NO HYPERBOLIC SETS IN J_∞ FOR INFINITELY RENORMALIZABLE QUADRATIC POLYNOMIALS

GENADI LEVIN AND FELIKS PRZYTYCKI

Abstract. Let f be an infinitely renormalizable quadratic polynomial and J_∞ be the intersection of forward orbits of "small" Julia sets of its simple renormalizations. We prove that J_∞ contains no hyperbolic sets.

1. Introduction

Let f be a rational function of degree at least 2 considered as a dynamical systems $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ on the Riemann sphere $\hat{\mathbb{C}}$. An f-invariant compact set $X \subset \hat{\mathbb{C}}$ is said to be *hyperbolic* if $f : X \to X$ is uniformly expanding , i.e., for some $C > 0$ and $\lambda > 1$, $|D(f^m)(x)| \geq C\lambda^m$ for all $x \in X$ and all $m \geq 0$ (here D stands for the spherical derivative and f^m is m-iterate of f). In particular, any repelling periodic orbit of f is a hyperbolic set. The closure of all repelling periodic orbits of f is the Julia set $J(f)$ of f. Hyperbolic sets of f are contained in $J(f)$. Apart of repelling periodic orbits, f admits plenty of infinite (Cantor) hyperbolic sets [30]. Attracting periodic orbits (if any) along with their basins are contained in the complement $\hat{\mathbb{C}} \setminus J(f)$ (which is called the Fatou set of f). See e.g. [3] for an introduction to complex dynamics and [31] for a recent survey.

If $J(f)$ is a hyperbolic set by itself, i.e., $f : J(f) \to J(f)$ is uniformly expanding, then f is called a hyperbolic rational map. Equivalently, all critical points of f are in basins of attracting cycles. Hyperbolic rational maps are analogous to Axiom A diffeomorphisms and their dynamics has been intensively studied and very well understood. The famous 'Density of Hyperbolicity Conjecture (DHC)' in holomorphic dynamics - sometimes also called the Fatou conjecture - asserts that any rational map (polynomial) can be approximated by hyperbolic rational maps (polynomials) of the same degree.

Date: June 22, 2022.

In what follows f (unless mentioned explicitly) is a quadratic polynomial $f_c(z) = z^2 + c$. The DHC (as well as a more general MLC: Mandelbrot set Locally Connected) is widely open for the quadratic family f_c, too (DHC for f_c as strongly believed accumulates in itself the essence of the general DHC). After a breakthrough work of Yoccoz [10] on the MLC, the only obstacle for proving DHC for quadratic polynomials are so-called infinitely renormalizable ones, see [28].

Somewhat informally, a quadratic polynomial f_c with connected Julia set is called renormalizable if, for some topological disks U, V around the critical point 0 of f_c and for some $p \geq 2$ (called period of the renormalization), the restriction $f_c^p : U \to V$ is conjugate to another quadratic polynomial $f_{c'}$ with connected Julia set (see [6] for exact definitions and the theory of polynomial-like mappings). The map $F := f_c^p : U \to V$ is then a *renormalization* of f_c and the set $K(F) = \{z \in U : F^n(z) \in U \text{ for all } n \geq 1\}$ is a *"small" (filled in) Julia set* of f_c. If $f_{c'}$ is renormalizable by itself, then f_c is called twice renormalizable, etc. If f_c admits infinitely many renormalizations, it is called *infinitely renormalizable.* Recall that the renormalization F is *simple* if any two sets $f^i(K(f))$, $f^j(K(F))$, $0 \leq i < j \leq p-1$, are either disjoint or intersect each other at a unique point which does not separate either of them.

To state our main result - which is Theorems 1.1 - let $f(z) = z^2+c$ be infinitely renormalizable. Let $1 = p_0 < p_1 < ... < p_n < ...$ be the sequence of consecutive periods of simple renormalizations of f and J_n denotes the "small" Julia set of the n-renormalization (where $J_0 = J(f)$). Then p_{n+1}/p_n is an integer, $f^{p_n}(J_n) = J_n$, for any n, and $\{J_n\}_{n=1}^{\infty}$ is a strictly decreasing sequence of continua without interior, all containing 0. Let

$$J_\infty = \cap_{n\geq 0} \cup_{j=0}^{p_n-1} f^j(J_n)$$

be the intersection of orbits of the "small" Julia sets. J_∞ is a compact f-invariant set which contains the omega-limit set $\omega(0)$ of 0. Each component of J_∞ is wandering, in particular, J_∞ contains no periodic orbits of f. Note that a hyperbolic set in J_∞ (if existed) could not be repelling, that is any forward orbit of a point sufficiently close to this set must be in the set itself, since otherwise shadowing periodic orbits must be in J_∞.

It is shown in [23] that the low Lyapunov exponent of the critical value $c \in J(f_c)$ is always non-negative. In the considered case, $c \in J_\infty$. We prove:

Theorem 1.1. *J_∞ contains no hyperbolic sets.*

Combined with the Fatou-Mane theorem [25] Theorem 1.1 immediately implies

Corollary 1.1. *$\omega(x) \cap \omega(0) \neq \emptyset$, for the omega-limit set $\omega(x)$ of every $x \in J_\infty$.*

The conclusion of Theorem 1.1 would obviously hold provided

$$J_\infty \text{ is totally disconnected.} \tag{1.1}$$

(1.1) is true indeed for many classes of maps (including real ones) where it follows from 'complex bounds' [33] (meaning roughly that the sequence of renormalizations is compact) [17], [9], [24], [13], [14], [15]. See also [11], [12]. However, (1.1) breaks down in general: see [26], [32] for the existence of such maps and [18], [19], [20] (see also [5]) for explicit combinatorial conditions on f_c for (1.1) to fail. Yoccoz [35] posed a problem to find a necessary and sufficient condition on the combinatorics of f_c for (1.1) to hold. At present, the gap between known sufficient and necessary conditions is still very big.

Another well-known open problem is to give necessary and sufficient conditions so that the Julia set $J(f)$ is locally-connected. For example, if (1.1) does not hold then $J(f)$ is not locally-connected. Theorem1.1 implies

Theorem 1.2. *Let $f(z) = z^2 + c$ and f has no irrational indifferent periodic orbits. Then $J(f)$ is locally-connected at every point of any hyperbolic set X of f. In particular, there are at least one and at most finitely many external rays landing at each $x \in X$.*

Remark 1.2. The case that f does have an irrational cycle seems to be open and requires a separate consideration, see [4] though. Note also that Theorem 1.2 removes the only restriction in Proposition 2.11 of [1] for degree 2 polynomials without irrational cycles.

Theorem 1.2 has been known for the following quadratic maps f. If f has an attracting cycle, then f is hyperbolic and the whole $J(f)$ is locally-connected. The same conclusion holds if f has a parabolic cycle [6]. The first part of Yoccoz's result (see e.g., [10]) says that $J(f)$ is locally-connected if f has no indifferent irrational cycles and at most finitely many times renormalizable. This allows us to reduce the proof of Theorem 1.2 to the case of f as in Theorem 1.1, hence, by the latter, to the case when X is disjoint from J_∞ in which case it is well-known that Yoccoz puzzle pieces shrink to each point of X [26], [21]. This shows that $J(f)$ is locally connected at points of X. The last claim follows then from [16], see also [34] and [21].

Acknowledgment. We thank Weixiao Shen for helpful comments. We thank the referee for many useful remarks that helped to improve the paper.

2. Preliminaries

Here we collect, for further references, necessary notations and general facts which are either well-known [28], [27] or follow readily from the known ones. Let $f(z) = z^2 + c$ be infinitely renormalizable. We keep the notations of the Introduction.

(A). Let G be the Green function of the basin of infinity $A(\infty) = \{z|f^n(z) \to \infty, n \to \infty\}$ of f with the standard normalization at infinity $G(z) = ln|z| + O(1/|z|)$. The external ray R_t of argument $t \in \mathbf{S^1} = \mathbf{R}/\mathbf{Z}$ is a gradient line to the level sets of G that has the (asymptotic) argument t at ∞. $G(z)$ is called the (Green) level of $z \in A(\infty)$ and the unique t such that $z \in R_t$ is called the (external) argument (or angle) of z. A point $z \in J(f)$ is accessible if there is an external ray R_t which lands at (i.e., converges to) z. Then t is called an (external) argument (angle) of z.

Let $\sigma : \mathbf{S^1} \to \mathbf{S^1}$ be the doubling map $\sigma(t) = 2t(mod 1)$. Then $f(R_t) = R_{\sigma(t)}$.

(B). Given a small Julia set J_n containing 0, sets $f^j(J_n)$ ($0 \le j < p_n$) are called small Julia sets of level n. Each $f^j(J_n)$ contains $p_{n+1}/p_n \ge 2$ small Julia sets $f^{j+kp_n}(J_{n+1})$, $0 \le k < p_{n+1}/p_n$, of level $n+1$. We have $J_n = -J_n$. Since all renormalizations are simple, for $j \neq 0$, the symmetric companion $-f^j(J_n)$ of $f^j(J_n)$ can intersect the orbit $orb(J_n) = \cup_{j=0}^{p_n-1} f^j(J_n)$ of J_n only at a single point which is preperiodic. On the other hand, since only finitely many external rays converge to each periodic point of f, the set J_∞ contains no periodic points. In particular, each component K of J_∞ is wandering, i.e., $f^i(K) \cap f^j(K) = \emptyset$ for all $0 \le i < j < \infty$. All this implies that $\{x, -x\} \subset J_\infty$ if and only if $x \in K_0 := \cap_{n=1}^{\infty} J_n$.

Given $x \in J_\infty$, for every n, let $j_n(x)$ be the unique $j \in \{0, 1, \cdots, p_n - 1\}$ such that $x \in f^{j(x)}(J_n)$. Let $J_{x,n} = f^{j_n(x)}(J_n)$ be a small Julia set of level n containing x and $K_x = \cap_{n \ge 0} J_{x,n}$, a component of J_∞ containing x.

In particular, $K_0 = \cap_{n\ge 0} J_n$ is the component of J_∞ containing 0 and $K_c = \cap_{n=1}^{\infty} f(J_n)$, the component containing c.

The map $f : K_x \to K_{f(x)}$ is one-to-one if $x \notin K_0$ while $f : K_0 \setminus \{0\} \to K_c \setminus \{c\}$ is two-to-one. Moreover, for every $y \in J_\infty$, $f^{-1}(y) \cap J_\infty$ consists of two points if $y \in K_c \setminus \{c\}$ and consists of a single point otherwise.

(C). Given $n \geq 0$, the map $f^{p_n} : f(J_n) \to f(J_n)$ has two fixed points: the separating fixed point α_n (that is, $f(J_n) \setminus \{\alpha_n\}$ has at least two components) and the non-separating β_n (so that $f(J_n) \setminus \{\beta_n\}$ has a single component).

For every $n > 0$, there are two rays R_{t_n} and $R_{\tilde{t}_n}$ $(0 < t_n < \tilde{t}_n < 1)$ to the non-separating fixed point $\beta_n \in f(J_n)$ of f^{p_n} such that the component Ω_n of $\mathbf{C} \setminus (R_{t_n} \cup R_{\tilde{t}_n} \cup \beta_n)$ which does not contain 0 has two characteristic properties:

(i) Ω_n contains c and contains no the forward orbit of β_n,

(ii) for every $1 \leq j \leq p_n$, consider arguments (angles) of the the external rays which land at $f^{j-1}(\beta_n)$. The angles split $\mathbf{S^1}$ into finitely many arcs. Then the arc

$$S_{n,1} = [t_n, \tilde{t}_n] = \{t : R_t \subset \Omega_n\}$$

has the smallest length among all these arcs.

Denote

$$t'_n = t_n + \frac{\tilde{t}_n - t_n}{2^{p_n}}, \quad \tilde{t}'_n = \tilde{t}_n - \frac{\tilde{t}_n - t_n}{2^{p_n}}.$$

The rays $R_{t'_n}$, $R_{\tilde{t}'_n}$ land at a common point $\beta'_n \in f^{-p_n}(\beta_n) \cap \Omega_n$. Introduce an (unbounded) domain U_n with the boundary to be two curves $R_{t_n} \cup R_{\tilde{t}_n} \cup \beta_n$ and $R_{t'_n} \cup R_{\tilde{t}'_n} \cup \beta'_n$. In other words, U_n is a component of $f^{-p_n}(\Omega_n)$ which is contained in Ω_n. Then $c \in U_n$ and $f^{p_n} : U_n \to \Omega_n$ is a two-to-one branched covering so that

$$f(J_n) = \{z | f^{kp_n}(z) \in \overline{U}_n, G(f^{kp_n}(z)) \leq 10, k = 0, 1, ...\}. \tag{2.1}$$

Moreover, for any n, the closure of U_{n+1} is contained in U_n. We denote

$$s_{n,1} = [t_n, t'_n] \cup [\tilde{t}'_n, \tilde{t}_n].$$

Then $s_{n,1} \subset S_{n,1}$ and

$$\sigma^{p_n} : s_{n,1} \to S_{n,1}$$

so that σ^{p_n} is a homeomorphism of each component of $s_{n,1}$ onto $S_{n,1}$. End points $t_n, \tilde{t}_n$ of $S_{n,1}$ are fixed points of σ^{p_n}. It's important to note that $S_{n+1,1} \subset S_{n,1}$, $s_{n,1} \subset s_{n+1,1}$ for all n and the length $(\tilde{t}_n - t_n)/2^{p_n}$ of each of the two components of $s_{n,1}$ tends to zero as $n \to \infty$ (while the length $|t_n - \tilde{t}_n|$ of $S_{n,1}$ can stay away from zero).

From now on, *given a compact set $Y \subset J(f)$ denote by $\tilde{Y}$ the set of arguments of the external rays which have their limit sets in Y.*

For each $k \geq 0$ the boundary of the set $\{z : f^{kp_n}(z) \in \overline{U}_{n,}\}$ consists of rays and (pre)periodic points. It follows that if a ray has at least one limit point in $f(J_n)$ then all its limit points are in $f(J_n)$, and (2.1) implies that

$$\widetilde{f(J_n)} = \{t|\sigma^{jp_n}(t) \in s_{n,1}, j = 0, 1, ...\}. \tag{2.2}$$

So $\widetilde{f(J_n)}$ is a Cantor set, in particular, closed. Let us show that

$$\tilde{K}_c = \cap_{n=1}^{\infty} s_{n,1}. \tag{2.3}$$

Indeed, $t \in \tilde{K}_c$ implies $t \in \widetilde{f(J_n)} \subset s_{n,1}$, for each n. Vice versa, let $t \in \cap_{n=1}^{\infty} s_{n,1}$. It is enough to show that $t \in \widetilde{f(J_n)}$ for each n (which would indeed imply that the limit set of the ray R_t belongs to $f(J_n)$ for all n, i.e., $t \in \tilde{K}_c$). Fix n and find a sequence $t_m \in \partial s_{m,1}$, such that $t_m \to t$ as $m \to \infty$. On the other hand, $\partial s_{m,1} \subset \widetilde{f(J_m)} \subset \widetilde{f(J_n)}$ because $f(J_m) \subset f(J_n)$ for $m > n$. As $\widetilde{f(J_n)}$ is closed, $t \in \widetilde{f(J_n)}$. This proves (2.3).

It implies that $\tilde{K}_c$ is either a single-point set or a two-point set. In particular, K_c contains at most two different accessible points. As $f : K_0 \setminus \{0\} \to K_c \setminus \{c\}$ is two-to-one, $\tilde{K}_0 = \sigma^{-1}(\tilde{K}_c)$ consists of either 2 or 4 points.

Let us give, for completeness of the picture, a similar description of $\tilde{K}$ for each component K of J_∞ (Lemma 2.1, see below). It will be needed for the proof of Lemma 4.2, part (ii). Note, however, that part (ii) of Lemma 4.2 is not used in the proof of the main result.

Let

$$s_{n,j} = \sigma^{j-1}(s_{n,1}) = [t_{n,j}, t'_{n,j}] \cup [\tilde{t}'_{n,j}, \tilde{t}_{n,j}], 1 \leq j \leq p_n,$$

where $t_{n,j} = \sigma^{j-1}(t_n)$, $\tilde{t}_{n,j} = \sigma^{j-1}(\tilde{t}_n)$, $t'_{n,j} = \sigma^{j-1}(t'_n)$, and $\tilde{t}'_{n,j} = \sigma^{j-1}(\tilde{t}'_n)$. Then

$$t'_{n,j} - t_{n,j} = \tilde{t}_{n,j} - \tilde{t}'_{n,j} = \frac{\tilde{t}_n - t_n}{2^{p_n - j+1}} < \tilde{t}_n - t_n < 1/2. \tag{2.4}$$

So $\sigma^{j-1} : s_{n,1} \to s_{n,j}$ is a homeomorphism and $s_{n,j}$ has two components ('windows') $[t_{n,j}, t'_{n,j}]$ and $[\tilde{t}'_{n,j}, \tilde{t}_{n,j}]$ of equal length. However, $\sigma^j(S_{n,1})$ can cover the whole circle $\mathbf{S^1}$ for some $j < p_n$. For this reason, an analogue of (2.1) breaks down if j is big. For $j = 1, 2, ..., p_n$, let $U_{n,j} = f^{j-1}(U_n)$ and $\beta_{n,j} = f^{j-1}(\beta_n)$. The domain $U_{n,j}$ is bounded by two rays $R_{t_{n,j}} \cup R_{\tilde{t}_{n,j}}$ converging to $\beta_{n,j}$ and completed by $\beta_{n,j}$ along with two rays $R_{t'_{n,j}} \cup R_{\tilde{t}'_{n,j}}$ completed by their common limit point $f^{j-1}(\beta'_n)$. Let $U^1_{n,j}$ be a component of $f^{-(p_n-j+1)}(U_n)$ which is contained in $U_{n,j}$. Then

$$f^{p_n} : U^1_{n,j} \to U_{n,j} \tag{2.5}$$

is a two-to-one branched covering, and

$$f^{j-1}(J_n) = \{z | f^{kp_n}(z) \in \overline{U}^1_{n,j}, G(f^{kp_n}(z)) \leq 10, k = 0, 1, ...\}. \tag{2.6}$$

Note that this is consistent with (2.1) for $j = 1$. Similar to $j = 1$, if a ray has at least one limit point in $f^{j-1}(J_n)$ then all its limit points are there.

Let $s^1_{n,j}$ be the set of arguments of rays entering $\overline{U}^1_{n,j}$. Then $s^1_{n,j} \subset s_{n,j}$ consists of two pairs of components (1-'windows') where each pair is adjacent to two end points of one of the 'windows' of $s_{n,j}$. Moreover, $\sigma^{p_n} : s^1_{n,j} \to s_{n,j}$ is a two-to-one covering which maps each of four 1-'windows' of $s^1_{n,j}$ homeomorphically onto one of the two 'windows' of $s_{n,j}$. Correspondingly, for each $j = 1, 2, ..., p_n$,

$$\widetilde{f^j(J_n)} = \{t | \sigma^{kp_n}(t) \in s^1_{n,j}, k = 0, 1, ...\}. \tag{2.7}$$

As (2.6) is consistent with (2.1) for $j = 1$, (2.7) is consistence with (2.2) for $j = 1$.

Given n, j, denote by $\Delta_{n,j}$ the length of each 'window' of $s_{n,j}$ and by $\Delta^1_{n,j}$ the length of each 'window' of $s^1_{n,j}$. By (2.4), $\Delta_{n,j} = \frac{\tilde{t}_n - t_n}{2^{p_n-j+1}}$ and $\Delta^1_{n,j} = \frac{\Delta_{n,j}}{2^{p_n}}$. So $\Delta^1_{n,j} < 2^{-p_n} \to 0$ uniformly in j as $n \to \infty$.

Let K be a component of J_∞:

$$K = \cap_{n=1}^\infty f^{j_n}(J_n),$$

where $1 \leq j_n \leq p_n$ and $f^{j_{n+1}}(J_{n+1}) \subset f^{j_n}(J_n)$.

Lemma 2.1.

$$\tilde{K} = \cap_{n=1}^\infty s^1_{n,j_n} \tag{2.8}$$

where $\{s^1_{n,j_n}\}_{n=1}^\infty$ *is a decreasing sequence of compacts each consisting of four* 1-*'windows' with equal lengths tending to zero as* $n \to \infty$. *In particular,* $\tilde{K}$ *consists of at most* 4 *points. There is an alternative (1)-(2):*

(1) *either* $p_n - j_n \to \infty$ *as* $n \to \infty$ *so that the length of each 'window' of* s_{n,j_n} *tends to zero, moreover,*

$$\tilde{K} = \cap_{n=1}^\infty s_{n,j_n} \tag{2.9}$$

so that $\#\tilde{K} \in \{1, 2\}$ *in this case.*

(2) *there is* $N \geq 0$ *such that* $p_n - j_n = N$ *for all* n, *so that* $f^N(K) = K_0$.

Proof. (2.8) is very similar to the proof of (2.3) and is left to the reader. As for the alternative (1)-(2), assume that there is $N \geq 0$ such that, for an infinite subsequence $(n') \subset \mathbb{N}$, $p_{n'} - j_{n'} = N$. Then $f^N(K) = \cap_{n'} f^{n'}(J_{n'}) = \cap_n f^n(J_n) = K_0$, hence, $p_n - j_n = N$ for all n. This explains why either $p_n - j_n \to \infty$ or $p_n - j_n = N$ for some $N \geq 0$ and all n. Consider the case $p_n - j_n \to \infty$. Then by (2.4) the length Δ_{n,j_n} of each 'window' of s_{n,j_n} tends to zero uniformly in j_n. Repeating again the proof of (2.3) we get that $\cap_{n=1}^{\infty} s_{n,j_n} = \cap_{n=1}^{\infty} s'_{n,j_n}$. This settles the alternative. □

(D1). Consider in more detail the case when $\tilde{K}_c = \{\tau_1, \tau_2\}$, $\tau_1 \neq \tau_2$. Let S_c be the shortest arc in $\mathbf{S^1}$ with the end points τ_1, τ_2. It follows from (C):

1_∞: $\sigma^k(\tau_i) \notin S_c$, for $i = 1, 2$ and all $k > 0$,

2_∞: for each $k > 0$, the length of arcs with the end points $\sigma^k(\tau_1)$, $\sigma^k(\tau_2)$ is bigger than or equal to the length of S_c,

3_∞: (unlinking) for each positive $j \neq k$, one of the two arcs $\mathbf{S^1} \setminus \{\sigma^k(\tau_1), \sigma^k(\tau_2)\}$ contains both points $\sigma^j(\tau_1), \sigma^j(\tau_2)$. Furthermore, as $\tilde{K}_0 = \sigma^{-1}(\{\tau_1, \tau_2\})$, the set $\tilde{K}_0$ splits the unit circle $\mathbf{S^1}$ into 4 arcs in such a way that, for each $j \geq 0$, both points $\sigma^j(\tau_1), \sigma^j(\tau_2)$ have to lie in one and only one of these arcs. In particular, $\sigma^j(\tau_1), \sigma^j(\tau_2)$ lie in one and only one of semicircles $\mathbf{S^1} \setminus \sigma^{-1}(\tau_i)$, for each $i = 1, 2$.

Remark 2.2. Let us put the 'unlinking' property in the context of Thurston's 'laminations' [34]. Consider a topological model of $J(f)$ by shrinking all components of J_∞ as well as all their preimages to points. More formally, let's build a lamination (a special equivalent relation on $\mathbf{S^1}$) as follows [34], [16]: as all periodic points of f are repelling, each (pre)periodic point is a landing point of at least one and at most finitely many rays. Let us identify arguments of such rays for each such point and take a closure of this partial relation to the whole $\mathbf{S^1}$. The resulting relation is invariant under the map $\sigma : \mathbf{S^1} \to \mathbf{S^1}$. For visualization, this relation is usually extended to the closed unit disk by taking the closed convex hull in the Euclidean plane of each equivalence class. Then obviously different convex hulls (classes) are disjoint. For example, $\sigma^{-1}(\{\tau_1, \tau_2\})$ is a single class, and $\sigma^j(\tau_1), \sigma^j(\tau_2)$ is the other one, for each $j \geq 0$, so their convex hulls are disjoint. The property $\#\tilde{K} \leq 2$ (which is proved in Lemma 2.1) for any component K of J_∞ other than preimages of K_0 is, in fact, a very particular case of the fundamental 'no wandering triangle' property of unicritical laminations [34], [21].

(D2). Given $\nu \in [0,1)$ there exists a unique *minimal* rotation set $\Lambda_\nu \subset \mathbf{S^1}$ of $\sigma : \mathbf{S^1} \to \mathbf{S^1}$ with the rotation number ν [2]. Recall that a closed subset Λ of $\mathbf{S^1}$ is a rotation set of σ with the rotation number ν if $\sigma(\Lambda) \subset \Lambda$ and $\sigma : \Lambda \to \Lambda$ extends to a map of $\mathbf{S^1}$ which lifts to an orientation preserving non-decreasing continuous map $F : \mathbf{R} \to \mathbf{R}$ with $F - \mathrm{Id}$ to be 1-periodic and the fractional part of the rotation number of F to be equal to ν. Then [2] ν is irrational if and only if Λ_ν is infinite, in this case there is a unique closed semi-circle containing Λ_ν so that the end points of this semi-circle belong to Λ_ν. Finally, any closed σ-invariant set $\Lambda \subset \mathbf{S^1}$ which is contained in a semi-circle is a rotation set of σ.

3. Accessibility

We define a *telescope* following essentially [29]. Given $x \in J(f)$, $r > 0$, $\delta > 0$, $k \in \mathbf{N}$ and $\kappa \in (0,1)$, an (r, κ, δ, k)-telescope at $x \in J$ is collections of times $0 = n_0 < n_1 < ... < n_k = n$ and disks $B_l = B(f^{n_l}(x), r)$, $l = 0, 1, ..., k$ such that, for every $l > 0$: (i) $l/n_l > \kappa$, (ii) there is a univalent branch $g_{n_l} : B(f^{n_l}(x), 2r) \to \mathbf{C}$ of f^{-n_l} so that $g_{n_l}(f^{n_l}(x)) = x$ and, for $l = 1, ..., k$, $d(f^{n_{l-1}} \circ g_{n_l}(B_l), \partial B_{l-1}) > \delta$ (clearly, here $f^{n_{l-1}} \circ g_{n_l}$ is a branch of $f^{-(n_l - n_{l-1})}$ that maps $f^{n_l}(x)$ to $f^{n_{l-1}}(x)$). The trace of the telescope is a collection of sets $B_{l,0} = g_{n_l}(B_l)$, $l = 0, 1, ..., k$. We have: $B_{k,0} \subset B_{k-1,0} \subset ... \subset B_{1,0} \subset B_{0,0} = B_0 = B(x, r)$. By the *first point of intersection* of a ray R_t, or an arc of R_t, with a set E we mean a point of $R_t \cap E$ with the minimal level (if it exists).

Theorem 3.1. [29] *Given $r > 0$, $\kappa \in (0,1)$, $\delta > 0$ and $C > 0$ there exist $M > 0$, $\tilde{l}, \tilde{k} \in \mathbb{N}$ and $K > 1$ such that for every (r, κ, δ, k)-telescope the following hold. Let $k > \tilde{k}$. Let $u_0 = u$ be any point at the boundary of B_k such that $G(u) \geq C$. Then there are indexes $1 \leq l_1 < l_2 < ... < l_j = k$ such that $l_1 < \tilde{l}$, $l_{i+1} < K l_i$, $i = 1, ..., j-1$ as follows. Let $u_k = g_{n_k}(u) \in \partial B_{k,0}$ and let γ_k be an infinite arc of an external ray through u_k between the pint u_k and ∞. Let $u_{k,k} = u_k$ and, for $l = 1, ..., k-1$, let $u_{k,l}$ be the first point of intersection of γ_k with $\partial B_{l,0}$. Then, for $i = 1, ..., j$,*

$$G(u_{k,l_i}) > M 2^{-n_{l_i}}.$$

Applying Theorem 3.1 as in [29] (where the existence of a ray to x is proved assuming $(r, \kappa, \delta, \infty)$-telescope to x) we obtain the following statement. See also [1] for a direct proof of part (1).

Corollary 3.1. *Let X be a hyperbolic set for f. (1) To every point $x \in X$ one can assign a non-empty set $A_x \subset \mathbf{S}^1$ such that for every*

$t \in A_x$ the external ray R_t lands at x. (2) The set $\mathcal{A} = \{(x,t) : x \in X, t \in A_x\}$ is closed in $\mathbb{C} \times \mathbf{S}^1$. (3) Moreover, for each $\mu > 0$ there is $C(\mu) > 0$ such that for all $x \in X$ and all $t \in A_x$, the first intersection of R_t with $\partial B(x,\mu)$ has the level at least $C(\mu)$.

Proof. As $f : X \to X$ is expanding, there exist $\lambda > 1$, $\rho > 0$ and j_0 such that, for every $x \in X$ and every $k > 0$, there exists a (univalent) branch $g_{k,x} : B(f^{kj_0}(x), \rho) \to \mathbf{C}$ of f^{-kj_0} with $g_{k,x}(f^{kj_0}(x)) = x$ and $|g'_{k,x}(y)/g'_{k,x}(z)| < 2$ for $y, z \in B(f^{kj_0}(x), \rho)$. Moreover, $|g'_{k,x}(z)| < \lambda^{-k}$ for all $k > 0$ and $x \in X$. Therefore, there are $r > 0$, $\delta > 0$ and $\kappa = 1/j_0$ such that for any $k > 1$, every point $x \in X$ admits an (r, κ, δ, k)-telescope with $n_k = kj_0$. In fact, $n_i = ij_0$ for $i = 0, 1, \cdots, k$. Let $B_{k,0}(x) \subset B_{k-1,0}(x) \subset \cdots \subset B_{1,0}(x) \subset B_{0,0}(x)$ be the corresponding trace. Define $C_0 = \inf_{y \in J(f)} \max_{z \in B(y,r)} G(z)$. It is easy to see that $C_0 > 0$. For each $x \in X$ we choose a point $u(x) \in \partial B(x,r)$ such that $G(u(x)) \geq C_0$. By Theorem 3.1, there are $\tilde{l}$ and $\tilde{k}$ such that for each $k > \tilde{k}$ and each $x \in X$ the following hold. There are $1 \leq l_{1,k}(x) < l_{2,k}(x) < \cdots < l_{j_k^x,k}(x) = k$ such that $l_{1,k}(x) < \tilde{l}$, $l_{i+1,k}(x) < K l_{i,k}(x)$, $i = 1, \cdots, j_k^x - 1$. Let $\gamma_k(x)$ be an arc of an external ray between the point $u_k(x) = g_{k,x}(u(f^{kj_0}(x))$ and ∞. Let $u_{k,l}(x)$ be the first intersection of $\gamma_k(x)$ with $\partial B_{l,0}(x)$. Then, for $i = 1, \cdots, j_k^x - 1$,

$$G(u_{k,l_{i,k}}(x)) > M 2^{-l_{i,k}(x) j_0}. \tag{3.1}$$

For all $i = 1, \cdots, j_k^x - 1$,

$$i \leq l_{i,k}(x) < K^i \tilde{l}. \tag{3.2}$$

Denote by $t_k(x)$ the argument of an external ray that contains the arc $\gamma_k(x)$. It is enough to prove in the situation above

Lemma 3.2. *(i) If $(x_m)_{m>0} \subset X$, $x_m \to y$ and $t_{k_m}(x_m) \to \tau$ for some $k_m \to \infty$, then R_τ lands at y. (ii) Moreover, for each $\mu > 0$ there is $C(\mu) > 0$ such that for all pairs (y, τ) like this the first intersection of R_τ with $\partial B(y,\mu)$ has the level at least $C(\mu)$.*

Indeed, assuming this lemma, we can define A_x as the set of all angles t so that there exist $x_m \in X$ and $k_m \to \infty$ with $x_m \to x$ and $t_{k_m}(x_m) \to t$. The set A_x is not empty because one can take $x_m = x$ for all m and $k_m \to \infty$ such that $\{t_{k_m}(x)\}$ converges. By Lemma 3.2(i), the ray R_t indeed lands at the point x. It is then an elementary exercise to check that the set $\mathcal{A}$ is closed. Claim (ii) implies obviously the part (3).

So, let's prove Lemma 3.2. Let (y, τ) be as in the lemma. Pick any $\mu \in (0, r)$. Fix l_0 so that $\lambda^{-l_0} r < \mu/2$ and let

$$C(\mu) = \frac{M}{2^{\tilde{l} j_0 K^{l_0}}}. \tag{3.3}$$

There is m_0 such that for all $m > m_0$ and all $l > l_0$, $B_{l_0,0}(x_m) \subset B(y, \mu)$. Then, by (3.1)-(3.2), for every $m > m_0$,

$$G(u_{k_m, l_{l_0,k_m}}(x_m)) > C(\mu). \tag{3.4}$$

Hence, for every $m > m_0$ the first intersection of $\gamma_{k_m}(x_m)$ with $B(y, \mu)$ has level at least $C(\mu)$. This implies that, given $C \in (0, C(\mu))$, for any $m > m_0$, an arc of the ray $R_{t_{k_m}(x_m)}$ between the levels C and $C(\mu)$ does not exit $B(y, \mu)$. As this sequence of arcs tend uniformly, as $m \to \infty$, to an arc of R_τ between the levels C and $C(\mu)$, this latter arc is contained in $B(y, \mu)$. As $C > 0$ can be chosen arbitrary small, the ray R_τ must land at y and its first intersection with $B(y, \mu)$ has the level at least $C(\mu)$. Lemma 3.2 is proved. □

4. A combinatorial property

Let f be an infinitely-renormalizable quadratic polynomial. First, we prove the following combinatorial fact (a maximality property) about f.

In the course of the further proofs the following well-known easy statement about expanding maps is used (for a more general theorem about expansive maps, see e.g. [8]):

Proposition 4.1. *Let h be a homeomorphism of a compact metric space (S, d) onto itself which is expanding in the following sense: there are $\delta > 0$ and $\lambda > 1$ such that $d(h(x), h(x')) \geq \lambda d(x, x')$ whenever $d(x, x') < \delta$. Then S is a finite set.*

Let $\omega(t)$ be the omega-limit set of $t \in \mathbf{S}^1$ under $\sigma : t \mapsto 2t (mod 1)$ and $\omega(E) = \cup_{t \in E} \omega(t)$.

Lemma 4.2. *(i)* $\sigma^{-1}(\tilde{K}_c) \subset \omega(\tilde{K}_c)$
(ii) $\tilde{J}_\infty = \omega(\tilde{K}_c)$.

Remark 4.3. Only part (i) is used in the proof of Theorem 1.1. Part (ii) seems to be of an independent interest and is included for completeness.

Proof of (i). (a) Consider $\sigma : \tilde{J}_\infty \to \tilde{J}_\infty$. Each $t \in J_\infty$ belongs to $\tilde{K}$, for one and only one component K of J_∞. By (B), one and only one component K^{-1} of $f^{-1}(K)$ is a component of J_∞, moreover, $f : K^{-1} \to K$ is a homeomorphism if $K \neq K_c$ and $f : K_0 \setminus \{0\} \to K_c \setminus \{c\}$ is two-to-one. Hence, $\sigma^{-1}(t) \cap \tilde{J}_\infty$ is a single point for $t \notin \tilde{K}_c$ and two different points for $t \in \tilde{K}_c$. On the other hand, for each $t \in \mathbf{S}^1$, σ maps $\omega(t)$ onto itself. Since $\sigma : \tilde{J}_\infty \to \tilde{J}_\infty$ has no periodic points, by Proposition 4.1, for each $t \in \tilde{J}_\infty$, the expanding map $\sigma : \omega(t) \to \omega(t)$ is not injective. Therefore, $\omega(t) \cap \tilde{K}_c \neq \emptyset$ for all $t \in \tilde{J}_\infty$.

(b) If $\tilde{K}_c$ consists of a single angle τ_c, then (a) implies that there is a single point $t \in \omega(\tau_c)$ such that $\sigma^{-1}(t) \subset \omega(\tau_c)$ and, moreover, $t = \tau_c$. Therefore, $\sigma^{-1}(\tau_c) \subset \omega(\tau_c)$ and we are done in this case.

(c) It remains to deal with a two-point set $\tilde{K}_c = \{\tau_1, \tau_2\}$. Let us assume the contrary, i.e., $\sigma^{-1}(\tau_1) \cup \sigma^{-1}(\tau_2)$ is not a subset of $\omega(\tilde{K}_c) = \omega(\tau_1) \cup \omega(\tau_2)$. Hence, by (a) and by the assumption, either $\sigma^{-1}(\tau_1)$ or $\sigma^{-1}(\tau_2)$ is a subset of each $\omega(\tau_i)$, $i = 1, 2$. Let, say, $\sigma^{-1}(\tau_1) \subset \omega(\tau_1) \cap \omega(\tau_2)$. By the assumption, there is $\tau \in \sigma^{-1}(\tau_2)$ such that $\tau \notin \omega(\{\tau_1, \tau_2\})$. Let L be a (open) semi-circles $\mathbf{S^1} \backslash \sigma^{-1}(\tau_1)$ that contains τ. We claim that it is enough to show that for each p_n and some $j_n > 0$ the arc L contains $\sigma^{j_n p_n - 1}(\tau_1)$. Indeed, assume this is the case. Then, by (D1)3_∞, Sect. 2, L must contain one of the arcs $\mathbf{S^1} \setminus \{\sigma^{j_n p_n - 1}(\tau_1), \sigma^{j_n p_n - 1}(\tau_2)\}$ and, by (D1)2_∞, the lengths of all such arcs are uniformly away from 0. Hence, there is a subsequence $n_i \to \infty$ such that the sequences $\sigma^{j_{n_i} p_{n_i} - 1}(\tau_1)$ and $\sigma^{j_{n_i} p_{n_i} - 1}(\tau_2)$ converge to points a_1 and a_2 respectively, where $a_1 \neq a_2$ and $a_1, a_2 \in \overline{L}$. On the other hand, $a_1, a_2 \in \tilde{K}_0$. But, from the assumption, $\tilde{K}_0 \cap L = \{\tau\}$. Therefore $\{a_1, a_2\} \subset \{\tau, \sigma^{-1}(\tau_1)\}$. Since $a_1, a_2 \in \omega(\{\tau_1, \tau_2\})$ while $\tau \notin \omega(\{\tau_1, \tau_2\})$, we conclude that $\{a_1, a_2\} = \sigma^{-1}(\tau_1)$. But then $\sigma^{j_{n_i} p_{n_i}}(\tau_1)$ and $\sigma^{j_{n_i} p_{n_i}}(\tau_2)$ converge to the same point τ_1 which is a contradiction with (D1)2_∞.

(d) To show that, for each n, the arc L contains a point $\sigma^{j_n p_n - 1}(\tau_1)$, for some $j_n > 0$, let us assume the contrary. So, we fix $n > 0$ and assume that $\Lambda := \{\sigma^{j p_n - 1}(\tau_1) : j > 0\} \subset L' := \mathbf{S^1} \backslash L$. The idea is to show that the set Λ corresponds to a rotation set Λ_0 of $\sigma : \mathbf{S^1} \to \mathbf{S^1}$ and use a certain structure of later sets [2], see (D2) of Sect. 2, to arrive at a contradiction.

Following the notations of (C), let $\Omega_n^0 = f^{-1}(\Omega_n)$ and $U_n^0 = f^{-1}(U_n)$. To connect it to other notations introduced in (C), $\Omega_n^0 = U_{n,p_n} = f^{p_n - 1}(U_n)$ and $U_n^0 = U_{n,p_n}^1$. Then $J_n = \{z | f^{j p_n}(z) \in \overline{U_n^0}, G(f^{j p_n}(z)) \leq 10, j = 0, 1, \cdots\}$. The domain Ω_n^0 is bounded by two bi-infinite curves Γ_n, Γ_n' (components of $f^{-1}(\partial \Omega_n)$) and two angular (closed) "arcs at infinity" which are components $S_{n,0}, S_{n,0}'$

of $\sigma^{-1}(S_{n,1})$ where $S_{n,1}=[t_n,\tilde{t}_n]$, $0<t_n<\tilde{t}_n<1$. In fact, $S_{n,0},S'_{n,0}$ are two components of $s_{n,p_n}=\sigma^{p_n-1}(s_{n,1})$. Arguments of rays entering $\overline{U_n^0}$ form the set $s^1_{n,p_n}\subset s_{n,p_n}$ consisting of four 1-'widows' such that σ^{p_n} is a homeomorphism of each of them onto either $S_{n,0}$ or $S'_{m,0}$. By (2.7), $\widetilde{J_n}=\widetilde{f^{p_n}(J_n)}=\{t|\sigma^{kp_n}(t)\in s^1_{n,p_n}, k=0,1,...\}$.

Let us specify Γ_n to be such component of $f^{-1}(\partial\Omega_n)$ that contains $f^{p_n-1}(\beta_n)$ (in other words, $f^{p_n}(\Gamma_n)=\Gamma_n$), and $S_{n,0}$ to be the first "arc" as one goes from Γ_n to Γ'_n inside of Ω_n^0 in the counterclockwise direction along the "circle at infinity". In other words, $S_{n,0}$ denotes the component of $\sigma^{-1}(S_{n,1})$ that contains $\sigma^{p_n-1}(t_n)$ in its boundary. Now, let $\epsilon(t)=0$ if $t\in S_{n,0}$ and $\epsilon(t)=1$ if $t\in S'_{n,0}$. To every $t\in\widetilde{J_n}$ we assign a point $\theta(t)\in\mathbf{S^1}$ as follows:

$$\theta(t)=\sum_{j=0}^{\infty}\frac{\epsilon(\sigma^{jp_n}(t))}{2^{j+1}}.$$

Then $\theta(\widetilde{J_n})=\mathbf{S^1}$, $\theta\circ\sigma^{p_n}=\sigma\circ\theta$, and $\theta:\widetilde{J_n}\to\mathbf{S^1}$ extends to a continuous monotone degree one map $\mathbf{S^1}\to\mathbf{S^1}$, see [7], [27], [22, Theorem 3]. Moreover, θ is injective on a subset $T=\{t\in\widetilde{J_n}|\sigma^{kp_n}(t)\notin\partial(S_{n,0}\cup S'_{n,0}), k\geq 0\}$. Note that $\widetilde{J_\infty}\subset T$ where $\widetilde{J_\infty}$ is closed. Besides, $\tilde{K}_0=\sigma^{-1}(\{\tau_1,\tau_2\})$ and $\Lambda=\{\sigma^{jp_n-1}(\tau_1):j>0\}$ are subsets of $\widetilde{J_\infty}\subset T$. If $\theta_0:=\theta(\tau_1/2)$ then $\theta_1:=\theta(\tau_1/2+1/2)=\theta_0+1/2$. Therefore, from the assumption, the set $\Lambda_0:=\theta(\Lambda)$ is a subset of a semi-circle C_{θ_0} with end points θ_0 and θ_1. As $\sigma^{p_n}(\Lambda)\subset\Lambda$, then $\sigma(\Lambda_0)\subset\Lambda_0$. It follows [2] (see (D2) of Sect. 2 in the present paper), that the set $\overline{\Lambda_0}$ is a rotation set of $\sigma:\mathbf{S^1}\to\mathbf{S^1}$. Let $E\subset\overline{\Lambda_0}$ be a closed subset such that $\sigma:E\to E$ is minimal. As $\overline{\Lambda}\subset\widetilde{J_\infty}\subset T$ where $\widetilde{J_\infty}$ contains no periodic points of σ^{p_n}, E contains no periodic points of σ. Hence, E is infinite. By [2], see (D2), for every closed infinite minimal rotation set of σ there is a unique closed semi-circle containing it and in this case the end points of the semi-circle belong to the set. Thus $\theta_0,\theta_0+1/2\in E\subset\overline{\Lambda_0}\subset\overline{C_{\theta_0}}$. Coming back to the f-plane, we find two sequences $j_i,j'_i\to\infty$ so that $\sigma^{j_ip_n-1}(\tau_1)\to\tau_1/2$ and $\sigma^{j'_ip_n-1}(\tau_1)\to\tau_1/2+1/2$ inside of the same semi-circle L' with the end points $\tau_1/2$, $\tau_1/2+1/2$. But then $\sigma^{j_ip_n}(\tau_1)$ and $\sigma^{j'_ip_n}(\tau_1)$ both tend to τ_1 from different sides, in a contradiction with (D1)1_∞. This completes the proof of part (i) of the statement.

Let us prove part (ii). Obviously, $\omega(\tilde{K}_c)\subset\tilde{J}_\infty$. To show the opposite, given $t\in\tilde{J}_\infty$ let K be a unique component of J_∞ such that $t\in\tilde{K}$. Let $K=\cap_{n\geq 1}f^{j_n}(J_n)$. By Lemma 2.1, there is an alternative: either (1) $p_n - j_n \to \infty$ as $n \to \infty$ and $\tilde{K} = \cap_{n=1}^{\infty} s_{n,j_n}$ or (2) there is $N \geq 0$ such that $f^N(K) = K_0$, i.e., $p_n - j_n = N$ for all n. In case (2), $f^N : K \to K_0$ is a homeomorphism, which implies that $\sigma^N : \tilde{K} \to \tilde{K}_0$ is also a homeomorphism. Hence, if $s \in \tilde{K}_0$ is a point of $\omega(\tilde{K}_c)$, then $\sigma^N|_{\tilde{K}}^{-1}(s)$ is also a point of $\omega(\tilde{K}_c)$. On the other hand, by part (i), $\tilde{K}_0 = \sigma^{-1}(\tilde{K}_c) \subset \omega(\tilde{K}_c)$. Therefore, $t \in \omega(\tilde{K}_c)$ in case (2). In case (1), $\{t\} = \cap_{n\geq 1} s_{n,j_n}^{(t)}$ where $s_{n,j_n}^{(t)}$ is one of the two 'windows' of s_{n,j_n}, and $\sigma^{j_n} : s_{n,1}^{(t)} \to s_{n,j_n}^{(t)}$ is a homeomorphism where of course $s_{n,1}^{(t)}$ is one of the two 'windows' of $s_{n,1}$. Hence, it's enough to show that each 'window' of $s_{n,1}$ contains points of the orbit of $\tilde{K}_c$. As $\tilde{K}_0 = \cap_n s_{n,1}$, in the case $\#\tilde{K}_c = 2$, for all n big enough, each 'window' of $s_{n,1}$ contains one and only one of the two points of $\tilde{K}_c$, and we are done in this case. In the remaining case, $\tilde{K}_c$ is a single point which lies in one of the two 'widows' of $s_{n,1}$, for each n. But, as σ^{p_n} maps each 'window' of $s_{n,1}$ onto $S_{n,1} \supset s_{n,1}$ homeomorphically it is obvious that each 'window' of $s_{n,1}$ contains infinitely many points of the σ^{p_n}-orbit of any $t_0 \in \widetilde{f(J_n)}$ provided no point of this orbit hits $\partial S_{n,1}$. In particular, this applies to points of $\tilde{K}$ This completes the proof. □

Remark 4.4. In case (1) we proved more: if $t \in \tilde{J}_\infty$ is such that $\sigma^N(t) \notin \tilde{K}_0$ for all $N \geq 0$, then $t \in \omega(\tau)$ for each $\tau \in \tilde{K}_c$.

5. Proof of Theorem 1.1

1. Assume the contrary and let $X \subset J_\infty$ be a compact f-invariant hyperbolic set. In particular, Corollary 3.1 applies.

2. Replacing X by its subset if necessary we can assume that $f : X \to X$ is a minimal map.

3. $0 \notin X$, hence, $c \notin X$, too.

4. As J_∞ contains no cycles, X is an infinite set. If we assume that $f : X \to X$ is one-to-one, then $f : X \to X$ is an expanding homeomorphism of a compact set, therefore, X is finite, a contradiction with Proposition 4.1.

5. Thus $f : X \to X$ is not one-to-one. Then, by (B), $X_c := X \cap K_c \neq \emptyset$. On the other hand, by step 3, $c \notin X_c$.

6. By (C), $\tilde{K}_c$ consists of either one or two arguments. As any point of X is accessible, X_c consists of either one or two different points. Let $x_1 \in X_c$ and $x_2 \in J(f)$ be any other point. Given $r > 0$ and n, let $W_n(x_1, r)$ be a component of $B(x_1, r) \cap \Omega_n$ (see

(C), Sect. 2, where Ω_n is defined) that contains the point x_1. We use repeatedly the following

Claim 1. *Given $\hat{r} > 0$ and $\hat{C} > 0$, there is $\hat{n} \in \mathbf{N}$ as follows. For $k = 1, 2$, suppose that, for some angles $\hat{t}_k$, the ray $R_{\hat{t}_k}$ lands at x_k and let z_k be the first intersection of $R_{\hat{t}_k}$ with $\partial B(x_k, \hat{r}/2)$. Assume: (a) $G(z_k) > \hat{C}$ for $k = 1, 2$, (b) $x_2 \in \Omega_n$ and $|x_1 - x_2| < \hat{r}/3$, (c) one of the following holds: (i) $t_n - \tilde{t}_n \to 0$ as $n \to \infty$, or (ii) $\hat{t}_1, \hat{t}_2$ belong to a single component of $s_{n,1} = [t_n, t'_n] \cup [\tilde{t}'_n, \tilde{t}_n]$. Then $x_2 \in W_n(x_1, \hat{r})$ for each $n > \hat{n}$.*

Indeed, the length of each component of $s_{n,1}$ goes to zero as $n \to \infty$. Hence, as $\hat{r}$ and $\hat{C}$ are constants and n is big enough, condition (c) implies that a curve which consists of an arc of $R_{\hat{t}_1}$ from x_1 to z_1, then the shortest arc of the equipotential containing z_1 from z_1 to the first intersection u_2 with $R_{\hat{t}_2}$ and then back along $R_{\hat{t}_2}$ from u_2 to x_2 belongs to Ω_n and $B(x_1, \hat{r})$ simultaneously.

7. Fix $r > 0$ small enough. Let $a \in X_c$ and $a_{-1} \in X$ be such that $f(a_{-1}) = a$. As $a_{-1} \in K_0 \cap X$, there is its uniquely defined backward orbit $\{a_{-m}\}_{m=1}^{\infty} \subset X$, $f(a_{-m-1}) = a_{-m}$, $m \geq 1$. Let a' be a limit point of the sequences a_{-p_n}, i.e, $a' = \lim_{i\to\infty} a_{-p_{n_i}}$. As $a_{-p_n} \in f(J_n)$, a' belongs to K_c and X at the same time, that is, $a' \in X_c$.

Claim 2. *For all i large enough, $a_{-p_{n_i}} \in W_{n_i}(a', r)$.*

Indeed, by Corollary 3.1 there is a subsequence (n'_i) of (n_i) and a converging sequence $t_i \in A_{a_{-(p_{n'_i})}}$ such that $t := \lim_{i\to\infty} t_i$ and $t \in A_{a'}$. We have: $t_i \in s_{n'_i}$ for all i. Then Claim 1 of Step 6 applies for each i big enough, with $\hat{r} = r$, $\hat{C} = C(r/2)$, $x_1 = a'$, $x_2 = a_{-p_{n'_i}}$, $\hat{t}_1 = t$, $\hat{t}_2 = t_i$ and z_1, z_2 defined by this data as in Claim 1. Indeed, (a) holds by Corollary 3.1(3) and (b) is obvious (note that $a_{-p_n} \in f(J_n) \setminus \{\beta_n\} \subset \Omega_n$). Moreover, if (ci) breaks down, since $t_i \to t$, then t_i and t must lie at the same component of $s_{n'_i}$.

By the conclusion of Claim 1, $a_{-p_{n'_i}} \in W_{n'_i}(a', r)$ for each i big enough. Finally, as $A_{a'}$ consists of at most four points (therefore, the sequence (a_{-n_i}) has at most four limit points), Claim 2 follows.

8. Consider the case $X_c = \{a\}$. Let $f^{-1}(a) = \{a_{-1}, a^*_{-1}\}$. By steps 2, 3 and 5, $a_{-1} \neq a^*_{-1}$ and $a_{-1}, a^*_{-1} \in X$. As $X_c = \{a\}$, there is a subsequence (n_i) such that backward images $a_{-p_{n_i}}$, $a^*_{-p_{n_i}}$ of a_{-1}, a^*_{-1} respectively tend to the same point a. By Claim 2 Step 7, for each i large, $a_{-p_{n_i}}$, $a^*_{-p_{n_i}} \in W_{n_i}(a, r)$. Therefore, the following two sets (which are preimages of $W_{n_i}(a, r)$ by $f^{p_{n_i}}$): $V_{n_i} :=$

$g_{p_{n_i},a_{-(p_{n_i})}}(W_{n_i}(a,r))$ and $V^*_{n_i} := g_{p_{n_i},a^*_{-(p_{n_i})}}(W_{n_i}(a,r))$, are disjoint with their closures (because preimages of $B(a,r)$ along points of X shrink exponentially) and both are contained in $W_{n_i}(a,r)$. Fix such $n = n_i$. Denote for brevity $J_{c,n} = f(J_n)$. Then we get that, for every $j > 0$, 2^j preimages of $a \in J_{c,n}$ by the map $f^{jp_n} : J_{c,n} \to J_{c,n}$ are contained in the (disjoint) closures of V_n and V^*_n. As the set of all those preimages are dense in $J_{c,n}$, we get a contradiction with the fact that $J_{c,n}$ is a continuum.

9. Consider the remaining case $X_c = \{a, b\}$, $a \neq b$. As $\#\tilde{K}_c \leq 2$, each point a and b is accessible by a single ray $R_{t(a)}$ and $R_{t(b)}$ respectively. Hence, any point u from the *grand orbits* of a and b is a landing point of precisely one ray R_{t_u}. Let us prove that $f^{-1}(X_c) \subset X$. Let $f(w) = x \in X_c$. By Lemma 4.2(i), one can find $y \in X_c$ and $m_i \to \infty$ such that $\sigma^{m_i}(t_y) \to t_w$ and $f^{m_i}(y)$ tends to some point $\tilde{w} \in X$. By Corollary 3.1, $t_w \in A_{\tilde{w}}$. But $\sigma(t_w) = t_x$, hence, $f(\tilde{w}) = x$ and $A_{\tilde{w}} = \{t_w\}$. Thus $w = \tilde{w} \in X$.

10. We have just proved that $\{a_{-1}, a^*_{-1}\} = f^{-1}(a) \subset X$ and $\{b_{-1}, b^*_{-1}\} = f^{-1}(b) \subset X$. Now, we repeat the consideration as in Step 8. The sequences $a_{-(p_n)}$, $a^*_{-(p_n)}$, $b_{-(p_n)}$, $b^*_{-(p_n)}$ must have all their limit points in X_c. As $r > 0$ is small enough, $\overline{B(a,r)} \cap \overline{B(b,r)} = \emptyset$. By Claim 2 of Step 7, for each n large, all 4 points $a_{-(p_n)}, a^*_{-(p_n)}, b_{-(p_n)}, b^*_{-(p_n)}$ are in $W_n(a,r) \cup W_n(b,r)$. Fixing n large, for each disk $B(x,r)$, $x \in \{a,b\}$, there are two univalent branches of f^{-p_n} which are defined in $B(x,r)$ such that they map $W_n(x,r)$ inside $W_n(a,r) \cup W_n(b,r)$. Hence, for every $j > 0$, 4^j preimages of $X_c \in J_{c,n}$ by the map $f^{j2p_n} : J_{c,n} \to J_{c,n}$ are contained in the (disjoint) closures of $W_n(a,r)$ and $W_n(b,r)$. As the set of all those preimages are dense in $J_{c,n}$, we again get a contradiction with the fact that $J_{c,n}$ is a continuum.

Remark 5.1. The combinatorial property for quadratic polynomials of Lemma 4.2 implies that if $X \subset J_\infty$ is a minimal hyperbolic set then $f^{-1}(X) \cap J_\infty = X$ provided f is quadratic, and this leads to a contradiction. Therefore, a small modification of the proof gives us the following claim for infinitely-renormalizable unicritical polynomial $f(z) = z^d + c$ with any $d \geq 2$: J_∞ contains no hyperbolic sets X such that $f : X \to X$ is minimal and $f^{-1}(X) \cap J_\infty = X$.

6. Final remarks

A hyperbolic set of a rational map always carries an invariant measure with a positive Lyapunov exponent. Conjecturally, J_∞ as in Theorem 1.1 never carries such a measure. We cannot prove this conjecture in the full generality so far, but we can easily prove at least that $F := f|_{J_\infty}$ is not "chaotic". Namely,

1. Every F-invariant probability measure μ has zero entropy, $h_\mu(F) = 0$.

2. Topological entropy of F is zero, $h_{\rm top}(F) = 0$.

Proving it we can assume μ is ergodic due to Ergodic Decomposition Theorem, see e.g. [30, Theorem 2.8.11a)]. Start by observing that, for every n and $0 \le j < p_n$, $\mu(f^j(J_n)) = 1/p_n$, hence, μ has no atoms and $\mu(K) = 0$ for every component K of J_∞. Therefore, if $J'_\infty = J_\infty \setminus \cup_{n\in\mathbb{Z}} f^n(K_0)$ where K_0 is the component of J_∞ containing 0 (see (B) of Sect. 2), then $F : J'_\infty \to J'_\infty$ is an automorphism. Suppose to the contrary that $h_\mu(F) > 0$. Then 1. follows from Rokhlin entropy formula, [30, Theorem 2.9.7], saying that $h_\mu(F) = \int \log \mathrm{Jac}_\mu(F) d\mu$. Here Jac_μ is Jacobian with respect to μ, equal to 1 μ-a.e,. since μ must be supported on $J'_\infty \subset J_\infty$ where F is an automorphism. A condition to be verified to apply Rokhlin formula is the existence of a one-sided countable generator of bounded entropy, proved to exist by Mañé, see e.g. [30, Lemma 11.3.2] and inclusion [30, (11.4.8)], due to positive Lyapunov exponent $\chi_\mu(F) := \int \log |F'| d\mu \ge \frac{1}{2} h_\mu(F) > 0$ (Ruelle's inequality). Thus $h_\mu(F) > 0$ has led to a contradiction.

2. follows from 1. by variational principle $h_{\rm top}(F) = \sup_\mu h_\mu(F)$.

Compare item 4 in Section 5. Here finiteness of X is replaced by zero entropy.

The same proof with obvious modifications holds for $f(z) = z^d + c$, $d \ge 2$.

References

[1] Benini, A., Lyubich, M.: *Repelling periodic points and landing rays for post-singularly bounded exponential maps.* Ann. Inst. Fourier (Grenoble) 64 (2014), 4, 1493-1520.

[2] Bullett, S., Sentenac, P.: *Ordered orbits of the shift, square roots, and the devil's staircase.* Math. Proc. Cambridge Philos. Soc. 115 (1994), 3, 451-481.

[3] Carleson, L., Gamelin, T.: Complex Dynamics. Springer-Verlag, 1993

[4] Cheraghi, D.: *Typical orbits of quadratic polynomials with a neutral fixed point: Brjuno type.* Comm. Math. Phys. 322 (2013), 3, 999-1035.

[5] Cheraghi, D., Shishikura, M.: *Satellite renormalization of quadratic polynomials.* https://arxiv.org/abs/1509.07843

[6] Douady, A., Hubbard, J.H.: *On the dynamics of polynomial-like mappings.* Ann. Sci. Ec. Norm. Super. (4) 18 (1985), no. 2, 287-343.

[7] Douady, A.: *Algorithms for computing angles in the Mandelbrot set.* In: "Chaotic Dynamics and Fractals", ed. Barnsley Demko, Acad. Press (1986), pp. 155-168.

[8] W. H. Gottschalk and G. A. Hedlund: *Topological Dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 36, Providence, 1955.

[9] Graczyk J, Świątek G.: *Polynomial-like property for real polynomials.* Topol. Proc. 21 (1996), 33-122.

[10] Hubbard, J.H.: *Local connectivity of Julia sets and bifurcation coci: three theorems of J.-C. Yoccoz.* In: "Topological Methods in Modern Mathematics." Houston, TX:Publish or Perish, 1993.

[11] Kozlovski, O., Shen, W., Van Strien, S.: *Rigidity for real polynomials.* Ann. Math. 165 (2007), no. 3, 749-841.

[12] Kozlovski, O., Shen, W., Van Strien, S.: Density of hyperbolicity in dimension one. Ann. Math. 166 (2007), no. 1, 145-182.

[13] Kahn, J.: *A priori bounds for some infinitely renormalizable quadratics: I. Bounded primitive combinatorics.* https://arxiv.org/abs/math/0609045

[14] Kahn, J., Lyubich, M.: *A priori bounds for some infinitely renormalizable quadratics. II. Decorations.* Ann. Sci. Ec. Norm. Super. (4) 41 (2008), no. 1, 57-84.

[15] Kahn, J., Lyubich, M.: *A priori bounds for some infinitely renormalizable quadratics. III. Molecules.* Complex dynamics, 229-254, A K Peters, Wellesley, MA, 2009.

[16] Kiwi, J.: *Rational laminations of complex polynomials.* Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998), 111-154, Contemp. Math., 269, Amer. Math. Soc., Providence, RI, 2001.

[17] Levin, G., Van Strien, S.: *Local connectivity of the Julia set of real polynomials.* Ann. Math.147, 471-541 (1998)

[18] Levin, G.: *Multipliers of periodic orbits of quadratic polynomials and the parameter plane.* Israel Journal of Mathematics 170 (2009), 285-315.

[19] Levin, G.: *Rigidity and non-local connectivity of Julia sets of some quadratic polynomials.* Comm. Math. Phys. 304 (2011), 2, 295-328.

[20] Levin, G.: *Addendum to: Rigidity and non-local connectivity of Julia sets of some quadratic polynomials.* Comm. Math. Phys. 325 (2014), 3, 1171-1178.

[21] Levin, G.: *On backward stability of holomorphic dynamical systems.* Fund. Math. 158 (1998), no. 2, 97-107.

[22] Levin, G.: *Rays to renormalizations.* Bull. Polish Acad. Sci. Math., 68 (2020), no. 2.

[23] Levin, G., Przytycki, F., Shen, W.: *The Lyapunov exponent of holomorphic maps.* Invent. Math. 205 (2016), 363-382.

[24] Lyubich M, Yampolsky M.: *Dynamics of quadratic polynomials: complex bounds for real maps.* Ann. Inst. Fourier (Grenoble) 47 (1997), 1219-1255.

[25] Mane, M.: *On a theorem of Fatou.* Boletim da Sociedade Brasileira de Matemática, 24 (1993), 1–11.

[26] Milnor, J.: *Local connectivity of Julia sets: expository lectures.* The Mandelbrot set, theme and variations, 67-116, London Math. Soc. Lecture Note Ser., 274, Cambridge Univ. Press, Cambridge, 2000.

[27] Milnor, J.: *Periodic orbits, external rays and the Mandelbrot set: an expository account.* "Geometrie Complex et Systemes Dynamiques". Colloque en l'honnoeur d'Adrien Douady (Orsay, 1995). Asterisque 261, 277-331 (2000).

[28] McMullen, C: *Complex Dynamics and Renormalization.* Annals of Mathematical Studies, 135. Princeton Univ. Press, Princeton, New Jersey, 1994.

[29] Przytycki, F.: *Accessibility of typical points for invariant measures of positive Lyapunov exponents for iterations of holomorphic maps.* Fund. Math. 144 (1994), no.3, 259-278

[30] Przytycki, F., Urbański, M.: *Conformal Fractals: Ergodic Theory Methods.* Cambridge, 2010

[31] Rees, M.: *One hundred years of complex dynamics.* Proc. R. Soc. A 472 (2016)

[32] Sorensen, D.E.K.: *Infinitely renormalizable quadratic polynomials with non-locally connected Julia set.* J. Geom. Anal. 10 (2000), 169-208.

[33] Sullivan, D.: *Bounds, quadratic differentials and renormalization conjectures.* In: Mathematics into the Twenty-First Century, AMS Centennial Publications 2, Providence, RI: Amer. Math. Soc., 1991

[34] Thurston, W.: *On the combinatorics of iterated rational maps.* Princeton, NJ: Princeton University and IAS, 1985.

[35] Yoccoz, J.-Ch.: *Recent developments in dynamics.* Fields Medallists' Lecture, Proc. Int. Congr. Math., 1994.

Institute of Mathematics, The Hebrew University of Jerusalem, Givat Ram, Jerusalem, 91904, Israel

Email address: `levin@math.huji.ac.il`

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich St., 8, 00-956 Warsaw, Poland

Email address: `feliksp@impan.pl`