \sum_{A\in\mathbb{A}^n} -\mu(A) \log \mu(A)$, where the supremum is taken over finite partitions $\mathbb{A}$ of X, where $\mathbb{A}^n := \bigvee_{j=0,\dots,n} f^{-j}\mathbb{A}$. Notice that this resembles the sum $\sum_{i=1}^n -p_i \log p_i$ in Lemma 1.1.

Topological pressure can also be defined in other ways, e.g. by (6.2), and then its equality to the one given by (1.1) is called the variational principle. This explains the notation P_{var}. Any $\mu \in \mathcal{M}(f)$ for which the supremum in (1.1) is attained is called *equilibrium*, *equilibrium measure* or *equilibrium state*.

A model case is any map $f : U \to \mathbb{R}^n$ of class C^1, defined on a neighbourhood U of a compact set $X \subset \mathbb{R}^n$, *expanding* (another name: *uniformly expanding* or *hyperbolic* in dimension 1) that is there exist $C > 0, \lambda > 1$ such that for all positive integers n all $x \in X$ and all v tangent to $\mathbb{R}^n$ at x,

$$||Df^n(v)|| \geq C\lambda^n ||v||, \tag{1.2}$$

and *repelling* that is every forward trajectory sufficiently close to X must be entirely in X. Not assuming the differentiability of f one uses the notion of *distance expanding* meaning the increase of distances under the action of f by a factor at least $\lambda > 1$ for pairs of distinct points sufficiently close to each other. Repelling happens to be equivalent to the internal condition: $f|_X$ being an open map, provided f is open on a neighbourhood of X, see [63, Lemma 6.1.2]. Then the classical theorem holds, here in the version from [63, Section 5.1]:

Theorem 1.3. *Let $f : X \to X$ be a distance expanding, topologically transitive continuous open map of a compact metric space X and $\phi : X \to \mathbb{R}$ be a Hölder continuous potential. Then, there exists exactly one measure $\mu_\phi \in \mathcal{M}(f, X)$, called the Gibbs measure, satisfying*

$$C < \frac{\mu_\phi(f_x^{-n}(B(f^n(x), r_0))}{\exp(S_n\phi(x) - nP(\phi))} < C^{-1} \tag{1.3}$$

where f_x^{-n} is the branch of f^{-n} mapping $f^n(x)$ to x (locally making sense, since f is a local homeomorphism) and $S_n\phi(x) := \sum_{j=0}^{n-1} \phi(f^j(x))$.

The measure μ_ϕ is the only equilibrium state for ϕ. It is equivalent to the unique ϕ-conformal measure m_ϕ, that is a forward quasi-invariant Borel probability measure m_ϕ with Jacobian $\exp -(\phi - P(\phi))$. *Moreover, the limit $P(\phi) = P(f, \phi) :=$ $\lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in f^{-n}(x_0)} \exp S_n\phi(x)$ exists and is equal to $P_{\mathrm{var}}(f, \phi)$ for every $x \in X$.*

This $P(\phi)$ is a normalizing quantity corresponding to $P(\phi_1, \dots, \phi_d)$ in Lemma 1.1 and the sum in the definition of $P(\phi)$ corresponds to the so called *statistical sum* over the space Ω_n of all admissible configurations over $\{0, 1, \dots, n-1\}$, as in the Ising model. Compare to the *tree pressure* defined in Definition 6.2.

So $\varsigma : \Sigma^d \to \Sigma^d$, the shift to the left on the space $\Sigma^d = \{(\alpha_0, \alpha_1, \dots) : \alpha_j \in \{1, \dots, d\}\}$, defined by $\varsigma((\alpha_n)) = (\alpha_{n+1})$, is an example where Theorem 1.3 holds. The sets $f_x^{-n}(B(f^n(x), r_0)$ correspond to *cylinders* of fixed $\{\alpha_j \in \{1, \dots, d\}, j = 0, \dots, n-1\}$. One can impose an admissibility condition: $\alpha_i\alpha_{i+1}$ admissible if the pair has the digit 1 attributed in a defining 0,1 $d \times d$ matrix. Then one calls the system a *one-sided topological Markov chain.*

The condition of openness of f can be replaced by a weaker one: the existence of a finite Markov partition, see [63].

The existence of a conformal measure follows from the existence of a fixed point in the convex weakly*-compact set of probability measures for the dual operator to the transfer (Perron-Frobenius-Ruelle) operator $\mathcal{L}$ divided by the norm, where for $u : X \to \mathbb{R}$ continuous one defines

$$\mathcal{L}(u)(x) := \sum_{y \in f^{-1}(x)} u(y) \exp \phi(y). \tag{1.4}$$

Indeed, for every Borel set $Y \subset X$ on which f is injective, denoting by I_Y indicator function: 1 on Y, 0 outside Y, due to an approximation by continuous functions, one has for every finite Borel measure ν on X

$$(\mathcal{L}^*(\nu))(Y)) = \int_X \mathcal{L}(I_Y)\, d\nu = \int_{f(Y)} \exp \phi \circ f|_Y^{-1}\, d\nu. \tag{1.5}$$

Hence the (positive) eigen-measure m_ϕ has Jacobian for $(f|_Y)^{-1}$ equal to $\exp(\phi \circ f|_Y^{-1})/\lambda$, hence f has Jacobian $\exp(-\phi)$ multiplied by an eigenvalue $\lambda := \exp P(\phi)$.

The proof of the existence of an invariant Gibbs measure equivalent to m_ϕ is harder. One first proves the existence of a positive eigenfunction u_ϕ for $\mathcal{L}$ and then defines $\mu_\phi = u_\phi m_\phi$. For a more complete introduction to this theory, see e.g. [63].

2. Introduction to dimension 1

Thermodynamic formalism is useful for studying properties of the underlying space X. In dimension 1, for f real of class $C^{1+\varepsilon}$ or f holomorphic, for an expanding repeller X, considering $\phi = \phi_t := -t \log |f'|$ for $t \in \mathbb{R}$, (1.3) gives

$$\begin{aligned} \mu_{\phi_t}(f_x^{-n}(B(f^n(x), r_0))) \approx \exp(S_n\phi(x) - nP(\phi_t)) \approx \\ \operatorname{diam} f_x^{-n}(B(f^n(x), r_0))^t \exp -nP(\phi_t). \end{aligned} \tag{2.1}$$

The latter follows from a comparison of the diameter with the inverse of the absolute value of the derivative of f^n at x, due to *bounded distortion*. Here, the symbol "$\approx$" denotes that the mutual ratios are bounded by a constant.

When $t = t_0$ is a zero of the function $t \mapsto P(\phi_t)$, this gives

$$\mu_{\phi_{t_0}}(B) \approx (\operatorname{diam} B)^{t_0} \tag{2.2}$$

for all small balls B (the t_0-Ahlfors measure property). We obtain the so-called Bowen's formula for Hausdorff dimension:

$$\mathrm{HD}(X) = t_0. \tag{2.3}$$

Moreover, the Hausdorff measure of X in this dimension is finite and nonzero.

A model example of application is the proof of

Theorem 2.1. *For $f_c(z) := z^2 + c$ for an arbitrary complex number $c \neq 0$ sufficiently close to 0, the invariant Jordan curve J (Julia set for f_c) is a fractal, i.e. has Hausdorff dimension bigger than 1.*

Sketch of Proof. $t_0 > 1$ yields $\mathrm{HD}(J) = t_0 > 1$ by (2.2) (one does not need to use the invariance of $\mu_{\phi_{t_0}}$).

The case $t_0 = 1$ yields by (2.2) finite Hausdorff measure in dimension 1, i.e. the rectifiability of J. To conclude that J is a circle and $c = 0$, one can use ergodic invariant measures in the classes of harmonic ones on J from inside and outside. They must coincide. This relies on Birkhoff's Ergodic Theorem, the heart of ergodic theory. This is an "echo" of the celebrated Mostov Rigidity Theorem. See [73] and [63, Theorem 9.5.5]. $\square$

In dimension 1 (real or complex), we call c a critical point if the derivative $f'(c) = 0$. The set of critical points will be denoted by $\mathrm{Crit}(f)$.

In this survey, we allow for the presence of critical points and concentrate mainly on two cases:

1. (Complex case) f is a rational mapping of degree at least 2 of the Riemann sphere $\overline{\mathbb{C}}$. We consider f acting on its Julia set $K = J(f)$.

For entire or meromorphic maps see e.g. [1, 2], compare Definition 5.2.

2. (Real case) f is a *generalized multimodal map* defined on a neighbourhood $U_K \subset \mathbb{R}$ of its compact invariant subset K. We assume that $f \in C^2$, is non-flat at all of its turning and inflection critical points, satisfies the *bounded distortion* property for iterates, abbr. BD, see [57], is topologically transitive and has positive topological entropy on K.

We assume that K is a maximal invariant subset of a finite union of pairwise disjoint closed intervals $\widehat{I} = I^1 \cup \cdots \cup I^k \subset U_K$ whose endpoints are in K. (This maximality corresponds to the Darboux property, compare [57, Appendix A] and [38, page 49].) We write $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$, with the subscript $+$ to mark positive entropy. In place of BD one can assume C^3 (and write $(f, K) \in \mathscr{A}_+^3$), and assume that all periodic orbits in K are hyperbolic repelling. Indeed, changing f outside K if necessary, one can get the corrected (f, K) in $\mathscr{A}_+^{\mathrm{BD}}$.

Recall the notions concerning periodic orbits: *Parabolic* means $f^n(p) = p$ with $(f^n)'(p)$ being a root of unity. For $|(f^n)'(p)| = 1$ the term *indifferent periodic* is used and for $|(f^n)'(p)| > 1$ the term *hyperbolic repelling.* If $|(f^n)'(p)| < 1$ the orbit is called *hyperbolic attracting.*

For the real setting, see [57], [16] and [55]. Examples are provided by basic sets in the spectral decomposition [10].

Question. Are there any other examples?

Problem. Generalize the real case theory, see further sections, to the piecewise continuous maps, that is allow the intervals I^j to have common ends (see [23] for some results in this direction).

In this survey, we compare equilibrium states to (refined) Hausdorff measures in the complex case. For the real case, we refer the reader to [22] and the references therein.

3. Hyperbolic potentials

For general $f : X \to X$ and $\phi : X \to \mathbb{R}$ as in Definition 1.2 the following conditions are of special interest [24],

1) $P(f, \phi) > \sup \phi$,
2) $P(f^n, S_n\phi) > \sup_X S_n\phi$ for an integer n,
3) $P(f, \phi) > \sup_{\nu \in \mathcal{M}(f)} \int \phi \, d\nu$,
4) For each equilibrium state μ for the potential ϕ, the entropy $h_\mu(f)$ is positive.

The conditions 2) – 4) are equivalent, see [24, Proposition 3.1]. Potentials ϕ satisfying them have been called in [24] *hyperbolic*. The condition 1) has longer traditions, see [12]. The intuitive meaning is that no minority of trajectories carries the full pressure.

For every $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ rational of degree at least 2 and $\phi : J(f) \to \mathbb{R}$ Hölder continuous, the following condition is also equivalent to 2) – 4), see [24]:

5) For each ergodic equilibrium state μ for ϕ, the Lyapunov exponent $\chi(\mu) := \int \log |f'| \, d\mu$ is positive, that is for μ-a.e. x, $\chi(\mu) = \chi(x) := \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| > 0$.

The conditions 2)-5) are also equivalent in the real case for $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ or $(f, K) \in \mathscr{A}_+^3$ and all periodic orbits hyperbolic repelling. The arguments in [24] work. See also [30].

Theorem 3.1. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational mapping as above. If ϕ is a Hölder continuous hyperbolic potential on $J(f)$, then there exists a unique equilibrium state μ_ϕ. For every Hölder $u : J(f) \to \mathbb{R}$, the Central Limit Theorem (abbr. CLT) for the sequence of random variables $u \circ f^n$ and μ_ϕ holds.*

For a proof, see [47] and preceding [12]. To find this equilibrium one can iterate the transfer operator proving $\mathcal{L}^n(1\!\!1)/\exp nP(f, \phi) \to u_\phi$. The convergence is uniformly $\exp -\sqrt{n}$ fast and the limit is Hölder continuous, [11]. Finally, define $\mu_\phi := u_\phi \cdot m_\phi$, as at the end of Section 1.

Remark 3.2. Given μ_ϕ a priori, an efficient way to study it is an inducing method, see [75], i.e. the use of a return map $A \ni x \mapsto f^{n(x)}(x) \in A$ for A and $n(x)$ adequate to μ_ϕ. Then one proves even an exponential convergence (with any u Hölder in place of $1\!\!1$), which yields exponential mixing, hence stochastic laws for $u \circ f^n$ for Hölder u, e.g. CLT, LIL, compare Sections 9 and 10. See also Remark 5.4. The key feature is the exponential decay of $\mu_\phi(A_n)$, where $A_n := \{x \in A : n(x) \geq n\}$.

See also [7] for the real case, and stronger [29] and [30] including also the complex case proving the exponential convergence to u_ϕ, hence CLT and LIL. See also [74] for endomorphisms f of higher dimensional complex projective space, where 1) is replaced by a stronger "gap" assumption.

4. Non-uniform hyperbolicity in real and complex dimension 1

Here we discuss a set of conditions, valid in both the real and complex situations. Below we concentrate on the case of complex rational maps with $K = J(f)$, only remarking differences in the real case.

(a) CE. *Collet-Eckmann condition.* There exist $\lambda_{CE} > 1$ and $C > 0$ such that for every critical point c in $J(f)$, whose forward orbit does not meet other critical points, for every $n \geq 0$ we have

$$|(f^n)'(f(c))| \geq C\lambda_{CE}^n.$$

Moreover, there are no parabolic (indifferent) periodic orbits.

(b) CE2(z_0). *Backward* or *second Collet-Eckmann condition at* $z_0 \in J(f)$. There exist $\lambda_{CE2} > 1$ and $C > 0$ such that for every $n \geq 1$ and every $w \in f^{-n}(z_0)$ (in a neighbourhood of K in the real case)

$$|(f^n)'(w)| \geq C\lambda_{CE2}^n.$$

(b') CE2. *The second Collet-Eckmann condition.* CE2(c) holds for all critical points c not in the forward orbit of any other critical point.

(c) TCE. *Topological Collet-Eckmann condition.* There exist $M \geq 0, P \geq 1$, $r > 0$ such that for every $x \in K$ there exists a strictly increasing sequence of positive integers n_j, $j = 1, 2, \ldots$, such that $n_j \leq P \cdot j$ and for each j (and discs $B(\cdot)$ below understood in $\overline{\mathbb{C}}$ or $\mathbb{R}$)

$$\#\{0 \leq i < n_j : \mathrm{Comp}_{f^i(x)}\, f^{-(n_j-i)}B(f^{n_j}(x), r)) \cap \mathrm{Crit}(f) \neq \emptyset\} \leq M, \tag{4.1}$$

where in general $\mathrm{Comp}_z V$ means for $z \in V$ the component of V containing z.

In the real case, one adds the condition that there are no parabolic periodic orbits, which is automatically true in the case of complex rational maps.

(d) ExpShrink. *Exponential shrinking of components.* There exist $\lambda_{\mathrm{Exp}} > 1$ and $r > 0$ such that for every $x \in K$, every $n > 0$ and every connected component W_n of $f^{-n}(B(x, r))$ for the disc (interval) $B(x, r)$ in $\overline{\mathbb{C}}$ (or $\mathbb{R}$), intersecting K

$$\mathrm{diam}(W_n) \leq \lambda_{\mathrm{Exp}}^{-n}. \tag{4.2}$$

(e) LyapHyp (*Lyapunov hyperbolicity*). There is a constant $\lambda_{\mathrm{Lyap}} > 1$ such that the Lyapunov exponent $\chi(\mu)$ of any ergodic measure $\mu \in \mathcal{M}(f, K)$ satisfies $\chi(\mu) \geq \log \lambda_{\mathrm{Lyap}}$.

(f) UHP. *Uniform Hyperbolicity on periodic orbits.* There exists $\lambda_{\mathrm{Per}} > 1$ such that every periodic point $p \in K$ of period $k \geq 1$ satisfies

$$|(f^k)'(p)| \geq \lambda_{\mathrm{Per}}^k.$$

We distinguish LyapHyp as the most adequate among these conditions to carry the name (strong) non-uniform hyperbolicity.[1]

Theorem 4.1. *1. The conditions (c)–(f) and else (b) for some z_0 are equivalent in the complex case. In the real case, the equivalence also holds under the assumption of weak isolation (see the definition below).*

2. In the complex case, the suprema over all possible constants λ_{Exp}, λ_{CE2} (supremum over all z_0), λ_{Per} and λ_{Lyap} coincide.

3. Both CE and CE2 imply (c)–(f).

4. If there is only one critical point in the Julia set in the complex case or if f is S-unimodal on $K = I$ in the real case, i.e. has just one turning critical point c and negative Schwarzian derivative on $I \setminus \{c\}$, then all conditions above are equivalent to each other.

For more details, see [58], [67] and [57].

Definition 4.2. $(f, K) \in \mathscr{A}$ is said to be *weakly isolated* if there exists an open neighbourhood U of K in the domain of f such that for every f-periodic orbit $O(p) \subset U$ is contained in K.

In the complex case, we can replace (4.1) by

$$\deg\Big(f^{n_j}|_{\mathrm{Comp}_x f^{-n_j}(B(f^{n_j}(x),r))}\Big) \le M'$$

for a constant M'. In the real case, this condition is weaker than (4.1) since f mapping W_{n+1} into W_n may happen not surjective. It can have folds, thus truncating backward trajectories of critical points acquired before when pulling back.

In the real case, the proof of CE⇒TCE can be found in [40]. For the complex case, we refer the reader to [60].

The implication TCE⇒CE was proved in the complex case in [52, Theorem 4.1]. The proof used the idea of the "reversed telescope" by [17]. In the real case, this implication was proved for S-unimodal maps in [41]. In presence of more than one critical point this implication may be false, see [58, Appendix C].

Question. Is this implication true for every $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ with one critical point, provided it is weakly isolated? See Definition 4.2. It seems that the answer is yes.

Since the condition TCE is stated in purely topological terms (in the class of maps without indifferent periodic orbits), it is invariant under topological conjugacy. So we obtain the following immediate corollary.

Corollary 4.3. *All equivalent conditions listed above are invariant under topological conjugacies between (f, K)'s).*

[1]Then all Hölder continuous potentials are hyperbolic, see Condition 5) in Section 3 and [24].

Another proof of the topological invariance of CE in the complex case was provided in [61] with the use of Heinonen and Koskela criterion for quasi-conformality, [21].

Note that this topological invariance is surprising, as all the conditions except TCE are expressed in geometric-differential terms. I do not know how to express CE for unimodal maps of interval in the (topological-combinatorial) kneading sequence terms.

An important lemma used here has been an estimate of an average distance in the logarithmic scale of every orbit from $\mathrm{Crit}(f)$, see [11]. Namely

Lemma 4.4.

$$\sum_{j=0}^{n}{}' - \log|f^j(x) - c| \le Qn \tag{4.3}$$

for a constant $Q > 0$ every $c \in \mathrm{Crit}(f)$, every $x \in K$ and every integer $n > 0$. Σ' means that we omit in the sum an index j of smallest distance $|f^j(x) - c|$.

An order of proving the equivalences in Theorem 4.1 is
$\mathrm{CE2}(z_0) \Rightarrow$ExpShrink$\Rightarrow$LyapHyp$\Rightarrow$UHP$\Rightarrow\mathrm{CE2}(z_0)$ and separately
ExpShrink$\Leftrightarrow$TCE. E.g. assumed UHP one proves $\mathrm{CE2}(z_0)$ by "shadowing", compare the beginning of Section 6.

5. Geometric pressure and equilibrium states

We go back to topological pressure, Definition 1.2, but for $\phi = -t\log|f'|$, $t \in \mathbb{R}$ in the complex $K = J(f)$ or real cases, where ϕ can attain the values $\pm\infty$ at the critical points of f. See the beginning of Section 2. We call it the geometric pressure, because it is useful in studying of geometry of the underlying space, e.g. as in (2.3) via equilibrium states for all t.

The definition of $P_{\mathrm{var}}(f, -t\log|f'|)$ in Definition 1.2 makes sense due to $\chi(\mu) \ge 0$ for all $\mu \in \mathcal{M}(f)$, in particular due to the integrability of $\log|f'|$, see [48] and [67, Appendix A] for a simpler proof. We conclude that it is convex and monotone decreasing. We start by defining a quantity occurring equal to $P(t) = P_{\mathrm{var}}(t) := P_{\mathrm{var}}(f, -t\log|f'|)$, to explain its geometric meaning, compare with Section 2.

Definition 5.1 (Hyperbolic pressure)**.**

$$P_{\mathrm{hyp}}(t) := \sup_{X\in\mathcal{H}(f,K)} P(f|_X, -t\log|f'|),$$

where $\mathcal{H}(f,K)$ is defined as the space of all compact forward f-invariant (that is $f(X) \subset X$) hyperbolic subsets of K, repellers in $\mathbb{R}$.

From this definition, it immediately follows that:

Proposition 5.2. (Generalized Bowen's formula, compare (2.3)) *The first zero t_0 of $t \mapsto P_{\mathrm{hyp}}(K,t)$ is equal to the hyperbolic dimension $\mathrm{HD}_{\mathrm{hyp}}(K)$ of K, defined by $\mathrm{HD}_{\mathrm{hyp}}(K) := \sup_{X\in\mathcal{H}(f,K)} \mathrm{HD}(X)$.*

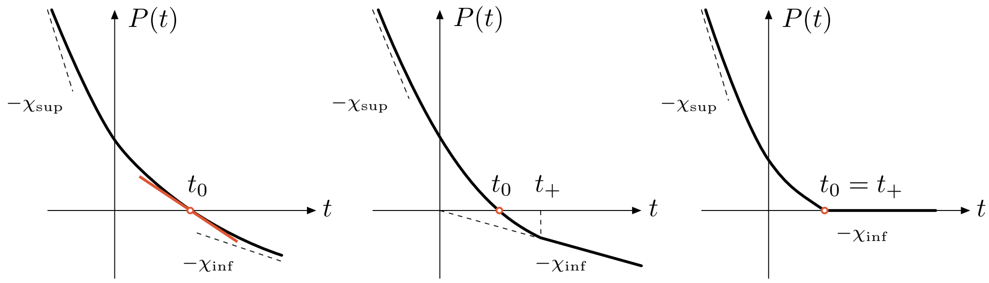

FIGURE 1. The geometric pressure: LyapHyp with $t_+ = \infty$, LyapHyp with $t_+ < \infty$, and non-LyapHyp. This Figure is taken from [16], see notation in Remarks below.

For the discussion $\mathrm{HD}_{\mathrm{hyp}}(J(f))$ vs $\mathrm{HD}(J(f))$, see [31, Section 2.13].

Below we state Theorem 5.3 proved in [56] in the complex setting and in [57] in the real setting. It extends [6, 43] and [25]. See also impressive [13].

Theorem 5.3. *1. Real case,* [57]. *Let $(f, K) \in \mathscr{A}_+^3$ and let all f-periodic orbits in K be hyperbolic repelling. Then $P(t)$ is real analytic on the open interval bounded by the "phase transition parameters" t_- and t_+. For every $t \in (t_-, t_+)$, the domain where*

$$P(t) > \sup_{\nu \in \mathcal{M}(f)} -t \int \log |f'| \, d\nu, \tag{5.1}$$

there is a unique invariant equilibrium state. It is ergodic and absolutely continuous with respect to an adequate conformal measure m_{ϕ_t} with the density bounded from below by a positive constant almost everywhere. If furthermore f is topologically exact on K (that is for every V an open subset of K there exists $n \geq 0$ such that $f^n(V) = K$), then this measure is mixing, has exponential decay of correlations and it satisfies the Central Limit Theorem for Lipschitz gauge functions.

2. Complex case, [56]. *The assertion is the same. One assumes a very weak expansion: the existence of arbitrarily small nice, or pleasant, couples and hyperbolicity away from critical points.*

Remarks. 1) t_- and t_+ are called the phase transition parameters. Since $P(0) = h_{\mathrm{top}}(f) > 0$, $t_- < 0 < t_+$, they need not exist; we say then they are equal to $-\infty$ and/or $+\infty$ respectively. $P(t)$ is linear to the left of t_- and to the right of t_+, equal to $t \mapsto -t\chi_{\sup}$ where $\chi_{\sup} := \sup_\nu \chi(\nu)$ and $t \mapsto -t\chi_{\inf}$, where $\chi_{\inf} := \inf_\nu \chi(\nu)$, respectively. Of course, $P(t)$ is not real-analytic at finite t_- and t_+.

2) For $f(z) = z^2 - 2$, $f : [-2, 2] \to [-2, 2]$ (the Tchebyshev polynomial), we have $f(2) = 2, f'(2) = 4, \chi(l) = \log 2$, where l is the normalized length measure. We have $P(t) = \log 2 - t \log 2$ for $t \geq -1$ and $P(t) = -t \log 4$ for $t \leq -1$, so $t_- = -1$, $P(t)$ is non-differentiable at t_- and for $t = -1$ there are two ergodic equilibrium states: Dirac at $z = 2$ and l.

3) For any f non-LyapHyp, $t_+ = t_0 < \infty$. However $t_+ < \infty$ can happen even for f LyapHyp, see [33] and [8, 9].

4) Notice that the condition (5.1) is similar to the condition 3) from Section 3. For f LyapHyp and $t > t_+$, no equilibrium state can exist, see [24].

5) For real f as in Theorem 5.3 satisfying LyapHyp and $K = \widehat{I}$, we have $t_0 = 1$ and for $-\log|f'|$ we conclude that a unique equilibrium state exists which is a.c.i.m.(that is: invariant absolutely continuous with respect to Lebesgue measure). In fact this assertions hold even for $t = t_0 = t_+ = 1$ with very weak hyperbolicity properties e.g. $|(f^n)'(f(c))| \to \infty$ for all $c \in \mathrm{Crit}(f)$, see [5] and [70]. For the complex case, see [18] and stronger [68].

Remark 5.4. In the proof of Theorem 5.3, we use (compare with the Remark 3.2) a return map $F(x) = f^{n(x)}$ to a "nice" (Markov) domain. However unlike in [75], we do not use in the construction of this set the equilibrium measure μ_ϕ because we do not know a priori that it exists. The construction is geometric. F is an infinite Iterated Function System, more precisely the family of all branches of F^{-1} is, see [35] and [42] and references therein, expanding due to the "acceleration" from f to F. Then we consider an equilibrium state $\mathtt{P}$ for (F, Φ) where $\Phi(x) := \sum_{j=0}^{n(x)-1} \phi_t(f^j(x))$, and consider an equivalent conformal measure. We propagate these measures to the Lai-Sang Young tower $\{(x,j) : 0 \le j < n(x)\}$ and project by $(x,j) \mapsto f^j(x)$ to K.[2]

Stochastic properties of $\mathtt{P}$ stay preserved along the construction to μ_ϕ. The analyticity of $P(t)$ follows from expressing $P(t)$ as zero of a pressure for F with potential depending on two parameters and Implicit Function Theorem. The latter idea came from [72].

Remark 5.5. For probability measures μ_n weakly* convergent to some $\widehat{\mu}$, in presence of critical points $\int \log|f'|\, d\mu_n$ need not converge to $\int \log|f'|\, d\widehat{\mu}$. Only upper semicontinuity holds. Therefore, for $t > 0$, the equilibrium states for $t_n \to t$ need not converge to an equilibrium state for t. A priori, the free energy in the Definition 1.2 can jump down. However, a modification of this method to prove existence of equilibria works, see [13].

Notice also that passing to a weak*-limit with averages of Dirac measures on $\{x, \dots, f^n(x)\}$ proves $\limsup_{n\to\infty} \sup_{x\in K} \frac{1}{n} S_n(\log|f'|)(x) \le \chi_{\max}$. However an analogous inequality $\liminf \dots \ge \chi_{\inf}$ is obviously false. These observations contribute to the understanding of Lyapunov spectrum.

Remarks on the Lyapunov spectrum. Theorem 5.3 allows us to express the so-called dimension spectrum for Lyapunov exponents with the

[2] For applications to decide the existence or nonexistence of a finite a.c.i.m. for maps of interval with flat critical points or for entire or meromorphic maps depending on the $\mathtt{P}$-integrability of the first return time, see papers by N. Dobbs, B. Skorulski, J. Kotus, G. Świątek.

use of Legendre transform, that is for all $\alpha > 0$ and $\mathcal{L}(\alpha) := \{x \in K : \chi(x) = \alpha\}$

$$\mathrm{HD}(\mathcal{L}(\alpha)) = \frac{1}{|\alpha|} \inf_{t\in\mathbb{R}} (P(t) + \alpha t)\,. \tag{5.2}$$

An ingredient is **Mañé's equality**

$$\mathrm{HD}(\mu) = h_\mu(f)/\chi(\mu) \tag{5.3}$$

provided $\chi(\mu) > 0$, [63], where $\mathrm{HD}(\mu) := \sup\{\mathrm{HD}(X) : \mu(X) = 1\}$, applied to μ_{ϕ_t}.

The equality (5.2) concerns regular x's, where $\chi(x) = \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|$ exists. It is also possible to provide formulas or at least estimates for Hausdorff dimension of the sets of irregular points $\mathcal{L}(\alpha, \beta) := \{x \in K : \underline{\chi}(x) = \alpha, \overline{\chi}(x) = \beta\}$ for lower and upper Lyapunov exponents where we replace lim by lim inf and lim sup respectively. See [15] and [16] for this theory in complex and real settings.

However, these papers give no information about the size of sets with zero (upper) Lyapunov exponent. Note at least that if $J(f) \neq \overline{\mathbb{C}}$ then $\mathrm{Leb}_2\{x \in J(f) : \overline{\chi}(x) > 0\} = 0$. This is so because $\overline{\chi}(x) > 0$ implies there exists $\mathcal{N} \subset \mathbb{Z}_+$ of positive upper density, such that for $n \in \mathcal{N}$, (4.2) and (4.1) hold, see [28, Section 3].

We do not know whether $\chi(x) = -\infty$ can happen for x not pre-critical, except there is only one critical point in K, where $\chi(x) > -\infty$ follows from (4.3), see [15, Lemma 6].

For x being a critical value we can prove (in analogy to $\chi(\mu) \geq 0$):

Theorem 5.6 ([28])**.** *If for a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ there is only one critical point c in $J(f)$ and no parabolic periodic orbits, then $\underline{\chi}(f(c)) \geq 0$.*

For S-unimodal maps of interval this was proved by [41].

6. Other definitions of geometric pressure

Definition 6.1 (safe)**.** See [63, Definition 12.5.7]. We call $z \in K$ *safe* if $z \notin \bigcup_{j=1}^\infty (f^j(\mathrm{Crit}(f)))$ and for every $\delta > 0$ and all n large enough $B(z, \exp(-\delta n)) \cap \bigcup_{j=1}^n (f^j(\mathrm{Crit}(f))) = \emptyset$.

Notice that this definition implies that all points except at most a set of Hausdorff dimension 0, are safe.

Definition 6.2 (Tree pressure)**.** For every $z \in K$ and $t \in \mathbb{R}$ define

$$P_{\mathrm{tree}}(z, t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{f^n(x)=z,\, x\in K} |(f^n)'(x)|^{-t}. \tag{6.1}$$

Compare with $P(f, \phi)$ from Theorem 1.3. Under suitable conditions, e.g. for z "safe" the limit exists, it is independent of z and equal to $P(t)$. See [51], [58] and [63] for the complex case and [57] and [55] for the real case.

A key is to extend all trajectories $T_n(x) = \{x, \dots, z\}$ backward and forward by time $m \ll n$ to get an Iterated Function System for f^{n+m} and to consider its limit set. Its trajectories for time n "shadow" $T_n(x)$. This proves $P_{\rm tree}(z,t) \le P_{\rm hyp}(t)$. The opposite inequality is immediate.

(Similarly one proves $P_{\rm var}(t) \le P_{\rm hyp}(t)$. Given μ with $\chi(\mu) > 0$ one captures a hyperbolic X by Pesin-Katok method.)

For a continuous potential $\phi : X \to \mathbb{R}$, consider

$$P_{\rm sep}(f,\phi) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \big(\sup_Y \sum_{y\in Y} \exp S_n\phi(y) \big), \tag{6.2}$$

where the supremum is taken over all (n,ε)-separated sets $Y \subset X$, that is such Y that for every distinct $y_1, y_2 \in Y$, $\rho_n(y_1,y_2) \ge \varepsilon$, where ρ_n is the metric defined by $\rho_n(x,y) = \max\{\rho(f^j(x), f^j(y)) : j = 0, \dots, n\}$.

For $\phi = -t\log|f'|$ for positive t, in presence of critical points for f, $P_{\rm sep}$ is always equal to ∞ by putting a point of a separated set at a critical point. So we replace it by the tree pressure. One can however use infimum over (n,ε)-spanning sets, thus defining $\underline{P_{\rm spanning}(f,\phi)}$. This is a valuable notion, often coinciding with other pressures. See [55] for an outline of a respective theory. Let me mention only that this is equal to $P(f, -t\log|f'|)$ for $t > 0$ in the complex case if

Definition 6.3. f is *weakly backward Lyapunov stable* which means that for every $\delta > 0$ and $\varepsilon > 0$ for all n large enough and every disc $B = B(x, \exp -\delta n)$ centered at $x \in K$, for every $0 \le j \le n$ and every component V of $f^{-j}(B)$ intersecting K, it holds that $\operatorname{diam} V \le \varepsilon$.

This holds for all rational maps with at most one critical point whose forward trajectory is in $J(f)$ or is attracted to $J(f)$, due to Theorem 5.6.

Question. Does backward weak Lyapunov stability hold for all rational maps?

Finally, *periodic pressure* $P_{\rm Per}$ is defined as $P_{\rm tree}$ with $x \in \mathrm{Per}_n$ (periodic of period n) rather than f^n-preimages of z. In [59], this was proved for rational f (see also [4] for a class of polynomials) on $K = J(f)$ that $P_{\rm Per}(t) = P(t)$ provided

Hypothesis H. For every $\delta > 0$ and all n large enough, if for a set $\mathcal{P} \subset \mathrm{Per}_n$ for all $p, q \in P$ and all $i : 0 \le i < n$ $\operatorname{dist}(f^i(p), f^i(q)) < \exp -\delta n$, then $\#\mathcal{P} \le \exp \delta n$.

Question. Does this condition always hold? In particular, can large bunches of periodic orbits exist with orbits exponentially close to a Cremer fixed point?

7. Geometric coding trees, limit sets, Gibbs meets Hausdorff

The notion of geometric coding tree, g.c.t., already appeared in the work [27], where in the expanding case the finite-to-one property of the resulting coding was proved. It was used later in [44, 45] and in a full strength in [64, 65] and papers following them. Similar graphs have since been constructed to analyse the topological aspects of non-invertible dynamics, see for instance [39, 19].

Definition 7.1. Let U be an open connected subset of the Riemann sphere $\overline{\mathbb{C}}$. Consider a holomorphic mapping $f : U \to \overline{\mathbb{C}}$ such that $f(U) \supset U$ and $f : U \to f(U)$ is a proper map. Suppose that $\mathrm{Crit}(f)$ is finite. Consider an arbitrary $z \in f(U)$. Let $z^1, z^2, \dots, z^d$ be some of the f-preimages of z in U with $d \geq 2$. Consider smooth curves $\gamma^j : [0,1] \to f(U)$, $j = 1, \dots, d$, joining z to z^j respectively (i.e. $\gamma^j(0) = z, \gamma^j(1) = z^j$), intersections allowed, such that $\gamma^j \cap f^n(\mathrm{Crit}(f)) = \emptyset$ for every j and $n > 0$.

For every sequence $\alpha = (\alpha_n)_{n=0}^{\infty} \in \Sigma^d$ (shift space with left shift map ς defined in Section 1) define $\gamma_0(\alpha) := \gamma^{\alpha_0}$. Suppose that for some $n \geq 0$, for every $0 \leq m \leq n$, and all $\alpha \in \Sigma^d$, curves $\gamma_m(\alpha) : [0.1] \to U$ are already defined. Suppose that for $1 \leq m \leq n$ we have $f \circ \gamma_m(\alpha) = \gamma_{m-1}(\varsigma(\alpha))$, and $\gamma_m(\alpha)(0) = \gamma_{m-1}(\alpha)(1)$. Define the curves $\gamma_{n+1}(\alpha)$ so that the previous equalities hold by taking respective f-preimages of curves γ_n. For every $\alpha \in \Sigma^d$ and $n \geq 0$ denote $z_n(\alpha) := \gamma_n(\alpha)(1)$.

The graph $\mathscr{T} = \mathscr{T}(z, \gamma^1, \dots, \gamma^d)$ with the vertices z and $z_n(\alpha)$ and edges $\gamma_n(\alpha)$ is called a *geometric coding tree* with the root at z. For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \geq 0$ is called an *infinite geometric branch* and denoted by $b(\alpha)$. It is called *convergent* if the sequence $\gamma_n(\alpha)$ is convergent to a point in $\mathrm{cl}\, U$. We define the *coding map* $z_\infty : \mathscr{D}(z_\infty) \to \mathrm{cl}\, U$ by $z_\infty(\alpha) := \lim_{n\to\infty} z_n(\alpha)$ on the domain $\mathscr{D} = \mathscr{D}(z_\infty)$ of all such α's for which $b(\alpha)$ is convergent.

Denote $\Lambda := z_\infty(\mathscr{D}(z_\infty))$. If the map f extends holomorphically to a neighbourhood of its closure $\mathrm{cl}\,\Lambda$ in $\overline{\mathbb{C}}$, then Λ is called a *quasi-repeller*, see [64].

A set formally larger than $\mathrm{cl}\,\Lambda$ is of interest, namely $\widehat{\Lambda}$ being the set of all accumulation points of $\{z_n(\alpha) : \alpha \in \Sigma^d\}$ as $n \to \infty$. If our g.c.t. is in Ω being an RB-domain, see Section 8, or f is just $R \circ g \circ R^{-1}$ defined only on Ω, see Remarks below, then it is easy to see that $\mathrm{cl}\,\Lambda = \widehat{\Lambda}$. I do not know how general this equality is.

Remarks. Given a Riemann map $R : \mathbb{D} \to \Omega$ to a connected simply connected domain $\Omega \subset \mathbb{C}$, (i.e. holomorphic bijection) we can consider a branched covering map, say $g(z) = z^d$ on $\mathbb{D}$, and $f = R \circ g \circ R^{-1}$. Then, chosen $z \in \Omega$ and γ^j joining it with its preimages in Ω (close to $\mathrm{Fr}\,\Omega$) we can consider the respective tree $\mathscr{T}$. Then instead of considering R and its radial limit $\overline{R}$, we can consider the limit (along branches) $z_\infty : \Sigma^d \to \mathrm{Fr}\,\Omega$. This provides a structure of symbolic dynamics useful to verify stochastic laws.

This is especially useful if considered measures come from $\partial\mathbb{D}$ via $\overline{R}$, rather than being some equilibrium states for potentials living directly on $\mathrm{Fr}\,\Omega$. This is the case of harmonic measure ω which is the $\overline{R}_*$-image of a length measure l. We can consider the lift of l to Σ^d via coding by the tree $\mathscr{T}' = R^{-1}(\mathscr{T})$ and next its projection by $(z_\infty)_*$ to $\mathrm{Fr}\,\Omega$.

Our g.c.t.'s are always available in presence of adequate holomorphic f, even in the absence of Ω, i.e. in the absence of a Riemann map. The tree with the coding it induces yields a discrete generalization/replacement of a Riemann map.

It was proved in [62] that $\mathscr{D}$ is the whole Σ^d except a "thin" set. In particular, for a Gibbs measure ν for a Hölder potential, $z_\infty(\alpha)$ exists for ν-a.e. α, hence the push forward measure $(z_\infty)_*(\nu)$ makes sense. Moreover, our codings ζ_∞ are always "thin"-to-one. This is a discrete generalization of Beurling's Theorem concerning the boundary behaviour of Riemann maps. "Thin" means of zero logarithmic capacity type, depending on the properties of the tree (the speed of the accumulation of γ^j by critical trajectories; the speed does not matter if we replace "thin" by zero Hausdorff dimension). In particular this coding preserves the entropies.

For appropriate $\nu \in \mathcal{M}(\varsigma, \Sigma^d)$ and $\psi : \Sigma^d \to \mathbb{R}$ with $\int \psi \, d\nu = 0$, consider the *asymptotic variance* (of course one can consider spaces more general than Σ^d)

$$\sigma^2 = \sigma^2_\nu(\psi) := \lim_{n\to\infty} \frac{1}{n} \int (S_n\psi)^2 \, d\nu. \tag{7.1}$$

Theorem 7.2. *Let Λ be a quasi-repeller for a geometric coding tree for a holomorphic map $f : U \to \overline{\mathbb{C}}$. Let ν be a ς-invariant Gibbs measure on Σ^d for a Hölder continuous real-valued function ϕ on Σ^d. Assume $P(\varsigma, \phi) = 0$. Consider $\mu := (z_\infty)_*(\nu)$.*

Then, for $\psi := -\mathrm{HD}(\mu)(\log|f'| \circ z_\infty)) - \phi$, we have $\int \psi \, d\nu = 0$.

If the asymptotic variance $\sigma^2 = \sigma^2_\nu(\psi)$ is positive, then there exists a compact f-invariant hyperbolic repeller X being a subset of Λ such that $\mathrm{HD}(X) > \mathrm{HD}(\mu)$. In consequence $\mathrm{HD}_{\mathrm{hyp}}(\Lambda) > \mathrm{HD}(\mu)$ (defined after (5.2)*).*

If $\sigma^2 = 0$ then ψ is cohomologous to 0. Then for each $x, y \in \mathrm{cl}\,\Lambda$ not postcritical, if $z = f^n(x) = f^m(y)$ for some positive integers n, m, the orders of criticality of f^n at x and f^m at y coincide. In particular all critical points in $\mathrm{cl}\,\Lambda$ are pre-periodic.

The latter condition happens only in special situations, see e.g. Theorem 7.3 below. See [75] for more details; ϕ lives there directly on $J(f)$, but it does not make substantial difference. See also Section 10.

Given a mapping $f : X \to X$, given two functions $u, v : X \to \mathbb{R}$ we call u *cohomologous* to v in class $\mathcal{C}$ if there exists $h : X \to \mathbb{R}$ belonging to $\mathcal{C}$ such that $u - v = h \circ f - h$. An important [64, Lemma 1] says that $\sigma^2 = 0$ above implies ψ cohomologous to 0 in $L^2(\mu)$ and often in a smaller class depending on ψ (Livšic type rigidity).

Notice that $\int \psi \, d\nu = -\mathrm{HD}(\mu)\chi(\mu) - \int \phi \, d\nu = -h_\mu(f) - \int \phi \, d\nu = -h_\nu(\varsigma) - \int \phi \, d\nu = P(\varsigma, \phi) = 0$. Now, to prove Theorem 7.2 note $2\chi(\mu) \geq h_\mu(f) = h_\nu(\varsigma) > 0$, see [63, Ruelle's inequality] (used also to 3)$\Rightarrow$5) in Section 3) and [44]. So considering the natural extension of $(\Sigma^d, \nu, \varsigma)$ (here two-sided shift space) and Katok-Pesin theory, we find hyperbolic X with $\mathrm{HD}(X) \geq \mathrm{HD}(\mu) - \varepsilon$ for an arbitrary $\varepsilon > 0$. Compare comments on shadowing in Section 6.

• The positive σ^2 yields by Central Limit Theorem large fluctuations of the sums $\sum_{j=0}^{n-1} \psi \circ \varsigma^j$ from $n \int \psi \, d\nu$ (here 0), allowing to find X with $\mathrm{HD}(X) > \mathrm{HD}(\mu)$.

A special care is needed to get $X \subset \Lambda$, see [53] (originated in [66]).

The above fluctuations were used by A. Zdunik to prove for constant ϕ

Theorem 7.3 ([76])**.** *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational mapping of degree $d \geq 2$. If $\sigma^2 > 0$, then for $\mu_{\max}(f)$ the measure of maximal entropy (equal $\log d$), $\mathrm{HD}(J(f)) > \mathrm{HD}(\mu_{\max}(f))$. Otherwise, f is postcritically finite with a parabolic orbifold,* [37].

She proved in fact the existence of a hyperbolic $X \subset J(f)$ satisfying $\mathrm{HD}(X) > \mathrm{HD}(\mu_{\max}(f))$, hence $\mathrm{HD}_{\mathrm{hyp}}(J(f)) > \mathrm{HD}(\mu_{\max}(f))$.

• In the $\sigma^2 = 0$ case, $v : J(f) \to \mathbb{R}$ satisfying the cohomology equation $\log |f'| = v \circ f - v + \mathrm{Const}$ on $J(f)$ extends to a harmonic function beyond $J(f)$ (Livšic rigidity) giving this equality on the union of real analytic curves containing $J(f)$ (called *real case*) or to $\overline{\mathbb{C}}$. In Theorem 7.2 on Λ and for the extension beyond, in Theorem 7.3, the "orders" of growth of $-\log |(f^n)'|$ at x and of $-\log |(f^m)'|$ at y must by cohomology equation be equal to the "order" of growth of v at z, so they must coincide (a phenomenon "conjugated" to the presence of an invariant line field). This implies parabolic orbifold for Theorem 7.3.

Theorem 7.3 applied to a polynomial f with connected Julia set, by $\mathrm{HD}(\mu_{\max}(f)) = 1$ [34], implies the following celebrated result:

Theorem 7.4 (A. Zdunik [76])**.** *For every polynomial f of degree at least 2, with connected Julia set, either $J(f)$ is a circle or an interval or else it is fractal, namely $\mathrm{HD}(J(f)) > 1$.*

8. Boundaries, radial growth, harmonic vs Hausdorff

For polynomials with connected Julia set the measure $\mu_{\max}(f)$ coincides with harmonic measure ω (viewed from ∞). This led to another proof of Theorem 7.4, especially the $\sigma^2 = 0$ part, see [77], in the language of boundary behaviour of Riemann map and harmonic measure (compare also model Theorem 2.1).

Theorem 7.4 has been strengthened from this point of view in [54], preceded by [66], as follows.

Theorem 8.1. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree at least 2 and Ω be a simply connected immediate basin of attraction to an attracting periodic orbit (that is a connected component of the set attracted to the orbit, intersecting it). Then, provided f is not a finite Blaschke product in some* *holomorphic coordinates, or a two-to-one holomorphic factor of a Blaschke product,* $\mathrm{HD}_{\mathrm{hyp}}(\mathrm{Fr}\,\Omega) > 1$.

The novelty was to show how to "capture" a large hyperbolic X in $\mathrm{Fr}\,\Omega$ in the case it was not the whole $J(f)$.

In fact the following "local" version of this theorem was proved in [54]

Theorem 8.2. *Assume that f is defined and holomorphic on a neighbourhood W of $\mathrm{Fr}\,\Omega$, where Ω is a connected, simply connected domain in $\overline{\mathbb{C}}$ whose boundary has at least 2 points. We assume that $f(W \cap \Omega) \subset \Omega$, $f(\mathrm{Fr}\,\Omega) \subset \mathrm{Fr}\,\Omega$ and $\mathrm{Fr}\,\Omega$ repels to the side of Ω, that is $\bigcap_{n=0}^{\infty} f^{-n}(W \cap \mathrm{cl}\,\Omega) = \mathrm{Fr}\,\Omega$. Then either $\mathrm{HD}_{\mathrm{hyp}}(\mathrm{Fr}(\Omega)) > 1$ or $\mathrm{Fr}\,\Omega$ is a real-analytic Jordan curve or arc.*

Ω with f as above has been called an *RB-domain* (repelling boundary), introduced in [45, 64]. Theorem 8.2 (at least the $\sigma^2 > 0$ part) follows directly from Theorem 7.2. Let $R : \mathbb{D} \to \Omega$ be a Riemann map and $g : W' \to \mathbb{D}$ be defined by $g := R^{-1} \circ f \circ R$ on $W' = R^{-1}(W \cap \Omega)$. We consider a g.c.t. $\mathscr{T} = \mathscr{T}(z, \gamma^1, \ldots, \gamma^d)$ with z and γ^j in $W \cap \Omega$, sufficiently close to $\mathrm{Fr}\,\Omega$ that the definition makes sense, and with $d = \deg f|_{W\cap\Omega}$, (the situation is the same as in Remarks in Section 7 above, but the order of defining f and g is different). Consider the g.c.t. $\mathscr{T}' = R^{-1}(\mathscr{T})$. The function g extends holomorphically beyond the circle $\partial\mathbb{D}$ and it is expanding. Hence $\phi : \Sigma^d \to \mathbb{R}$ defined by $\phi(\alpha) = -\log|g'| \circ (R^{-1}(z))_\infty(\alpha)$ for the tree $\mathscr{T}'$ is Hölder continuous. Let $\nu = \nu_\phi$.

Note that here $P(\phi) = 0$, e.g. since by expanding property of g on $\partial\mathbb{D}$ there exists $\widehat{l} \in \mathcal{M}(g)$, equivalent to length measure l (a.c.i.m.). Then ν is the lift of $\widehat{l}$ to Σ^d with the use of $\mathscr{T}'$. Note that our $\mu = z_\infty(\nu)$ is equal to $\widehat{\omega} = \overline{R}_*(\widehat{l})$ which is f-invariant, equivalent to harmonic measures ω on $\mathrm{Fr}\,\Omega$ viewed from Ω.

Note that $\mathrm{HD}(\widehat{\omega}) = 1$ due to Mañé's equality, (5.3), $h_{\widehat{\omega}}(f) = h_{\widehat{l}}(g)$, see [44, 45], and the equality of Lyapunov exponents $\int \log|f'|\, d\widehat{\omega} = \int \log|g'|\, d\widehat{l} > 0$. The latter equality holds due to the equality for almost every $\zeta \in \partial\mathbb{D}$:

$$\lim_{r\to 1} \frac{\log|(f^n)'(R(r\zeta))| - \log|(g^n)'(r\zeta))}{\log(1-r)} = \lim_{r\to 1} \frac{-\log|R'(r\zeta)|}{\log(1-r)} = 0. \tag{8.1}$$

The first equality is proved using $f \circ R = R \circ g$ in $\mathbb{D}$, first applying R close to $\partial\mathbb{D}$, next by iterating f applying R^{-1} well inside Ω, finally iterating g back. The latter equality relies on the harmonicity of $\log|R'|$ allowing to replace its integral on circles by its value at the origin. For details see [45]. Remind however that in fact $\mathrm{HD}(\omega) = 1$ holds in general, see [32].

The sketch of Proof of Theorem 8.2 for $\sigma^2 > 0$ is over. That $\sigma^2 = 0$ implies the analyticity of $\mathrm{Fr}\,\Omega$ was already commented at the beginning of this Section.

9. Law of Iterated Logarithm refined versions

Applying Law of Iterated Logarithm (abbr. LIL) to $\psi : \Sigma^d \to \mathbb{R}$ the fluctuations of $S_n\psi$ from 0 which follow lead to, see [64] and [63],

Theorem 9.1. *In the setting of Theorem 7.2 if $\sigma^2 = \sigma^2_\nu(\psi) > 0$, for $c(\mu) := \sqrt{2\sigma^2/\chi(\mu)}$, $\kappa := \mathrm{HD}(\mu)$ and $\alpha_c(r) := r^\kappa \exp(c\sqrt{\log 1/r \log\log\log 1/r})$*

1) $\mu \perp H_{\alpha_c}$, that is singular with respect to the refined Hausdorff measure, [63, Section 8.2] *for the gauge function α_c), for all $0 < c < c(\mu)$;*

2) $\mu \ll H_{\alpha_c}$, that is absolutely continuous, for all $c > c(\mu)$.

Indeed, substituting in LIL $n \sim (\log 1/r_n)/\chi(\mu)$ for $r_n = |(f^n)'(z)|^{-n}$, we get for μ-a.e. z

$$\limsup_{n\to\infty} \frac{\mu(B(z,r_n)}{\alpha_c(r_n)} = \infty \text{ for } 0 < c < c(\mu) \quad \text{and} \quad \cdots = 0 \text{ for } c > c(\mu). \tag{9.1}$$

This is called the Refined Volume Lemma, [64, Section 4] and, the harder case: $c > c(\mu)$, [65, Section 5].

We can apply the assertion of Theorem 9.1 for $\mu = \widehat{\omega} \in \mathcal{M}(f, \mathrm{Fr}\,\Omega)$ equivalent to a harmonic measure ω as in Section 8.

This yields refined information about the radial growth of the derivative of Riemann maps, following the proof of (8.1):

Theorem 9.2. *Let Ω be a simply connected RB-domain in $\overline{\mathbb{C}}$ with non-analytic boundary and $R : \mathbb{D} \to \Omega$ be a Riemann map. Then there exists $c(\Omega) > 0$ such that for Lebesgue a.e. $\zeta \in \partial\mathbb{D}$*

$$G^+(\zeta) := \limsup_{r\to 1} \frac{\log|R'(r\zeta)|}{\sqrt{\log(1/1-r)\log\log\log(1/1-r)}} = c(\Omega). \tag{9.2}$$

Similarly $G^-(\zeta) := \liminf \cdots = -c(\Omega)$. Finally $c(\Omega) = c(\widehat{\omega})$ in Theorem 9.1.

In fact Theorem 9.1 for $\mu = \widehat{\omega}$ and Theorem 9.2 hold for every connected, simply connected open $\Omega \subset \mathbb{C}$, together with $c(\Omega) = c(\widehat{\omega})$. No dynamics is needed. Of course one should add to both definitions $\operatorname{ess\,sup}$ over $\zeta \in \partial\mathbb{D}$ and over $z \in \mathrm{Fr}\,\Omega$ (for $c(z) = c(\omega)$ calculated from (9.1), see [63, Th. 8.6.1]) respectively, since in the absence of ergodicity these functions need not be constant. See [14, Th. VIII.2.1 (a)] and references to Makarov's breakthrough papers therein, in particular [32].

There is a universal Makarov's upper bound $C_{\mathrm{M}} < \infty$ for all $c(\Omega), c(\widehat{\omega})$'s in (9.2). The best upper estimate I found in literature is $C_{\mathrm{M}} \le 1.2326$, [20]. I proved in [46] a much weaker estimate, using a natural method of representing $\log|R'|$ by a series of weakly dependent random variables leading to a martingale on $\partial\mathbb{D}$, thus satisfying LIL. Unfortunately consecutive approximations resulted with looses in the final estimate.

For a holomorphic expanding repeller $f : X \to X$ and a Hölder continuous potential $\phi : X \to X$, the asymptotic variance for the equilibrium state

$\mu = \mu_{t_0\phi}$ for every $t_0 \in \mathbb{R}$ satisfies Ruelle's formula (see [63]):

$$\sigma_\mu^2(\phi - \int \phi \, d\mu) = \frac{d^2 P(t\phi)}{dt^2}\Big|_{t=t_0}. \tag{9.3}$$

Question. Does (9.3) hold for all rational maps and hyperbolic potentials on Julia sets? For all simply connected RB-domains, $f : \operatorname{Fr}\Omega \to \operatorname{Fr}\Omega$ and $\mu = \widehat{\omega}$?

For a simply connected RB-domain Ω for f and for $\phi = -\log|f'|$, if $g(z) = z^d$ (e.g. Ω being the basin of ∞ for a polynomial f), one considers the *integral means spectrum* depending only on Ω,

$$\beta_\Omega(t) := \limsup_{r\to 1} \frac{1}{|\log(1-r)|} \log \int_{\zeta\in\partial\mathbb{D}} |R'(r\zeta)|^t \, |d\zeta| \tag{9.4}$$

which happens to satisfy $\beta_\Omega(t) = t - 1 + \frac{P(t\phi)}{\log d}$, see e.g. [63, Eq. (9.6.2.)].

For $t_0 = 0$ we have $\mu = \widehat{\omega}$ and the left hand side of (9.3) can be written as $(\frac{1}{2}\log d)\sigma^2(\log R')$, see (7.1) and (8.1), where

$$\sigma^2(\log R') := \limsup_{r\to 1} \frac{\int_{\partial\mathbb{D}} |\log R'(t\zeta)|^2 \, |d\zeta|}{-2\pi \log(1-r)|}.$$

So (9.3) changes to $\sigma^2(\log R') = 2\frac{d^2\beta_\Omega(t))}{dt^2}|_{t=0}$, compare [26]. It has an analytic, non-dynamical, meaning. It is also related to the Weil-Petersson metric, see [36].

10. Accessibility

Let us recall the following theorem from [49].

Theorem 10.1. *Let Λ be a quasi-repeller for a geometric coding tree for a holomorphic map $f : U \to \overline{\mathbb{C}}$. Suppose that*

$$\operatorname{diam}(\gamma_n(\alpha)) \to 0, \ \text{as } n \to \infty \tag{10.1}$$

uniformly with respect to $\alpha \in \Sigma^d$. Then every good $q \in \widehat{\Lambda}$ (defined in Section 7) is a limit of a convergent branch $b(\alpha)$. So $q \in \Lambda$. In particular, this holds for every q with $\underline{\chi}(q) > 0$ and the local backward inviariance (explained below).

For the definition of "good", see [49, Definition 2.5]. It roughly says that there are many integers n (positive lower density) for which f^n properly map small domains $D_{n,0}$ in U close to q onto large $D_n \subset U$, giving "telescopes" Tel_k with "traces" $D_{n_k,0} \subset D_{n_{k-1},0} \subset \cdots \subset D_{n_1,0} \subset D_0$; for each k the choices may be different. A part of this condition that $D_{n,0} \subset U$ can be called a "local backward invariance" of U along the forward trajectory of q.

When U is an immediate basin of attraction of an attracting fixed point for a rational map f or just an RB-domain then this theorem asserts that q is an endpoint of a continuous curve in U. This is a generalization of the Douady-Eremenko-Levin-Petersen theorem where q is a repelling periodic point and the domain is completely invariant, e.g. basin of attraction to ∞ for f a polynomial.

Due to this theorem we can prove that invariant measures of positive Lyapunov exponents lift to Σ^d. More precisely, the following holds:

Corollary 10.2. *Every non-atomic hyperbolic probability measure μ (i.e. $\chi(\mu) > 0$), on $\widehat{\Lambda}$, is the $(z_\infty)_*$ image of a probability ς-invariant measure ν on Σ^d, assumed* (10.1)*, $\mathscr{T}$ has no self-intersections and else μ-a.e. local backward invariance of $U_{,}$. In particular, ν exists for every RB-domain which is completely (i.e. backward) invariant.*

Proof. (the lifting part missing in [49] and [54]). By Theorem 10.1 μ is supported on Λ i.e. on $z_\infty(\mathscr{D}(z_\infty))$. The lift of μ to μ' on the pre-image $\mathscr{B}'$ under z_∞ of the Borel σ-algebra of subsets of Λ can be extended to a ς-invariant ν on $\mathscr{B}$ the Borel σ-algebra of the subsets of Σ^d by using the fact that the set of at least triple points (limit points of at least three infinite branches of $\mathscr{T}$) is countable, hence $z_\infty^{-1}(x)$ of μ-a.e x contains at most 2 points. More precisely, let A_1 be the set of points having one z_∞-preimage, A_2 two preimages. They are both f-invariant (except measure 0), so are their z_∞-preimages A_1' and A_2' under ς. We extend μ' by distributing conditional measures on the two points preimages of points in A_2 half-half and Dirac on one point preimages.

$\square$

This allows to conclude Theorem 9.1 (a part relying on CLT) and Theorem 7.2 for equilibrium states for rational maps and Hölder potentials on $J(f)$ by lifting μ_ϕ to Σ^d as in [54]. However, this seems useless since the proof of CLT in [54] is done directly on $J(f)$ (seemingly also for LIL, for which one should however refer to the proofs in [64]) and there are direct proofs of LIL in [30] and [75].

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References

1. K. Barański, B. Karpińska, A. Zdunik, *Hyperbolic dimension of Julia sets of meromorphic maps with logarithmic tracts*, Int. Math. Res. Not. 4 (2009), 615 – 624.
2. K. Barański, B. Karpińska, A. Zdunik, *Bowen's formula for meromorphic functions*, Ergod. Th. & Dynam. Sys. 32 (2012), no. 4, 1165 – 1189.
3. R. Bowen, *Equlibrium states and the ergodic theory of Anosov diffeomorphisms*, Lect. Notes in Math. 470 (1975), Springer Verlag.
4. I. Binder, N. Makarov, S. Smirnov, *Harmonic measure and polynomial Julia sets*, Duke Math. J. 117.2 (2003), 343 – 365.
5. H. Bruin, J. Rivera-Letelier, W. Shen, S. van Strien, *Large derivatives, backward contraction and invariant densities for interval maps*, Invent. math. 172.3 (2008), 509 – 533.
6. H. Bruin, M. Todd, *equilibrium states for interval maps: the potential $-t\log|Df|$*, Ann. Scient. Éc. Norm. Sup. 4 serié **42** (2009), 559 – 600.

7. H. Bruin, M. Todd, *Equilibrium states for interval maps: potentials with* $\sup\phi - \inf\phi < h_\top(f)$, Comm. Math. Phys. 283.3 (2008), 579 – 611. Erratum: Comm. Math. Phys. 304.2 (2011), 583 – 584.
8. D. Coronel, J. Rivera-Letelier, *Low-temperature phase transitions in the quadratic family*, arXiv:1205.1833, to appear in Adv. Math.
9. D. Coronel, J. Rivera-Letelier, *High-order phase transitions in the quadratic family*, arXiv:1305.4971.
10. W. de Melo, S. van Strien, *One-dimensional Dynamics*, Springer-Verlag, 1994.
11. M. Denker, F. Przytycki, M. Urbański, *On the transfer operator for rational functions on the Riemann sphere*, Ergod. Th. & Dynam. Sys. **16** (1996), 255 – 266.
12. M. Denker, M. Urbański, *Ergodic theory of equilibrium states for rational maps*, Nonlinearity 4(1) (1991), 103 – 134.
13. N. Dobbs, M. Todd, *Free entropy jumps up*, arXiv:1512.09245.
14. J. B. Garnett, D. E. Marshall, *Harmonic Measure*, New Mathematical Monographs, Cambridge, 2005.
15. K. Gelfert, F. Przytycki, M. Rams, *On the Lyapunov spectrum for rational maps*, Math. Annalen **348** (2010), 965 – 1004.
16. K. Gelfert, F. Przytycki, M. Rams, *Lyapunov spectrum for multimodal maps*, Ergod. Th. & Dynam. Sys. **36** (2016), 1441 – 1493.
17. J. Graczyk, S. Smirnov, *Collet, Eckmann and Hölder*, Invent. Math. 133 (1998), 69 – 96.
18. J. Graczyk, S. Smirnov. *Non-uniform hyperbolicity in complex dynamics*, Invent. math. 175 (2009), 335 – 415.
19. P. Haissinsky, K. M. Pilgrim *Coarse Expanding Conformal Dynamics*, Asterisque 325, 2009.
20. H. Hedenmalm, I. Kayumov, *On the Makarov Law of the Iterated Logarithm*, Proc. Amer. Math. Soc. 135.7 (2007), 2235 – 2248.
21. J. Heinonen, P. Koskela, *Definitions of quasiconformality*, Invent. math. 120 (1995), 61 – 79.
22. F. Hofbauer, G. Keller, *Equilibrium states and Hausdorff measures for interval maps*, Math. Nachr. 164 (1993), 239 – 257.
23. F. Hofbauer, M. Urbański, *Fractal properties of invariant subsets for piecewise monotonic maps on the interval* Trans. Amer. Math. Soc. 343 (1994), 659 – 673.
24. I. Inoquio-Renteria, J. Rivera-Letelier, *A characterization of hyperbolic potentials of rational maps*, Bull. Braz. Math. Soc. (N.S.) 43.1 (2012), 99 – 127.
25. G. Iommi, M. Todd, *Natural equilibrium states for multimodal maps*, Communications in Math. Phys. **300** (2010), 65 – 94.
26. O. Ivrii, *On Makarov's principle in conformal mapping*, arXiv:1604.05619, to appear in Int. Math. Res. Not.
27. M. V. Jakobson, *Markov partitions for rational endomorphisms of the Riemann sphere*, Mnogokomponentnyje slucajnyje sistemy, pp. 303 – 319. Izd. Nauka, Moscow 1978 (In Russian)
28. G. Levin, F. Przytycki, W. Shen, *The Lyapunov exponent of holomorphic maps*, Inventiones math. 205 (2016), 363 – 382.
29. H. Li, J. Rivera-Letelier, *Equilibrium states of interval maps for hyperbolic potentials*, Nonlinearity 27.8 (2014), 1779 – 1804.
30. H. Li, J. Rivera-Letelier, *Equilibrium states of weakly hyperbolic one-dimensional maps for Hölder potentials* Comm. Math. Phys. 328.1 (2014), 397 – 419.
31. M. Lyubich, *Analytic low-dimensional dynamics: from dimension one to two* Proc. ICM, Seoul 2014, Vol. I, 443 – 474.
32. N. G. Makarov, *On the distortion of boundary sets under conformal mappings*, Proc. London Math. Soc. 3^d ser. 51 (1985), 369 – 384.

33. N. Makarov and S. Smirnov, *On thermodynamics of rational maps. II. Non-recurrent maps*, J. London Math. Soc. (2), 67(2) (2003), 417 – 432.
34. A. Manning, *The dimension of the maximal measure for a polynomial map*, Ann. of Math. 119 (1984), 425 – 430.
35. D. Mauldin, M. Urbański, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*, Cambridge University Press, 2003.
36. C. T. McMullen, *Thermodynamics, dimension and the Weil-Petersson metric*, Invent. math. 173 (2008), 365 – 428.
37. J. Milnor, *On Lattes maps. Dynamics on the Riemann sphere*, 9 – 43, Eur. Math. Soc., Zürich, 2006.
38. M. Misiurewicz, W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. **67** (1980), 45 – 63.
39. Nekrashevych, *Iterated monodromy groups*, Groups St Andrews 2009 in Bath, Vol. 1, London Mathematical Society, Lecture Note Series 387, Cambridge University Press, 2011, pp. 41 – 93.
40. T. Nowicki, F. Przytycki, *Topological invariance of the Collet-Eckmann property for S-unimodal maps*, Fund. Math. **155** (1998), 33 – 43.
41. T. Nowicki, D. Sands. *Non-uniform hyperbolicity and universal bounds for S-unimodal maps*, Invent. Math. 132 (1998), 633 – 680.
42. Y. Pesin, *On the work of Sarig on countable Markov chains and thermodynamic formalism*, J. Mod. Dyn. 8.1 (2014), 1 – 14.
43. Y. Pesin, S. Senti *Equilibrium measures for maps with inducing schemes*, Journal of Modern Dynamics **2.3** (2008), 397 – 430.
44. F. Przytycki, *Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map*, Invent. math. 80 (1985), 161 – 179.
45. F. Przytycki, *Riemann map and holomorphic dynamics*, Invent. Math. 85 (1986), 439 – 455.
46. F. Przytycki, *On the law of iterated logarithm for Bloch functions*, Studia Math. 93.2 (1989), 145 – 154.
47. F. Przytycki, *On the Perron-Frobenius-Ruelle operator for rational maps on the Riemann sphere and for Hölder continuous functions*, Bull. Braz. Math. Soc. 20.2 (1990), 95 – 125.
48. F. Przytycki, *Lyapunov characteristic exponents are nonnegative*, Proc. Amer. Math. Soc. **119** (1993), 309 – 317.
49. F. Przytycki, *Accessibility of typical points for invariant measures of positive Lyapunov exponents for iterations of holomorphic maps*, Fundamenta Mathematicae 144 (1994), 259 – 278.
50. F. Przytycki, *Iteration of holomorphic Collet-Eckmann maps: Conformal and invariant measures. Appendix: On non-renormalizable quadratic polynomials*, Trans. Amer. Math. Soc. **350** (1998), 717 – 742.
51. F. Przytycki, *Conical limit set and Poincaré exponent for iterations of rational functions*, Trans. Amer. Math. Soc. **351** (1999), 2081 – 2099.
52. F. Przytycki. *Hölder implies Collet-Eckmann.* Géométrie complexe et systèmes dynamiques (Orsay, 1995). Astérisque 261 (2000), 385 – 403.
53. F. Przytycki, *Expanding repellers in limit sets for iterations of holomorphic functions*, Fund. Math. 186.1 (2005), 85 – 96.
54. F. Przytycki, *On the hyperbolic Hausdorff dimension of the boundary of a basin of attraction for a holomorphic map and of quasirepellers*, Bull. Pol. Acad. Sci. Math. 54.1 (2006), 41 – 52.
55. F. Przytycki, *Geometric pressure in real and complex 1-dimensional dynamics via trees of pre-images and via spanning sets*, Monatshefte für Math., published online : 21 Nov. 2017, DOI 10.1007/s00605-017-1137-8.

56. F. Przytycki, J. Rivera-Letelier, *Nice inducing schemes and the thermodynamics of rational maps*, Communications in Math. Phys. **301.3** (2011), 661 – 707.
57. F. Przytycki and J. Rivera-Letelier, *Geometric pressure for multimodal maps of the interval*, `arXiv:1405.2443`, to appear in Memoirs of the Amer. Math. Soc.
58. F. Przytycki, J. Rivera-Letelier, S. Smirnov, *Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps*, Inventiones Mathematicae **151** (2003), 29 – 63.
59. F. Przytycki, J. Rivera-Letelier, and S. Smirnov, *Equality of pressures for rational functions*, Ergodic Theory Dynam. Systems **24** (2004), 891 – 914.
60. F. Przytycki, S. Rohde, *Porosity of Collet-Eckmann Julia sets*, Fund. Math. 155 (1998), 189 – 199.
61. F. Przytycki, S. Rohde. *Rigidity of holomorphic Collet-Eckmann repellers*, Arkiv för Mat. 37.2 (1999), 357 – 371.
62. F. Przytycki, J. Skrzypczak, *Convergence and pre-images of limit points for coding trees for iterations of holomorphic maps*, Math. Annalen 290 (1991), 425 – 440.
63. F. Przytycki, M. Urbański, *Conformal Fractals: Ergodic Theory Methods*, London Mathematical Society Lecture Note Series 371, Cambridge University Press, 2010.
64. F. Przytycki, M. Urbański, A. Zdunik, *Harmonic, Gibbs and Hausdorff measures for holomorphic maps, I*, Annals of Math. 130 (1989), 1 – 40.
65. F. Przytycki, M. Urbański, A. Zdunik, *Harmonic, Gibbs and Hausdorff measures for holomorphic maps, II*, Studia Math. 97.3 (1991), 189 – 225.
66. F. Przytycki, A. Zdunik, *Density of periodic sources in the boundary of a basin of attraction for iteration of holomorphic maps, geometric coding trees technique*, Fund. Math. 145 (1994), 65 – 77.
67. J. Rivera-Letelier, *Asymptotic expansion of smooth interval maps*, arXiv:1204.3071v2.
68. J. Rivera-Letelier, W. Shen, *Statistical properties of one-dimensional maps under weak hyperbolicity assumptions*, Ann. Sci. de l'ENS 47.6 (2014), 1027 – 1083.
69. D. Ruelle, *Thermodynamic Formalism*, Encyclopedia of Mathematics and Its Applications, vol. 5, Addison-Wesley Publ. Co., London, 1978.
70. W. Shen, S. van Strien, *Recent developments in interval dynamics*, Proc. ICM Seoul 2014, Vol III, (2014), 699 – 719.
71. Ya. G. Sinai, *Gibbs measures in ergodic theory*, (Russian) Uspehi Mat. Nauk 27, no. 4(166) (1972), 21 – 64. Russian Mathematical Surveys, 1972, 27:4, 21 – 69 (English).
72. B. O. Stratmann, M. Urbański. *Real analyticity of topological pressure for indifferentally semihyperbolic generalized polynomial-like maps*, Indag. Math. **14(1)**, (2003), 119 – 134.
73. D. Sullivan, *Seminar on Conformal and hyperbolic Geometry*, Notes by M. Baker and J. Seade, Preprint IHES, 1982.
74. M. Szostakiewicz, M. Urbański, A. Zdunik, *Stochastics and thermodynamics for equilibrium measures of holomorphic endomorphisms on complex projective spaces*, Monatsh. Math. 174.1 (2014), 141 – 162.
75. M. Szostakiewicz, M. Urbański, A. Zdunik, *Fine inducing and equilibrium measures for rational functions of the Riemann sphere*, Israel J. Math. 210.1 (2015), 399 – 465.
76. A. Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps*, Invent. math. 99 (1990), 627 – 649.
77. A. Zdunik, *Harmonic measure versus Hausdorff measures on repellers for holomorphic maps*, Trans. AMS 326.2 (1991), 633 – 652.

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