Thermodynamic formalism methods in one-dimensional real and complex dynamics

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1 – Introduction

- Pioneer of statistical physics: Marian Smoluchowski 1872 – 1917 Vienna-Lvov-Cracow.
- Application of thermodynamic methods to dynamics: Yakov Sinai, David Ruelle, Rufus Bowen 1960/70 -ties.

Lemma (finite variational principle)

For given real numbers $\phi_1, \dots, \phi_d$, the function

$$F(p_1, \dots p_d) := \underbrace{\sum_{i=1}^d -p_i \log p_i}_{entropy} + \underbrace{\sum_{i=1}^d p_i \phi_i}_{average\ \ potential}$$

on the simplex $\{(p_1, \dots, p_d) : p_i \geq 0, \sum_{i=1}^d p_i = 1\}$ attains its maximum, called pressure equal to $P(\phi) = \log \sum_{i=1}^d e^{\phi_i}$, at the equilibrium

$$\hat{p}_j = e^{\phi_j} / \sum_{i=1}^d e^{\phi_i}.$$

Hint: $\sum_{i=1}^d -p_i \log p_i + \sum_{i=1}^d p_i \phi_i = \sum_{i=1}^d p_i \log(e^{\phi_i}/p_i)$.

1 – Introduction: dynamics setting corresponding notions

$f : X \to X$ a contin. map for a compact metric space (X, ρ),
$\phi : X \to \mathbb{R}$ a continuous function (*potential*).

Definition (variational topological pressure)

$$P_{\mathrm{var}}(f, \phi) := \sup_{\mu \in \mathcal{M}(f)} \left(h_\mu(f) + \int_X \phi \, d\mu \right),$$

where $\mathcal{M}(f)$ is the set of all f-invariant Borel probability measures on X and $h_\mu(f)$ is measure-theoretical entropy.

Any measure where sup is attained is called *equilibrium state*.

Definition (topological pressure via separated sets)

$P_{\mathrm{sep}}(f, \phi) := \lim_{\varepsilon \to 0} \overline{\lim}_{n \to \infty} \frac{1}{n} \log \left(\sup_Y \sum_{y \in Y} \exp S_n\phi(y) \right)$,
supremum over all $Y \subset X$ such that for distinct $x, y \in Y$,
$\rho_n(x, y) := \max\{\rho(f^i(x), f^i(y)), 0 \le i \le n\} \ge \varepsilon$.

• $h_\mu(f) := \sup_{\mathscr{A}} \lim_{n\to\infty} \frac{1}{n+1} \sum_{A\in\mathscr{A}^n} -\mu(A)\log\mu(A)$,
supremum over finite partitions $\mathscr{A}$ of X,
$\mathscr{A}^n := \bigvee_{j=0,\dots,n} f^{-j}\mathscr{A}$.

Theorem (variational principle: Ruelle, Walters, Misiurewicz, Denker, ...)

$P_{\rm var}(f,\phi) = P_{\rm sep}(f,\phi)$.

FP & M. Urbański "Conformal Fractals: Ergodic Theory Methods" Cambridge 2010.

Theorem (Gibbs measure – uniform case)

Let $f : X \to X$ be a distance expanding, topologically transitive continuous open map of a compact metric space X and $\phi : X \to \mathbb{R}$ be a Hölder continuous potential. Then, there exists exactly one $\mu_\phi \in \mathcal{M}(f, X)$, called Gibbs measure, s.t.

$$C < \frac{\mu_\phi(f_x^{-n}(B(f^n(x), r_0)))}{\exp(S_n\phi(x) - nP)} < C^{-1},$$

called Gibbs property, where f_x^{-n} is the local branch of f^{-n} mapping $f^n(x)$ to x and $S_n\phi(x) := \sum_{j=0}^{n-1} \phi(f^j(x))$.

- *μ_ϕ is the unique equilibrium state for ϕ. It is equivalent to the unique $\exp -(\phi - P)$-conformal measure m_ϕ, that is an f-quasi-invariant measure with Jacobian $\exp -(\phi - P)$ for a constant P.*
- *$P = P(f, \phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in f^{-n}(x_0)} \exp S_n\phi(x)$. This normalizing limit exists and is equal $P_{\mathrm{sep}}(f, \phi)$ for every $x \in X$.*

2 – Introduction to dimension 1

Thermodynamic formalism is useful for studying properties of the underlying space X. In dimension 1, for f real of class $C^{1+\varepsilon}$ or f holomorphic (conformal) for an expanding repeller X, considering $\phi = \phi_t := -t \log|f'|$ for $t \in \mathbb{R}$, Gibbs property gives, as $\exp S_n(\phi_t) = |(f^n)'|^{-t}$,

$$\mu_{\phi_t}(f_x^{-n}(B(f^n(x), r_0))) \approx \exp(S_n\phi(x) - nP(\phi_t)) \approx \\ \operatorname{diam} f_x^{-n}(B(f^n(x), r_0))^t \exp -nP(\phi_t).$$

The latter follows from a comparison of the diameter with the inverse of the absolute value of the derivative of f^n at x, due to *bounded distortion*.

When $t = t_0$ is a zero of the function $t \mapsto P(\phi_t)$, this gives

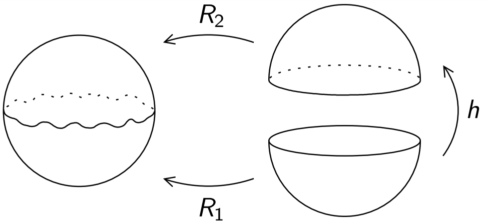
$$\mu_{\phi_{t_0}}(B) \approx (\operatorname{diam} B)^{t_0}$$

for all small balls B, hence $\mathrm{HD}(X) = t_0$. Moreover, the Hausdorff measure of X in this dimension is finite and nonzero.

A model application

Theorem (Bowen, Series, Sullivan)

For $f_c(z) := z^2 + c$ for an arbitrary complex number $c \neq 0$ sufficiently close to 0, the invariant Jordan curve J (Julia set for f_c) is fractal, i.e. has Hausdorff dimension bigger than 1.



If $\mathrm{HD}(J) = 1$, then $0 < H_1(J) < \infty$ and $h = R_2^{-1} \circ R_1$ on S^1 is absolutely continuous. $g_i := R_i^{-1} \circ f_c \circ R_i$ for $i = 1, 2$ preserve length ℓ on S^1 and are ergodic. Hence h preserves ℓ so it is a rotation, identity for appropriate R_1, R_2. Hence R_1 and R_2 glue together to a homography. Compare Mostov rigidity theorem.

In the complex case we consider f a rational mapping of degree at least 2 of the Riemann sphere $\overline{\mathbb{C}}$. We consider f acting on its Julia set $K = J(f)$ (generalizing the $z^2 + c$ model).

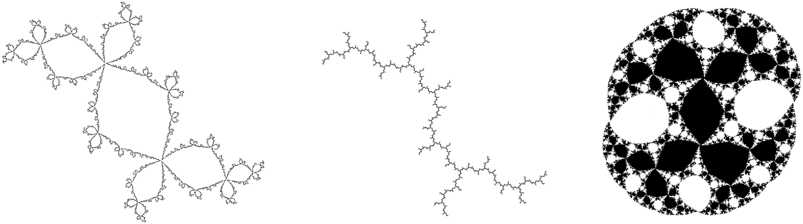


Figure: Douady's zoo: rabbit $f(z) = z^2 - 0.123 + 0.745i$, dendrite $f(z) = z^2 + i$, basilica mated with rabbit $f(z) = \frac{z^2+c}{z^2-1}$ for $c = \frac{1+\sqrt{-3}}{2}$ with $J(f)$ being the boundary of black and white

Definition (Real case, FP & Rivera-Letelier)

$f \in C^2$ is called a *generalized multimodal map* if defined on a neighbourhood of a compact invariant set K, critical points are not infinitely flat, *bounded distortion* property for iterates holds, abbr. BD, f is topologically transitive and has positive topological entropy on K.
Also K is a maximal forward invariant subset of a finite union of pairwise disjoint closed intervals whose endpoints are in K.

This maximality corresponds to Darboux property. We write $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$, where $+$ marks positive entropy. In place of BD one can assume C^3 (and write $(f, K) \in \mathscr{A}_+^3$) and assume that all periodic orbits in K are hyperbolic repelling. Then changing f outside K allows to get $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$.

Examples: Basic sets in spectral decomposition via renormalizations (de Melo, van Strien).

3 – Hyperbolic potentials

Call $\phi : K \to \mathbb{R}$ satisfying $P(f,\phi) > \sup_{\nu\in\mathcal{M}(f)} \int \phi\, d\nu$ *hyperbolic potential* (Inoquio-Renteria, Rivera-Letelier: BBMS 2012). Equiv. $P(f,\phi) > \sup_K \frac{1}{n} S_n\phi$ for some n.

Theorem (complex and real: Denker, Urbański, FP, Haydn, Rivera-Letelier, Zdunik, Szostakiewicz, H. Li, Bruin, Todd)

. If ϕ is a Hölder continuous hyperbolic potential, then there exists a unique equilibrium state μ_ϕ. For every Hölder $u : K \to \mathbb{R}$, the Central Limit Theorem (CLT) and Law of Iterated Logarithm (LIL) for the sequence of random variables $u \circ f^n$ and μ_ϕ hold.

CLT follows from sufficiently fast convergence of iteration of transfer operator (spectral gap). LIL is proved via LIL for a return map (inducing) to a nice domain related to μ_ϕ (Mañé, Denker, Urbański) providing a Markov structure (Infinite Iterated Function System) avoiding critical points, satisf. BD.

a) CE. *Collet-Eckmann condition*. There exists $\lambda > 1$, $C > 0$

$$|(f^n)'(f(c))| \geq C\lambda^n.$$

for all critical points $c \in K$ whose forward orbit is disjoint from $\mathrm{Crit}(f)$. Moreover there are no indifferent periodic orbits in K.

(b) CE2(z_0). *Backward* or *second Collet-Eckmann condition at $z_0 \in K$*. There exist $\lambda > 1$ and $C > 0$ such that for every $n \geq 1$ and every $w \in f^{-n}(z_0)$ (in a neighbourhood of K in the real case)

$$|(f^n)'(w)| \geq C\lambda^n.$$

(c) TCE. *Topological Collet-Eckmann condition* (FP & S. Rohde, Fund. Math. 1998).
There exist $M \geq 0, P \geq 1,\ r > 0$ such that for every $x \in K$ there exist increasing n_j, $j = 1, 2, \ldots,$ such that $n_j \leq P \cdot j$ and for each j and discs $B(\cdot)$ below understood in $\overline{\mathbb{C}}$ or $\mathbb{R}$.

$$\#\{0 \leq i < n_j : \mathrm{Comp}_{f^i(x)}\, f^{-(n_j-i)} B(f^{n_j}(x), r)) \cap \mathrm{Crit}(f) \neq \emptyset\} \leq M.$$

Each component of $f^{-n}(B)$ is called a pullback of B.

(d) ExpShrink. *Exponential shrinking of components*. There exist $\lambda > 1$ and $r > 0$ such that for every $x \in K$, every $n > 0$ and every connected component W_n of $f^{-n}(B(x,r))$ for the disc (interval) $B(x,r)$ in $\overline{\mathbb{C}}$ (or $\mathbb{R}$), intersecting K

$$\operatorname{diam}(W_n) \le \lambda^{-n}.$$

(e) LyapHyp. *Lyapunov hyperbolicity*. There is $\lambda > 1$ such that the Lyapunov exponent $\chi(\mu) := \int_K \log|f'|\, d\mu$ of any ergodic measure $\mu \in \mathcal{M}(f,K)$ satisfies $\chi(\mu) \ge \log\lambda$.

(f) UHP. *uniform hyperbolicity on periodic orbits*. There exists $\lambda > 1$ such that every periodic point $p \in K$ of period $k \ge 1$ satisfies

$$|(f^k)'(p)| \ge \lambda^k.$$

Theorem (..., Keller, Nowicki, Sands, FP, Rohde, Rivera-Letelier, Graczyk, Smirnov)

Assume there are no indifferent periodic orbits in K. Then

1. The conditions (c)–(f) and else (b) for some z_0 are equivalent (in the real case under the assumption of weak isolation: any periodic orbit close to K must be in K).

2. CE implies (b)–(f).

3. If there is only one critical point in the Julia set in the complex case or if f is S-unimodal on $K = I$ in the real case, then all conditions above are equivalent to each other.

4. TCE is topologically invariant; therefore all other conditions equivalent to it are topologically invariant.

For polynomials (b)-(f) are equivalent to $K = J(f) = \mathrm{Fr}\,\Omega_\infty(f)$, the basin of ∞, being Hölder (Graczyk, Smirnov).

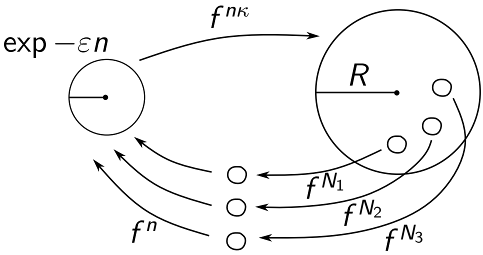
An order of proving the equivalences in Theorem above is, for z_0 safe,
CE2(z_0) $\Rightarrow$ExpShrink$\Rightarrow$LyapHyp$\Rightarrow$UHP$\Rightarrow$CE2(z_0)
Separately one proves ExpShrink$\Leftrightarrow$TCE using for $\Rightarrow$ the following

Lemma (Denker, FP, Urbański, ETDS 1996)

$$\sum_{j=0}^{n}{}' -\log|f^j(x)-c| \leq Qn$$

for a constant $Q>0$ every $c \in \mathrm{Crit}(f)$, every $x \in K$ and every integer $n>0$. Σ' means that we omit in the sum an index j of smallest distance $|f^j(x)-c|$.

Assumed UHP one proves CE2(z_0) for safe and hyperbolic z_0 by "shadowing".



Definition (safe)

We call $z \in K$ *safe* if $z \notin \bigcup_{j=1}^{\infty}(f^j(\mathrm{Crit}(f)))$ and for every $\epsilon > 0$ and all n large enough $B(z, \exp(-\epsilon n)) \cap \bigcup_{j=1}^{n}(f^j(\mathrm{Crit}(f))) = \varnothing$.

Notice that this definition implies that all points except at most a set of Hausdorff dimension 0, are safe.

5 – Geometric variational pressure and equilibrium states

For $\phi = \phi_t := -t \log |f'|$, the variational definition of pressure, here

$$P(t) := P_{\mathrm{var}}(f, \phi_t) = \sup_{\mu \in \mathcal{M}(f)} \left(h_\mu(f) - t \int_K \log |f'| \, d\mu \right)$$

still makes sense by the integrability of $\log |f'|$. Moreover $\int_K \log |f'| \, d\mu = \chi(\mu) \geq 0$,
for all ergodic μ even in presence of critical points where $\phi = \pm\infty$, [FP: PAMS 1993, Rivera-Letelier: arXiv 2012]. By this definition $t \mapsto P(t)$ is convex, monotone decreasing.

We usually assume $t > 0$ later on.

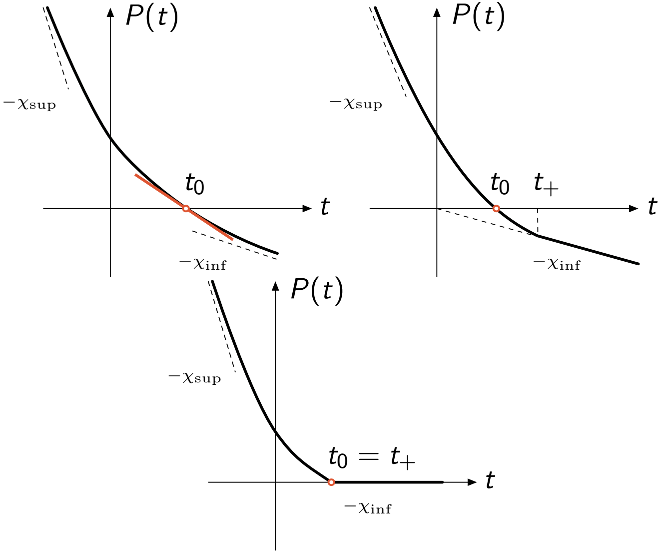


Figure: The geometric pressure: LyapHyp with $t_+ = \infty$, LyapHyp with $t_+ < \infty$, and non-LyapHyp.

$P(t)$ is equal to several other quantities (Complex: FP TAMS 1999, FP & Rivera-Letelier & Smirnov ETDS 2004). E.g.

Definition (hyperbolic pressure)

$$P_{\mathrm{hyp}}(t) := \sup_{X \in \mathscr{H}(f,K)} P(f|_X, -t \log|f'|),$$

where $\mathscr{H}(f, K)$ is defined as the space of all compact forward inv., i.e. $f(X) \subset X$, expanding subsets of K, repellers in $\mathbb{R}$.

Definition (hyperbolic dimension)

$$\mathrm{HD}_{\mathrm{hyp}}(K) := \sup_{X \in \mathscr{H}(f,K)} \mathrm{HD}(X).$$

For expanding $f : X \to X$, $t_0(X) = \mathrm{HD}(X)$. Passing to sup:

Proposition (Generalized Bowen's formula)

The first zero t_0 of $t \mapsto P_{\mathrm{hyp}}(K, t)$ is equal to $\mathrm{HD}_{\mathrm{hyp}}(K)$.

It may happen $\mathrm{HD}_{\mathrm{hyp}}(J(f)) < \mathrm{HD}(J(f)) = 2$ for f quadratic polynomials, Avila & Lyubich.

Theorem (FP & Rivera-Letelier)

1. Real case (arXiv 2014, to appear in Memoir of the AMS). Let $(f, K) \in \mathscr{A}_+^3$, f-periodic orbits in K be hyperbolic repelling. Then

• *$t \mapsto P(t)$ is real analytic on an open interval (t_-, t_+) defined by $P(t) > \sup_{\nu \in \mathcal{M}(f)} -t \int \log |f'| \, d\nu$*

• *For each t in this interval there is a unique invariant equilibrium state μ_{ϕ_t}. It is ergodic and absolutely continuous with respect to an adequate conformal measure m_{ϕ_t} with $d\mu_{\phi_t}/dm_{\phi_t} \geq \mathrm{Const} > 0$ a.e.*

• *If furthermore f is topologically exact on K (that is for every V an open subset of K there exists $n \geq 0$ such that $f^n(V) = K$), then this measure is mixing, has expon. decay of corr. and satisfies CLT for Lipschitz gauge functions.*

This generalizes results by Bruin, Iommi, Pesin, Senti, Todd.

Theorem (FP & Rivera-Letelier)

2. Complex case (Comm. Mat. Phys. 2011). The assertion is the same. One assumes a very weak expansion: the existence of arbitrarily small nice, or pleasant, couples and hyperbolicity away from critical points.

Remark. For real f satisfying LyapHyp and $K = \hat{I}$, we have the unique zero of pressure $t_0 = 1$ and for $-\log|f'|$ we conclude that a unique equilibrium state exists which is a.c.i.m. .

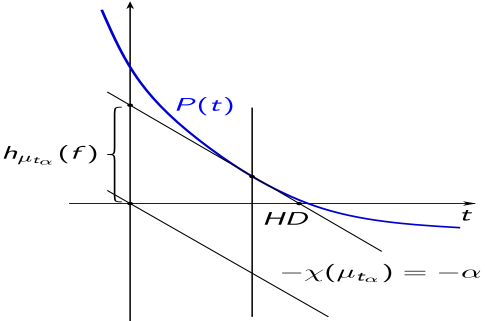
In general it holds assumed e.g. $|(f^n)'(f(c))| \to \infty$ for all $c \in \mathrm{Crit}(f)$ (Bruin & Rivera-Letelier & Shen & van Strien: Inv. math 2008). For $t > t_+$, LyapHyp, equilibria do not exist (Rivera-Letelier & Inoquio 2012).

Proofs use inducing (Lai-Sang Young towers). For a different proof, the real case, see a recent preprint by Dobbs and Todd.

$P_{\text{var}}(t)$ allows to study dimension spectrum for Lyapunov exponent via Legendre transformation, proving in particular

$$\mathrm{HD}(\{x \in K : \chi(x) = \alpha\}) = \frac{1}{|\alpha|} \inf_{t\in\mathbb{R}} (P(t) + \alpha t) .$$

Proof of $\geq$. Given α consider t where inf is attained. The tangent to $P(t)$ at t is parallel to $-\alpha t$ and for μ_t the equilibrium, it is $h_{\mu_t}(f) - t\chi(\mu_t)$. So the infimum is $h_{\mu_t}(f)$, see Fig. (By variational definition $P(t)$ and h_μ are mutual Legendre type transforms.) Dividing by α gives $\geq$ using Mañé's equality $\mathrm{HD}(\mu) = h_\mu(f)/\chi(\mu)$.



Proof of $\leq$ uses conformal measures.

Use of the Legendre transform of $P(t)$ allows also to give formulas for HD of irregular sets

$$\mathrm{HD}(\{\underline{\chi}(x) = \alpha, \overline{\chi}(x) = \beta\})$$

for $\beta > 0$ [Gelfert & FP & Rams: Math. Ann. 2010, ETDS 2016].

In analogy to $\chi(\mu) \geq 0$ one has:

Theorem (Levin & FP & Shen: Inv.math. 2016))

If for a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ there is only one critical point c in $J(f)$ and no parabolic periodic orbits, then $\underline{\chi}(f(c)) \geq 0$.

For S-unimodal maps of interval this was proved much earlier by Nowicki and Sands.

Definition (tree pressure)

For every $z \in K$ and $t \in \mathbb{R}$ define

$$P_{\mathrm{tree}}(z,t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{f^n(x)=z,\, x\in K} |(f^n)'(x)|^{-t}.$$

Theorem

$P_{\mathrm{tree}}(z,t)$ does not depend on z for z safe.

• In the complex case to prove $P_{\mathrm{tree}}(z_1,t) = P_{\mathrm{tree}}(z_2,t)$ one joins z_1 to z_2 with a curve not fast accumulated by critical trajectories, FP: TAMS 1999, FP & Rivera-Letelier & Smirnov: ETDS 2004.

• In the real case there is no room for such curves. Instead, one relies on topological transitivity, FP & Rivera-Letelier: arXiv 2014 & Memoir AMS 2019, FP: Monatsh. Math. 2018.

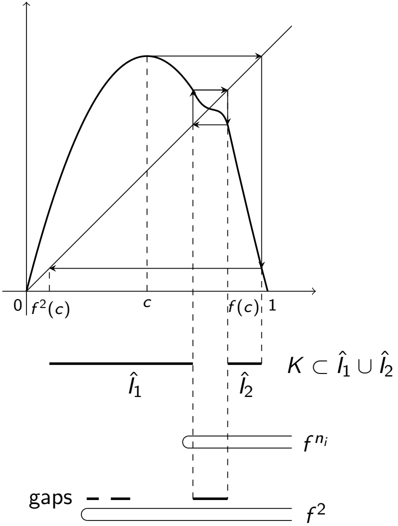
• For $\phi = -t \log |f'|$ pressure via separated sets does not make sense. Indeed, in presence of critical points for f, it is equal to $+\infty$. So it is replaced by P_{tree}.

• One can consider however *spanning geometric pressure* $P_{\mathrm{span}}(t)$ using (n, ε)-spanning sets and infimum.
Assumed weak backward Lyapunov stability it is indeed equal to $P(t)$ in the complex case (FP: Monatsh. Math. 2018).
• This is not so in the real case (where wbls always holds if all periodic orbits hyperbolic repelling). It happens $P_{\mathrm{span}}(t) = \infty$ if some x with big $|(f^n)'(x)|^{-1}$ is well ρ_n-isolated.

Definition (weak backward Lyapunov stability, wbls)

f is *weakly backward Lyapunov stable* if for every $\delta > 0$ and $\varepsilon > 0$ for all n large enough and every disc $B = B(x, \exp -\delta n)$ centered at $x \in K$, for every $0 \le j \le n$ and every component V of $f^{-j}(B)$ intersecting K, it holds that diam $V \le \varepsilon$.

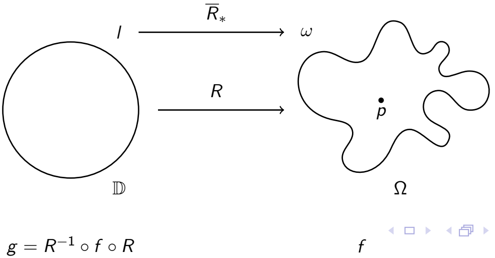
Question. Does wbls hold for all rational maps?



0
$f^2(c)$
c
$f(c)$
1
$K \subset \hat{I}_1 \cup \hat{I}_2$
$\hat{I}_1$
$\hat{I}_2$
f^{n_i}
gaps
f^2

7 – Boundary dichotomy

- Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with $\deg(f) \geq 2$. and let $\Omega = \Omega_p(f)$ be a simply connected immediate basin of attraction to a fixed point p. Let $R : \mathbb{D} \to \Omega$ be a Riemann map $R(0) = p$ and $g : \mathbb{D} \to \mathbb{D}$ defined by $g := R^{-1} \circ f \circ R$, extended conformaly beyond $\mathrm{Fr}\,\Omega$ (Schwarz symmetry), thus expanding on $\partial\mathbb{D}$.
- Consider harmonic measure $\omega = \overline{R}_*(l)$, where l is normalized length measure on $\partial\mathbb{D}$ and $\overline{R}$ is radial limit, defined l-a.e. l is g-invariant, hence ω is f-invariant. Denote by H_1 Hausdorff measure in dimension 1.



Theorem (FP, Urbański, Zdunik: 1985 – 2006)

For f, Ω as above, $HD(\omega) = 1$. One of two cases holds:
1) $\omega \perp H_1$, which implies $\mathrm{HD}_{\mathrm{hyp}}(\mathrm{Fr}\,\Omega) > 1$;
2) $\omega \ll H_1$ and f is a finite Blaschke product or a two-to-one holomorphic factor of a Blaschke product in some holomorphic coordinates on $\overline{\mathbb{C}}$.

Consider $\psi := \log|g'| - \log|f'| \circ \overline{R}$. Notice that

$$\int_{\partial\mathbb{D}} \psi \, dl = 0, \text{ hence } \mathrm{HD}(\omega) = 1.$$

The latter was proved in 1985 by Makarov without assuming existence of f.

Consider the asymptotic variance
$\sigma^2 = \sigma_\nu^2(\psi) := \lim_{n\to\infty} \frac{1}{n} \int_{\partial\mathbb{D}} (S_n\psi)^2 \, dl$.
Then $\omega \perp H_1$ is equivalent to $\sigma^2 > 0$ and equivalent to ψ not being cohomologous to 0 (not of the form $u \circ f - u$).

Theorem (LIL-refined-HD for harmonic measure, FP, Urbański, Zdunik: Ann. Math. 1989, Studia Math. 1991)

For f, Ω with $\sigma^2 > 0$, there exists $c(\Omega) > 0$, such that for
$\alpha_c(r) := r \exp(c\sqrt{\log 1/r \log\log\log 1/r})$
i) $\omega \perp H_{\alpha_c}$ for the gauge function α_c, for all $0 < c < c(\Omega)$;
ii) $\mu \ll H_{\alpha_c}$ for all $c > c_1(\Omega)$.

This theorem applies also e.g. to snowflake-type Ω's,

Proofs.

We can find X with $\mathrm{HD}(X) \geq \mathrm{HD}(\omega) - \epsilon$ by Katok method and using $\mathrm{HD} = h/\chi$. But we can do better:

$\sigma^2 > 0$ yields by CLT large fluctuations of the sums $\sum_{j=0}^{n-1} \psi \circ \varsigma^j$ from 0, allowing to find expanding X with $\mathrm{HD}(X) > \mathrm{HD}(\omega)$. One builds an iterated function system, for which X is the limit set. A special care is needed to get $X \subset \mathrm{Fr}\,\Omega$.

Substituting in LIL $n \sim (\log 1/r_n)/\chi(\omega)$ for $r_n = |(f^n)'(x)|^{-n}$, comparing $\log |(g^n)'| - \log |(f^n)'| \circ \overline{R}$ with $\sqrt{2\sigma^2 n \log\log n}$ for a sequence of n's, we get

Lemma (Refined Volume Lemma)

For ω-a.e. x

$$\limsup_{n\to\infty} \frac{\omega(B(x,r_n))}{\alpha_c(r_n)} = \begin{cases} \infty, & \text{for } 0 < c < c(\omega), \\ 0, & \text{for } c > c(\omega). \end{cases}$$

Using $R = f^{-n} \circ R \circ g^n$ one obtains

Theorem (radial growth)

For Lebesgue a.e. $\zeta \in \partial\mathbb{D}$

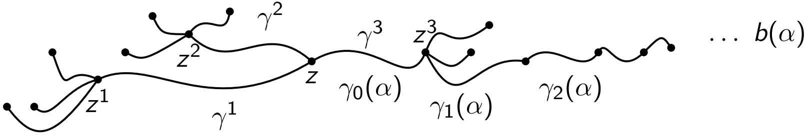
$$G^+(\zeta) := \limsup_{r \nearrow 1} \frac{\log|R'(r\zeta)|}{\sqrt{\log(1/1-r)\log\log\log(1/1-r)}} = c(\Omega).$$

Similarly $G^-(\zeta) := \liminf \cdots = -c(\Omega)$.

Above theorems hold for every connected, simply connected open $\Omega \subset \mathbb{C}$, different from $\mathbb{C}$, without existence of f. Of course one should add $\mathrm{ess\,sup}$ over $\zeta \in \partial\mathbb{D}$ and over $z \in \mathrm{Fr}\,\Omega$ in Refined Volume Lemma and reformulate the case i) . There is a universal Makarov's upper bound $C_{\mathrm{M}} < \infty$ for all $c(\Omega)$, $C_{\mathrm{M}} \leq 1.2326$ (Hedenmalm, Kayumov: PAMS 2007). In 1989 I gave a weaker estimate.

Geometric coding trees, g.c.t.

- Above theorems hold in an abstract setting of a geometric coding tree in U for $f : U \to \overline{\mathbb{C}}$, $f(U) \supset U$ proper, giving a coding $\pi : \Sigma^d \to \Lambda$ to the limit set Λ (in place of $\overline{R} : \partial\mathbb{D} \to \mathrm{Fr}\,\Omega$), provided f extends holomorphically beyond $\mathrm{cl}\,\Lambda$ called then a *quasi-repeller*.



Curves $\gamma^j : [0,1] \to f(U)$, $j = 1, \dots, d$, join z to z^j

$$\gamma_0(\alpha) := \gamma^{\alpha_0},$$
$$f \circ \gamma_n(\alpha) = \gamma_{n-1}(\varsigma(\alpha)),$$
$$\gamma_n(\alpha)(0) = \gamma_{n-1}(\alpha)(1).$$

• For a Hölder potential $\phi : \Sigma^d \to \mathbb{R}$ (in place of $-\log|g'|$) and Gibbs measure μ_ϕ one gets a dichotomy for $\mu := \pi_*(\mu_\phi)$ on Λ.

• For a constant potential $\mu = \mu_{\max}$ a measure of maximal entropy on Julia set $J(f)$ for $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ rational. Then

1) If $\sigma^2 > 0$ then $\mathrm{HD}_{\mathrm{hyp}}(J(f)) > \mathrm{HD}(\mu_{\max})$.

2) If $\sigma^2 = 0$ then for each $x, y \in J(f)$ not postcritical, if $z = f^n(x) = f^m(y)$ for some positive integers n, m, the orders of criticality of f^n at x and f^m at y coincide. In particular all critical points in $J(f)$ are pre-periodic, f is postcritically finite with parabolic orbifold, in particular z^d, Chebyshev or some Lattès maps, (Zdunik, Inv. math. 1990).

• In the Ω version it is sufficient to assume f is defined only in a neighbourhood of $\partial\Omega$ repelling on the side of Ω, called RB-domain.

• This applies to f polynomial and simply connected $\Omega = \Omega_\infty$ giving again the dichotomy on $\mathrm{Fr}\,\Omega$.

- For a simply connected domain $\Omega \subset \mathbb{C}$ one considers the *integral means spectrum*:

$$\beta_\Omega(t) := \limsup_{r \nearrow 1} \frac{1}{|\log(1-r)|} \log \int_{\zeta \in \partial\mathbb{D}} |R'(r\zeta)|^t \, |d\zeta|.$$

This, in presence of f, e.g. for an RB-domain Ω and for $\phi = -\log|f'|$ for $g(z) = z^d$, e.g. Ω being a simply connected basin of ∞ for a polynomial of degree d, satisfies

$\beta_\Omega(t) = t - 1 + \frac{P(t\phi)}{\log d}$. (Makarov, FP & Rohde)

One considers

$$\sigma^2(\log R') := \limsup_{r \nearrow 1} \frac{\int_{\partial\mathbb{D}} |\log R'(t\zeta)|^2 \, |d\zeta|}{-2\pi \log(1-r)|}.$$

It holds $\sigma^2(\log R') = 2\frac{d^2\beta_\Omega(t))}{dt^2}|_{t=0}$ (O. Ivrii). It is related to the Weil-Petersson metric (McMullen).

Recall $\sigma^2_\mu(t\phi) = \frac{d^2 P(f,t\phi)}{dt^2}$ for μ Gibbs in expanding case, Ruelle: Thermodyn. Formalism, FP & Urbański: Conformal Fractals.

8. Accessibility

Theorem (Douady-Eremenko-Levin-Petersen, accessibility of periodic sources; FP, the general case: Fund. Math. 1994)

Let Λ be a limit set for a g.c.t. $\mathscr{T}$ for holomorphic $f : U \to \overline{\mathbb{C}}$. Assume $\mathrm{diam}(\gamma_n(\alpha)) \to 0$, as $n \to \infty$, uniform shrinking with respect to $\alpha \in \Sigma^d$. Then every good $q \in \mathrm{cl}\,\Lambda$ is a limit of a convergent branch $b(\alpha)$, i.e. $q \in \Lambda$. In particular, this holds for every q with $\underline{\chi}(q) > 0$ satisfying a local backward invariance.

Corollary (lifting of measure, FP 1994 & Proc. ICM18)

Every non-atomic hyperbolic probability measure μ, i.e. $\chi(\mu) > 0$, on $\mathrm{cl}\,\Lambda$, is the π_* image of a probability ς-invariant measure ν on Σ^d, assumed uniform shrinking, $\mathscr{T}$ has no self-intersections and μ-a.e. local backward invariance of U. In part. a lift ν exists for every completely invariant RB-domain, e.g. for μ on $\mathrm{Fr}\,\Omega_\infty$ for f polynomial.

THANK YOU FOR YOUR ATTENTION