# LYAPUNOV SPECTRUM FOR RATIONAL MAPS

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Abstract. We study the dimension spectrum of Lyapunov exponents for rational maps on the Riemann sphere.

Contents



## 1. Introduction

Let $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function of degree $d \geq 2$ on the Riemann sphere and let $J = J(f)$ be its Julia set. Our goal is to study the spectrum of Lyapunov exponents of $f|_J$. Given $x \in J$ we denote by $\underline{\chi}(x)$ and $\overline{\chi}(x)$

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the *lower* and *upper Lyapunov exponent* at $x$, respectively, where

$$\underline{\chi}(x) \stackrel{\text{def}}{=} \liminf_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|, \quad \overline{\chi}(x) \stackrel{\text{def}}{=} \limsup_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|.$$

If both values coincide then we call the common value the *Lyapunov exponent* at $x$ and denote it by $\chi(x)$. For given numbers $0 \le \alpha \le \beta$ we consider also the following level sets

$$\mathcal{L}(\alpha,\beta) \stackrel{\text{def}}{=} \left\{x \in J \colon \underline{\chi}(x) = \alpha,\, \overline{\chi}(x) = \beta\right\}.$$

We denote by $\mathcal{L}(\alpha) \stackrel{\text{def}}{=} \mathcal{L}(\alpha,\alpha)$ the set of *Lyapunov regular points* with exponent $\alpha$. If $\alpha < \beta$ then $\mathcal{L}(\alpha,\beta)$ is contained in the set of so-called *irregular points*

$$\mathcal{L}_{\rm irr} \stackrel{\text{def}}{=} \left\{x \in J \colon \underline{\chi}(x) < \overline{\chi}(x)\right\}.$$

Recall that it follows from the Birkhoff ergodic theorem that $\mu(\mathcal{L}_{\rm irr}) = 0$ for any $f$-invariant probability measure $\mu$.

While the first results on the multifractal formalism go already back to Besicovitch [4], its systematic study has been initiated by work of Collet, Lebowitz and Porzio [5]. The case of spectra of Lyapunov exponents for conformal uniformly expanding repellers has been covered for the first time in [2] building also on work by Weiss [26] (see [13] for more details and references). To our best knowledge, the first results on irregular parts of a spectrum were obtained in [4]. Its first complete description (for digit expansions) was given by Barreira, Saussol, and Schmeling [3].

In this work we will formulate our results on the spectrum of Lyapunov exponents in terms of the topological pressure $P$. For any $\alpha > 0$ let us first define

$$F(\alpha) \stackrel{\text{def}}{=} \frac{1}{\alpha} \inf_{d\in\mathbb{R}} \left(P_{f|J}(-d\log|f'|) + d\alpha\right).$$

and

$$F(0) \stackrel{\text{def}}{=} \lim_{\alpha\to 0+} F(\alpha).$$

Let $[\alpha^-,\alpha^+]$ be the interval on which $F \ne -\infty$ (a more formal and equivalent definition is given in (4) below).

Before stating our main results, we recall what is already known in the case that $J$ is a uniformly expanding repeller with respect to $f$, that is, $J$ is a compact $f$-forward invariant (i.e., $f(J) = J$) isolated set such that $f|_J$ is uniformly expanding. Recall that $f$ is said to be *uniformly expanding* or *uniformly hyperbolic* on a set $\Lambda$ if there exist $c > 0$ and $\lambda > 1$ such that for every $n \ge 1$ and every $x \in \Lambda$ we have $|(f^n)'| \ge c\lambda^n$. Recall that a set $\Lambda$ is said to be *isolated* if there exists an open neighborhood $U \subset \overline{\mathbb{C}}$ of $\Lambda$ such that $f^n(x) \in U$ for every $n \ge 0$ implies $x \in \Lambda$. In our setting the Julia set $J$ is a uniformly expanding repeller if it does not contain any critical point nor parabolic point. Here a point $x$ is said to be *critical* if $f'(x) = 0$ and to be *parabolic* if $x$ is periodic and its multiplier $(f^{\mathrm{per}(x)})'(x)$ is a root of 1. If $J$ is a uniformly expanding repeller then for every $\alpha \in [\alpha^-, \alpha^+]$ we have

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha) = F(\alpha)$$

(see [2, 26]) and $\mathcal{L}(\alpha) = \varnothing$ if and only if $\alpha \notin [\alpha^-, \alpha^+]$ [23]. This gives the full description of the regular part of the Lyapunov spectrum. Moreover, in this setting of a uniformly expanding repeller $J$, the interval $[\alpha^-, \alpha^+]$ coincides with the closure of the range of the function $\alpha(d) = -\frac{d}{ds} P_{f|J}(-s \log |f'|)|_{s=d}$ and the spectrum can be written as

$$F(\alpha(d)) = \frac{1}{\alpha(d)} \big(P_{f|J}(-d \log |f'|) + d\alpha(d)\big) = \frac{h_{\mu_d}(f)}{\alpha(d)},$$

where $\alpha(d)$ is the unique number satisfying

$$-\frac{d}{ds} P_{f|J}(-s \log |f'|)|_{s=d} = \int \log |f'| \, d\mu_d = \alpha(d)$$

and where $\mu_d$ is the unique equilibrium state corresponding to the potential $-d \log |f'|$. If $\log |f'|$ is not cohomologous to a constant then we have $\alpha^- < \alpha^+$, and $\alpha \mapsto \alpha F(\alpha)$ and $d \mapsto P_{f|J}(-d \log |f'|)$ are real analytic strictly convex functions that form a Legendre pair.

We now state our first main result.

**Theorem 1.** *Let $f$ be a rational function of degree $\geq 2$ with no critical points in its Julia set $J$. For any $\alpha^- \leq \alpha \leq \beta \leq \alpha^+$, $\beta > 0$, we have*

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) = \min\{F(\alpha), F(\beta)\}\,.$$

*In particular, for any $\alpha \in [\alpha^-, \alpha^+] \setminus \{0\}$ we have*

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha) = F(\alpha)\,.$$

*If there exists a parabolic point in $J$ (and hence $F(0) > -\infty$) then*

$$\dim_{\mathrm{H}} \mathcal{L}(0) = \dim_{\mathrm{H}} J = F(0)\,.$$

*Moreover,*

$$\big\{x \in J \colon \underline{\chi}(x) < \alpha^-\big\} = \big\{x \in J \colon \overline{\chi}(x) > \alpha^+\big\} = \varnothing.$$

We denote by Crit the set of all critical points of $f$. Following Makarov and Smirnov [10, Section 1.3], we will say that $f$ is *exceptional* if there exists a finite, nonempty set $\Sigma_f \subset \overline{\mathbb{C}}$ such that

$$f^{-1}(\Sigma_f) \setminus \mathrm{Crit} = \Sigma_f\,.$$

This set need not be unique. We will further denote by $\Sigma$ the largest of such sets (notice that it has no more than 4 points).

**Theorem 2.** *Let $f$ be a rational function of degree $\geq 2$. Assume that $f$ is non-exceptional or that $f$ is exceptional but $\Sigma \cap J = \varnothing$. For any $0 < \alpha \leq \beta \leq \alpha^+$ we have*

$$\min\{F(\alpha), F(\beta)\} \leq \dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) \leq \max_{\alpha \leq q \leq \beta} F(q)\,.$$

*In particular, for any* $\alpha \in [\alpha^-, \alpha^+] \setminus \{0\}$ *we have*

$$\dim_{\rm H} \mathcal{L}(\alpha) = F(\alpha)$$

*and*

$$\dim_{\rm H} \mathcal{L}(0) \geq F(0)\,.$$

*Moreover,*

$$\big\{x \in J\colon -\infty < \chi(x) < \alpha^-\big\} = \big\{x \in J\colon \overline{\chi}(x) > \alpha^+\big\} = \varnothing$$

*and*

$$\dim_{\rm H} \big\{x \in J\colon 0 < \overline{\chi}(x) < \alpha^-\big\} = 0.$$

If $f$ is exceptional and $\Sigma \cap J \neq \varnothing$ (this happens, for example, for Chebyshev polynomials and some Lattès maps) then the situation can be much different from the above-mentioned cases. For example, the map $f(x) = x^2 - 2$ possesses countably many points with Lyapunov exponent $-\infty$, two points with Lyapunov exponent $2\log 2$, a set of dimension 1 of points with Lyapunov exponent $\log 2$, and no other Lyapunov regular points. Hence, for this map the Lyapunov spectrum is not complete in the interval $[\alpha^-, \alpha^+] = [\log 2, 2\log 2]$.

The present paper does not provide a complete description of the irregular part of the Lyapunov spectrum even in the case $\Sigma \cap J = \varnothing$. We do not know how big the set $\mathcal{L}(-\infty)$ is except in the case when $f$ has only one critical point in $J$ (in which case $\mathcal{L}(-\infty)$ consists only of the backward orbit of this critical point). Moreover, we do not know whether the set $\{x \in J\colon \underline{\chi}(x) < \alpha^-\}$ contains any points other than the backward orbits of critical points contained in $J$ and we only have some estimation for the Hausdorff dimension of the set $\mathcal{L}(\alpha, \beta)$ even for values $\alpha$, $\beta \in [\alpha^-, \alpha^+]$.

The paper is organized as follows. In Section 2 we introduce the tools we are going to use in this paper. In particular, we construct a family of uniformly expanding Cantor repellers with pressures pointwise converging to the pressure on $J$ (Proposition 1). Section 3 discusses general properties of the spectrum of exponents. In Section 4 we obtain upper bounds for the Hausdorff dimension. Here we use conformal measures to deal with conical points (Proposition 2) and we prove that the set of non-conical points with positive upper Lyapunov exponent is very small using the pullback construction (Proposition 3). In Section 5 we derive lower bounds for the dimension. To do so, we first consider the interior of the spectrum and we will use the sequence of Cantor repellers from Section 2 to obtain for any $\alpha \in (\alpha^-, \alpha^+)$ a big uniformly expanding subset of points with Lyapunov exponent $\alpha$ from which we derive our estimates. We finally study the boundary of the spectrum and the irregular part of the spectrum using a construction, that generalizes the w-measure construction from [8].

## 2. Tools for non-uniformly hyperbolic dynamical systems

2.1. **Topological pressure.** Given a compact $f$-invariant set $\Lambda \subset J$, we denote by $\mathcal{M}(\Lambda)$ the family of $f$-invariant Borel probability measures supported on $\Lambda$. We denote by $\mathcal{M}_{\rm E}(\Lambda)$ the subset of ergodic measures. Given $\mu \in \mathcal{M}_{\rm E}(\Lambda)$, we denote by

$$\chi(\mu) \stackrel{\rm def}{=} \int_\Lambda \log|f'|\,d\mu$$

the Lyapunov exponent of $\mu$. Notice that we have $\chi(\mu) \geq 0$ for any $\mu \in \mathcal{M}(J)$ [17].

Given $d \in \mathbb{R}$, we define the function $\varphi_d \colon J \setminus \mathrm{Crit} \to \mathbb{R}$ by

$$\varphi_d(x) \stackrel{\rm def}{=} -d \log|f'(x)|. \tag{1}$$

Given a compact $f$-invariant uniformly expanding set $\Lambda \subset J$, the *topological pressure* of $\varphi_d$ (with respect to $f|_\Lambda$) is defined by

$$P_{f|\Lambda}(\varphi_d) \stackrel{\rm def}{=} \max_{\mu\in\mathcal{M}(\Lambda)} \left( h_\mu(f) + \int_\Lambda \varphi_d\,d\mu \right), \tag{2}$$

where $h_\mu(f)$ denotes the entropy of $f$ with respect to $\mu$. We simply write $\mathcal{M} = \mathcal{M}(J)$ and $P(\varphi_d) = P_{f|J}(\varphi_d)$ if we consider the full Julia set $J$ and if there is no confusion about the system. A measure $\mu \in \mathcal{M}$ is called *equilibrium state* for the potential $\varphi_d$ (with respect to $f|_J$) if

$$P(\varphi_d) = h_\mu(f) + \int_J \varphi_d\,d\mu.$$

For every $d \in \mathbb{R}$ we have the following equivalent characterizations of the pressure function (see [21], where further equivalences are shown). We have

$$\begin{aligned} P(\varphi_d) &= \sup_{\mu\in\mathcal{M}_{\rm E}^+} \left( h_\mu(f) + \int_J \varphi_d\,d\mu \right) \\ &= \sup_\Lambda P_{f|\Lambda}(\varphi_d). \end{aligned} \tag{3}$$

Here in the first equality the supremum is taken over the set $\mathcal{M}_{\rm E}^+$ of all ergodic $f$-invariant Borel probability measures on $J$ that have a positive Lyapunov exponent and are supported on some $f$-invariant uniformly expanding subset of $J$. In the second equality the supremum is taken over all uniformly expanding repellers $\Lambda \subset J$. In fact, in the second equality it suffices to take the supremum over all uniformly expanding Cantor repellers, that is, uniformly expanding repellers that are limit sets of finite graph directed systems satisfying the strong separation condition with respect to $f$, see Section 2.2.

Let us introduce some further notation. Let

$$\begin{aligned} \alpha^- &\stackrel{\text{def}}{=} \lim_{d\to\infty} -\frac{1}{d}P(\varphi_d) = \inf_{\mu\in\mathcal{M}} \chi(\mu), \\ \alpha^+ &\stackrel{\text{def}}{=} \lim_{d\to-\infty} -\frac{1}{d}P(\varphi_d) = \sup_{\mu\in\mathcal{M}} \chi(\mu), \end{aligned} \tag{4}$$

where the given characterizations follow easily from the variational principle.

Recall that, given $\alpha > 0$, we define

$$F(\alpha) \stackrel{\text{def}}{=} \frac{1}{\alpha} \inf_{d\in\mathbb{R}} \left(P(\varphi_d) + \alpha d\right) \tag{5}$$

and

$$F(0) \stackrel{\text{def}}{=} \lim_{\alpha\to 0+} F(\alpha). \tag{6}$$

Note that

$$F(0) = d_0 \stackrel{\text{def}}{=} \inf\{d \colon P(\varphi_d) = 0\}.$$

2.2. **Building bridges between unstable islands.** We describe a construction of connecting two given hyperbolic subsets of the Julia set by "building bridges" between the sets.[1]

We call a point $x \subset J$ *non-immediately post-critical* if there exists some preimage branch $x_0 = x = f(x_1)$, $x_1 = f(x_2)$, $\ldots$ that is dense in $J$ and disjoint from Crit. If $f$ is non-exceptional or if it is exceptional but $\Sigma \cap J = \varnothing$ then for every hyperbolic set all except possibly finitely many points (in particular, all periodic points) are non-immediately post-critical.

We will now consider a set $\Lambda$ that is an $f$-uniformly expanding Cantor repeller (ECR for short), that is, a uniformly expanding repeller and a limit set of a finite graph directed system (GDS) satisfying the strong separation condition (SSC) with respect to $f$. Recall that by a GDS satisfying the SSC with respect to $f$ we mean a family of domains and maps satisfying the following conditions (compare [12, pp. 3, 58]):

(i) There exists a finite family $\mathcal{U} = \{U_k \colon k = 1, \ldots, K\}$ of open connected (not necessarily simply connected) domains in the Riemann sphere with pairwise disjoint closures.

(ii) There exists a family $G = \{g_{k\ell} \colon k, \ell \in \{1, \ldots, K\}\}$ of branches of $f^{-1}$ mapping $\overline{U_\ell}$ into $U_k$ with bounded distortion (not all pairs $k, \ell$ must appear here).
Note that a general definition of GDS allows many maps $g$ from each $\overline{U_\ell}$ to each $U_k$. Here however there can be at most one, since we assume that $f$-critical points are far away from $\Lambda$ and that the maps $g$ are branches of $f^{-1}$.

[1]This is a precise realization of an idea of Prado [15].

(iii) We have

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{k_1,\ldots,k_n} g_{k_1k_2} \circ g_{k_2k_3} \circ \ldots \circ g_{k_{n-1}k_n}.$$

We assume that we have $f(\Lambda) = \Lambda$ and hence that for each $k$ there exists $\ell$ and for each $\ell$ there exists $k$ such that $g_{k\ell} \in G$.

We can view $k = 1, \ldots, K$ as vertices and $g_{k\ell}$ as edges from $\ell$ to $k$ of a directed graph $\Gamma = \Gamma(\mathcal{U}, G)$.

This definition easily implies that $f$ is uniformly expanding on the limit set $\Lambda$ of such a GDS, and that $\Lambda$ is a repeller for $f$. Clearly $\Lambda$ is a Cantor set. In fact a sort of converse is true (though we shall not use this fact in this paper, but it clears up the definitions). Namely we observe the following fact.

**Lemma 1.** *If $\Lambda \subset J$ is an $f$-invariant compact uniformly expanding set that is a Cantor set, then $\Lambda$ is contained in the limit set of a GDS satisfying the SSC (this limit set can be chosen to be contained in an arbitrarily small neighborhood of $\Lambda$). Hence, $\Lambda$ is contained in an $f$-ECR set.*

*Proof.* We can multiply the standard sphere Riemann metric by a positive smooth function such that with respect to this new metric $\rho_\Lambda$ we have $|f'| \geq \lambda > 1$ on $\Lambda$. It is easy to show that one can find an arbitrarily small number $r > 0$ such that the neighborhood $B(\Lambda, r) = \{z \in \overline{\mathbb{C}} \colon \rho_\Lambda(z, \Lambda) < r\}$ consists of a finite number of connected open domains $U_k$ with pairwise disjoint closures. We account for our GDS the branches of $g = f^{-1}$ on the sets $U_k$ such that each $g(U_k)$ intersects $\Lambda$. Then $g$ maps each $\overline{U_k}$ into some $U_\ell$ because it is a contraction (by the factor $\lambda^{-1}$). Hence the family of maps $g|_{U_k}$ satisfies the assumptions of a GDS with the SSC. □

In the proof of the following lemma we will "build bridges" between two ECR's.

**Lemma 2.** *For any two disjoint $f$-ECR sets $\Lambda_1$, $\Lambda_2 \subset J$ that both contain non-immediately postcritical points there exists an $f$-ECR set $\Lambda \subset J$ containing the set $\Lambda_1 \cup \Lambda_2$. If $f$ is topologically transitive on each $\Lambda_i$, $i = 1, 2$, then $f$ is topologically transitive on $\Lambda$.*

*Proof.* Let $\Lambda_1$, $\Lambda_2 \subset J$ be two sets satisfying the assumptions of the lemma and let $p_1 \in \Lambda_1$ and $p_2 \in \Lambda_2$ be two non-immediately postcritical points. Consider a family $\mathcal{U}_i = \{U_{i,k}\}_{k=1}^{K_i}$ of open connected domains and a family $G_i = \{g_{i,k\ell}\}$ of branches of $f^{-1}$ mapping $U_{i,\ell}$ to $U_{i,k}$ that define the DGS's satisfying the SSC that have $\Lambda_i$ as their limit sets, for $i = 1, 2$, respectively. Let

$$D_i \stackrel{\text{def}}{=} \bigcup_{k=1}^{K_i} U_{i,k}.$$

We can assume that each $D_i$ is an arbitrarily small neighborhood of $\Lambda_i$, by replacing $\mathcal{U}_i$ by $G^m(\mathcal{U}_i)$, where by $G^m$ we denote the family of all compositions

$$G^m = \left\{g_{i,k_1k_2} \circ g_{i,k_2k_3} \circ \ldots \circ g_{i,k_{m-1}k_m} \colon g_{i,k_nk_{n+1}} \in G_i, n = 1, \ldots, m-1\right\}.$$

For each $i = 1, 2$ let us choose a backward trajectory $y_{i,t}$ of the point $p_i$ (the "bridge") such that

$$y_{i,0} = p_i, \quad f(y_{i,t}) = y_{i,t-1} \text{ for } t = 1, ..., t_i,$$

$y_{i,t} \notin \overline{D_1} \cup \overline{D_2}$ for all $t = 1, \ldots, t_i - 1$ and $y_{1,t_1} \in D_2$, $y_{2,t_2} \in D_1$. Let us denote by $h_{i,t}$ the branch of $f^{-t}$ that maps $p_i$ to $y_{i,t}$, that is, let

$$h_{i,t} \stackrel{\text{def}}{=} f^{-t}_{y_{i,t}}.$$

Let $V_i$ be an open disc centered at $p_i$ that is contained in $D_i$ and satisfies $h_{i,t}(\overline{V_i}) \cap (\overline{D_1} \cup \overline{D_2}) = \varnothing$ for all $t = 1, \ldots, t_i - 1$ (note that this is possible provided we choose $V_i$ small enough) and

$$h_{1,t_1}(\overline{V_1}) \subset D_2 \text{ and } h_{2,t_2}(\overline{V_2}) \subset D_1.$$

Let us consider an integer $N \geq 0$ such that the component of $f^{-N}(D_i)$ containing $p_i$ is contained in $V_i$, that is, that

$$f^{-N}_{p_i}(D_i) \subset V_i$$

for $i = 1, 2$. Now let us replace $\mathcal{U}_i$ by $\widehat{\mathcal{U}}_i \stackrel{\text{def}}{=} G^N(\mathcal{U}_i)$, let us replace each map $g_{i,k\ell}$ by the family of its restrictions to $\widehat{U} \in \widehat{\mathcal{U}}_i$ contained in $U_\ell$ and and let us denote by $\widehat{G}_i$ the union of those families. This defines a GDS with graph $\Gamma_i = \Gamma_i(\widehat{\mathcal{U}}_i, \widehat{G}_i)$, for $i = 1, 2$. Now we restrict each bridge $h_{i,t}$ to the element $\widehat{V}_i$ of $\widehat{\mathcal{U}}_i$ that contains $p_i$. As the next step we consider

$$\widehat{V}_{i,t} \stackrel{\text{def}}{=} h_{i,t}(\widehat{V}_i) \quad \text{ for } t = 1, ..., t_i + N - 1,$$

where for $t > t_i$ we choose an arbitrary prolongation of the bridge $y_{i,t}$ by maps $g_{j,k_tk_{t+1}}$. Finally, we consecutively thicken slightly $\widehat{V}_{i,t}$ along the bridges such that $f(\widehat{V}_{i,t}) \supset \overline{\widehat{V}_{i,t-1}}$. For each $t$ let us denote by $g_{i,t}$ the branch of $f^{-1}$ from $\widehat{V}_{i,t}$ to $\widehat{V}_{i,t+1}$ for $t = 0$, ..., $t_i + N - 1$. Let us denote by $H_i$ the family of all these branches. By construction the family of domains

$$\mathcal{U} \stackrel{\text{def}}{=} \widehat{\mathcal{U}}_1 \cup \widehat{\mathcal{U}}_2 \cup \bigcup_{i=1,2} \bigcup_{t=1,\ldots,t_i+N-1} \widehat{V}_{i,t}$$

and the family of maps $G \stackrel{\text{def}}{=} \widehat{G}_1 \cup \widehat{G}_2 \cup H_1 \cup H_2$ form our desired GDS with a graph $\Gamma = \Gamma(\mathcal{U}, G)$ satisfying the SSC and hence defines an $f$-ECR set $\Lambda \subset J$ that containes $\Lambda_1 \cup \Lambda_2$.

Finally notice that this system has topologically transitive limit set $\Lambda$ since its transition graph $\Gamma$ is transitive. This follows from the assumption that due to topological transitivity of $f|_{\Lambda_i}$ the graphs $\Gamma_i$ are transitive and from the construction of the bridges. □

2.3. **Hyperbolic subsystems and approximation of pressure.** Our approach is to "exhaust" the Julia set $J$ by some family of subsets $\Lambda_m \subset J$ and to show that the corresponding pressure functions converge towards the pressure of $f|_J$. In particular, in order to be able to conclude convergence of associated spectral quantities, it is crucial that each such $\Lambda_m$ is an invariant uniformly expanding and topologically transitive set.

We start by stating a classical result from Pesin-Katok theory. In follows for example from [22, Theorems 10.6.1 and 11.2.3]. Recall that an iterated function system (IFS) is a GDS that is given by a complete graph.

**Lemma 3.** *For every ergodic $f$-invariant measure $\mu$ that is supported on $J$ and has a positive Lyapunov exponent, for every continuous function $\phi\colon J \to \mathbb{R}$ and for every $\varepsilon > 0$, there exist an integer $n > 0$ and an $f^n$-ECR set $\Lambda \subset J$ that is topologically transitive and a limit set of an IFS, such that $\dim_{\rm H} \Lambda \geq \dim_{\rm H} \mu - \varepsilon$*

$$P_{f^n|\Lambda}(S_n\phi) \geq h_\mu(f^n) + n\int \phi\, d\mu - n\varepsilon, \tag{7}$$

*where we use the notation $S_n\phi(x) \stackrel{\rm def}{=} \phi(x) + \phi(f(x)) + \ldots + \phi(f^{n-1}(x))$, and in particular*

$$P_{f|\bigcup_{k=0}^{n-1} f^k(\Lambda)}(\phi) \geq h_\mu(f) + \int \phi\, d\mu - \varepsilon. \tag{8}$$

Our aim is to apply Lemma 3 to potentials $\phi = \varphi_d$ and to use the resulting ECR sets to construct a sequence of $f^{a_m}$-ECR sets $\Lambda_m$ on which the pressure function $\frac{1}{a_m} P_{f^{a_m}|\Lambda_m}(S_{a_m}\varphi_d)$ converges pointwise to $P(\varphi_d)$. If the ECR sets generated by Lemma 3 are pairwise disjoint, we can simply build bridges between such sets applying Lemma 2. In the general situation we start with the following lemma.

**Lemma 4.** *Let $\Lambda$ be a topologically transitive $f^n$-ECR set. Let $\phi_i\colon \Lambda \to \mathbb{R}$ be a finite number of continuous functions. Then for any open disc $D$ intersecting $\Lambda$ and for any $\varepsilon > 0$ one can find a set $\Lambda' \subset D \cap \Lambda$ and a natural number $\ell > 0$ such that $\Lambda'$ is an $f^\ell$-ECR set and for every $\phi_i$ satisfies*

$$\frac{1}{\ell} P_{f^\ell|\Lambda'}(S_\ell \phi_i) \geq \frac{1}{n} P_{f^n|\Lambda}(S_n\phi_i) - \varepsilon.$$

*Proof.* Consider in $\Lambda$ a clopen set $C$ contained in $D$. Let $x \in C$. Let us choose $N$ large enough such that in the case that $\ell \geq N$ and $f_x^{-\ell}(C) \cap C \neq \varnothing$ the pullback satisfies $f_x^{-\ell}(C) \subset C$. Note that it is enough to take

$$N > \frac{\log \operatorname{diam} \Lambda}{\rho_\Lambda(C, \Lambda \setminus C) \log \lambda},$$

where $\lambda$ is the expanding constant on $\Lambda$ in the metric $\rho_\Lambda$ defined as in the proof of Lemma 1. Note that the topological transitivity of $f^n|_\Lambda$ implies that every pullback of $C$ can be continued by a bounded number of consecutive pullbacks until it hits $C$, say this number is bounded by a constant $N'$. This way we obtain an IFS for $f^\ell$, where $N \le \ell \le N+N'$, with its limit set $\Lambda'$ contained in $C$.

Recall the equivalent definition of tree pressure established in [21, Theorems A, A.4]. Due to topological transitivity in the definition of pressure we can consider separated sets that are contained in the set of preimages $f^{-N}(x)$. Therefore, for any $\phi_i$ the pressures with respect to $f|_{\Lambda'}$ and with respect to $f|_\Lambda$ differ by at most $O(\frac{N'}{N})$ from each other. As $N$ can be chosen arbitrarily big, this proves the lemma. □

The following approximation results are fundamental for our approach.

**Proposition 1.** *Assume that $f$ is non-exceptional or that $f$ is exceptional but $\Sigma \cap J = \varnothing$. Then there exists a sequence $\{a_m\}_m$ of positive integers and a sequence $\Lambda_m \subset J$ of $f^{a_m}$-invariant uniformly expanding topologically transitive sets such that for every $d \in \mathbb{R}$, we have*

$$P(\varphi_d) = \lim_{m\to\infty} \frac{1}{a_m} P_{f^{a_m}|\Lambda_m}(S_{a_m}\varphi_d) = \sup_{m\ge1} \frac{1}{a_m} P_{f^{a_m}|\Lambda_m}(S_{a_m}\varphi_d). \tag{9}$$

*For every $\alpha \in (\alpha^-, \alpha^+)$ we have*

$$F(\alpha) = \lim_{m\to\infty} F_m(\alpha) = \sup_{m\ge1} F_m(\alpha) \tag{10}$$

*and*

$$\lim_{m\to\infty} \alpha_m^- = \inf_{m\ge1} \alpha_m^- = \alpha^-, \quad \lim_{m\to\infty} \alpha_m^+ = \sup_{m\ge1} \alpha_m^+ = \alpha^+, \tag{11}$$

*where $F_m$ and $\alpha_m^\pm$ are defined as in* (5) *and* (4) *but with $\frac{1}{a_m}P_{f^{a_m}|\Lambda_m}(S_{a_m}\phi)$ instead of $P_{f|J}(\phi)$.*

*Proof.* To prove (9) it is enough to construct an $f^{a_m}$-ECR set $\Lambda_m \subset J$ such that

$$\frac{1}{a_m} P_{f^{a_m}|\Lambda_m}(S_{a_m}\varphi_d) \ge P(\varphi_d) - \frac{1}{m} \tag{12}$$

for all $d \in [-m, m]$. Recall that we have (3). As $d \mapsto P(\varphi_d)$ is uniformly Lipschitz continuous, we only need to check (12) for a finite number of potentials $\phi_i = \varphi_{d_i}$. Given $m$, we apply Lemma 3 to potentials $\phi_i$, obtaining a family of $f^{n_i}$-ECR sets $\Lambda_{m,i}$ on which the pressure $\frac{1}{n_i}P_{f^{n_i}|\Lambda_{m,i}}(S_{n_i}\phi_i) \ge P(\phi_i) - \frac{1}{2m}$. We then apply Lemma 4 to construct a family of pairwise disjoint $f^{\ell_i}$-ECR sets $\Lambda'_{m,i}$ that satisfy $\frac{1}{\ell_i}P_{f^{\ell_i}|\Lambda'_{m,i}}(S_{\ell_i}\phi_i) \ge P(\phi_i) - \frac{1}{m}$. Those sets $\Lambda'_{m,i}$ are also disjoint $f^{a_m}$-ECR sets for $a_m = \prod_i \ell_i$ and by our assumption contain non-immediately post-critical points. Hence we can consecutively apply Lemma 2 to them. We obtain an $f^{a_m}$-ECR set satisfying (12) for all $\phi_i$. This proves (9).

As $d \mapsto P(\varphi_d)$ and $\alpha \mapsto \alpha F(\alpha)$ form a Legendre pair, (10) and (11) follow from (9) by a result of Wijsman [27]. □

**Remark 1.** It is enough for us to work with hyperbolic sets for some iterations of $f$ instead of hyperbolic sets for $f$ (in other words, to use (7) instead of (8)). However, notice that we could instead also extend the set $\bigcup_{i=0}^{a_m-1} f^i(\Lambda_m)$ to an $f$-ECR set using Lemma 1.

2.4. **Conformal measures.** The dynamical properties of any measure $\nu$ with respect to $f|_J$ are captured through its Jacobian. The *Jacobian* of $\nu$ with respect to $f|_J$ is the (essentially) unique function $\mathrm{Jac}_\nu f$ determined through

$$\nu(f(A)) = \int_A \mathrm{Jac}_\nu f \, d\nu \tag{13}$$

for every Borel subset $A$ of $J$ such that $f|_A$ is injective. In particular, its existence yields the absolute continuity $\nu \circ T \prec \nu$.

A probability measure $\nu$ that satisfies

$$\mathrm{Jac}_\nu f = e^{P(\varphi_d)-\varphi_d}$$

is called $e^{P(\varphi_d)-\varphi_d}$-*conformal measure*. If $d \geq 0$ then one can always find a $e^{P(\varphi_d)-\varphi_d}$-conformal measure $\nu_d$ that is positive on each open set intersecting $J$, see [21]. When $d < 0$ such a measure can always be found if $f$ is not an exceptional map or if $f$ is exceptional but $\Sigma \cap J = \varnothing$ (see [21, Appendix A.2]).

2.5. **Hyperbolic times and conical limit points.** When $f|_J$ is not uniformly expanding, we can still observe a slightly weaker form of non-uniform hyperbolicity. We recall two concepts that have been introduced.

A number $n \in \mathbb{N}$ is called a *hyperbolic time* for a point $x$ with exponent $\sigma$ if

$$|(f^k)'(f^{n-k}(x))| \geq e^{k\sigma} \text{ for every } 1 \leq k \leq n.$$

It is an immediate consequence of the Pliss lemma (see, for example, [14]) that for a given point $x \in J$, for any $\sigma < \overline{\chi}(x)$ there exist infinitely many hyperbolic times for $x$ with exponent $\sigma$.

We denote by

$$\mathrm{Dist}\, g|_Z \stackrel{\mathrm{def}}{=} \sup_{x,y \in Z} \frac{|g'(x)|}{|g'(y)|}$$

the maximal distortion of a map $g$ on a set $Z$. After [6], we will call a point $x \in J$ *conical* if there exist a number $r > 0$, a sequence of numbers $n_i \nearrow \infty$ and a sequence $\{U_i\}_i$ of neighborhoods of $x$ such that $f^{n_i}(U_i) \supset B(f^{n_i}(x), r)$ and that $\mathrm{Dist}\, f^{n_i}|_{U_i}$ is bounded uniformly in $i$.

## 3. On the completeness of the spectrum

In the following two lemmas we will investigate which numbers can occur at all as upper/lower Lyapunov exponents.

**Lemma 5.** *We have*

$$\{x \in J \colon \overline{\chi}(x) > \alpha^+\} = \varnothing.$$

*If $J$ does not contain any critical point of $f$ then we have*

$$\{x \in J \colon \underline{\chi}(x) < \alpha^-\} = \varnothing.$$

*Proof.* Consider an arbitrary $x \in J$ and a sequence $n_i \nearrow \infty$ such that

$$\lim_{i\to\infty} \frac{1}{n_i} \sum_{j=0}^{n_i-1} \log |f'(f^j(x))| = \overline{\chi}(x)$$

and

$$\mu_i \stackrel{\text{def}}{=} \frac{1}{n_i} \sum_{j=0}^{n_i-1} \delta_{f^j(x)} \to \mu$$

in the weak$*$ topology. The limit measure $\mu$ is $f$-invariant, [25, Theorem 6.9]. Define

$$g_N \stackrel{\text{def}}{=} \max\{\log |f'|, -N\}.$$

Notice that $\{g_N\}_N$ is a monotonically decreasing sequence of continuous functions that converge pointwise to $\log |f'|$. Hence we obtain

$$\overline{\chi}(x) \le \lim_{N\to\infty} \int g_N \, d\mu = \int \log |f'| \, d\mu = \chi(\mu) \le \alpha^+,$$

where the equality follows from the Lebesgue monotone convergence theorem. This proves the first statement.

The second statement follows simply from the fact that $\log |f'|$ is continuous on $J$ if $f$ has no critical points in $J$. □

**Remark 2.** We remark that the same method of proof gives a slightly stronger result than the fact that $\overline{\chi}(x) \le \alpha^+$ for every $x$. Namely we have

$$\lim_{n\to\infty} \sup_{z\in\mathbb{C}} \frac{1}{n} \log |(f^n)'(z)| \le \alpha^+,$$

see [19, Proposition 2.3. item 2].

Recall that on a set of total probability we have $\chi(x) \ge 0$ [17]. In fact, a better estimate can be given for any Lyapunov regular point $x$ (compare the proof of [20, Proposition 4.1]).

**Lemma 6.** *If $x \in J$ has a finite Lyapunov exponent $\chi(x)$ then $\chi(x) \ge \alpha^-$, that is, we have*

$$\{x \in J \colon -\infty < \chi(x) < \alpha^-\} = \varnothing.$$

*If there are no critical points in $J$ then $\mathcal{L}(-\infty)$ is empty. If there is only one critical point in $J$ then $\mathcal{L}(-\infty)$ consists only of this critical point and its preimages.*

*Proof.* Let $x \in J$ be a Lyapunov regular point with exponent $\chi(x)$ and assume that $\chi(x) < \alpha^-$. It is enough for us to prove that there exists a periodic point with Lyapunov exponent arbitrarily close to $\chi(x)$, the contradiction will then follow from the definition of $\alpha^-$.

Note first that if $\chi(x)$ exists and is finite then $\frac{1}{n}\log|(f^n)'(x)|$ must be a Cauchy sequence and hence satisfies

$$\liminf_{n\to\infty} \frac{1}{n} \log \rho(f^n(x), \mathrm{Crit}) = 0. \tag{14}$$

By [17, Corollary to Lemma 6], we then can conclude that $\chi(x) \geq 0$. Choose now a small number $\delta > 0$. Let $n$ be a hyperbolic time for $x$ with exponent $-\delta/2$ (recall that $x$ has infinitely many hyperbolic times with that exponent, and hence that $n$ can be chosen arbitrarily big). Because of (14), there exists an integer $n_0 > 0$ such that for all $k \geq n_0$ we have

$$\rho(f^k(x), \mathrm{Crit}) > \exp(-k\delta) \tag{15}$$

and we can assume that $n > n_0$.

We start with a construction that is standard in Pesin theory. For each $k = 0, \ldots, n - n_0 - 1$ we define

$$B_k \overset{\text{def}}{=} B\left(f^{n-k}(x), e^{(-(d+1)n+k)\delta}\right),$$

where $d$ is the greatest degree of a critical point of $f$. This way we define a sequence of balls that are centered at points of the backward trajectory of $f^n(x)$ and that have diameters shrinking slower than the derivative of $f^{-k}$ along this branch and at the same time have diameters much smaller than their distance from any critical point. As we have $f^{n-k}(x) \in B_k$, (15) implies that for $n$ big enough for any $k < n - n_0$ the set $f^{-1}_{f^{n-k-1}(x)}(B_k)$ does not contain any critical point and that

$$\log \mathrm{Dist}\, f^{-1}_{f^{n-k-1}(x)}|_{B_k} \leq K_1 \frac{\mathrm{diam}\, B_k}{\rho(f^{n-k}(x), \mathrm{Crit})}, \tag{16}$$

where $K_1$ is some constant that depends only on $f$.

**Claim:** If $n$ is sufficiently big then for any $k = 0, \ldots, n$ the map $f^k$ is univalent and has bounded distortion $\leq \exp(\delta/2)$ on the set $f^{-k}_{f^{n-k}(x)}(B_0)$ and satisfies $f^k(B_k) \supset B_0$.

To prove the above claim, note that the ball $B_{n-k}$ shrinks as $n$ increases. Hence, it is enough to prove the statement for $k < n - n_0$. The statement for the initial finitely many steps $k = n - n_0, \ldots, n$ is then automatically provided $n$ is big enough. Let us assume that $n$ is sufficiently big such that also

$$K_1(n - n_0)e^{-dn\delta} < \frac{\delta}{2}.$$

By construction of the family $\{B_k\}_k$ and by (15), for each $\ell \geq 1$ we have

$$\sum_{k=0}^{\ell-1} \frac{\operatorname{diam} B_k}{\rho(f^{n-k}(x), \operatorname{Crit})} < \sum_{k=0}^{\ell-1} \frac{e^{(-(d+1)n+k)\delta}}{e^{(-n+k)\delta}} = \ell e^{-dn\delta}.$$

Hence, if for $\ell \leq n - n_0$ we have $f^k(B_k) \supset B_0$ for every $k = 1, \ldots, \ell - 1$ then (16) implies

$$\log \operatorname{Dist} f^{-\ell}_{f^{n-\ell}(x)}|_{B_0} \leq K_1 \ell e^{-dn\delta} < \frac{\delta}{2}.$$

On the other hand, recall that $n$ is a hyperbolic time for $x$ with exponent $-\delta/2$. Hence, if

$$\log \operatorname{Dist} f^{-\ell}_{f^{n-\ell}(x)}|_{B_0} < \frac{\delta}{2}$$

then $f^\ell(B_\ell) \supset B_0$. The above claim now follows by induction over $\ell$.

Let us consider the set

$$E \stackrel{\text{def}}{=} \bigcap_{k\geq 1} \bigcup_{\ell > k} E_\ell, \quad \text{where } E_\ell \stackrel{\text{def}}{=} \bigcup_{j=1}^{\ell} B\big(f^j(\operatorname{Crit}), e^{-2d\ell\delta}\big).$$

Notice that $B_0 \setminus E_n$ is nonempty whenever the hyperbolic time $n$ is big enough. For such $n$, let $y \in B_0 \setminus E_n$. From our distortion estimations for $f_x^{-n}|_{B_0}$ in the Claim we obtain

$$\log \frac{|(f_x^{-n})'(f^n(x))|}{\big|(f_x^{-n})'(y)\big|} < \frac{\delta}{2}. \tag{17}$$

In the remaining proof we will follow closely techniques in [20, Section 3]. Let us fix some arbitrary $f$-invariant uniformly expanding set $\Lambda \subset J$ that has positive Hausdorff dimension (the existence of such a set follows for example from Lemma 3). As Crit is a finite set, we have $\dim_\Lambda E = 0$ and we can find a point $z \in \Lambda \setminus E$. In particular $z \in \Lambda \setminus E_n$ for $n$ large. Note that in particular for every $n$ large enough we have

$$\rho(z, f^n(\operatorname{Crit})) > e^{-2dn\delta}. \tag{18}$$

and hence on the disk $B(z, e^{-2dn\delta})$ any pull-back $f^{-n}$ is univalent.

By [18, Lemma 3.1], there exists a number $K_2 > 0$ depending only on $f$ and a sequence of disks $\{D_i\}_{i=1,\ldots,K}$ such that $\bigcup_{i=1}^K D_i$ is connected, $y$ is the center of disk $D_1$, $z$ is the center of disk $D_K$,

$$\rho\Big(D_i, \bigcup_{j=1}^{n} f^j(\operatorname{Crit})\Big) \geq \operatorname{diam} D_i,$$

and the number of disks is bounded by $K \leq K_2(n\delta)^{1/2}$. By the Koebe distortion lemma, for each branch of $f^{-n}$ and for every $D_i$ we have

$$\operatorname{Dist} f^{-n}|_{D_i} < K_3,$$

where $K_3 > 0$ is some constant. Hence, in particular

$$\left|\log|(f_w^{-n})'(z)| - \log|(f_x^{-n})'(y)|\right| \le K \log K_3$$

for some $w \in f^{-n}(z)$. Together with (17), this implies that

$$\begin{aligned}\frac{1}{n}\Big|\log|(f^n)'(x)| - \log|(f^n)'(w)|\Big| &\le \frac{1}{n}\left(\frac{\delta}{2} + K \log K_3\right)\\ &\le \frac{1}{n}\left(\frac{\delta}{2} + K_2(n\delta)^{1/2}\log K_3\right) < \frac{\delta}{2},\end{aligned}$$

whenever $n$ has been chosen large enough.

By hyperbolicity of $f|_\Lambda$, there exist positive constants $\Delta$ and $K_4$ such that for all $\ell \ge 0$ the map $f^{-\ell}$ is univalent and has bounded distortion $\le K_4$ on $B(f^\ell(z), \Delta)$. Recall that the Julia set $J$ has the property that there exists $m = m(\Delta) > 0$ such that the image $f^m(B(f^\ell(z), \Delta) \cap J)$ is equal to $J$.

Let $\ell$ be the smallest positive integer such that $|(f^\ell)'(z)| \ge K_4 \Delta e^{2d(n+m)\delta}$. Hence $f_z^{-\ell}(B(f^\ell(z), \Delta)) \subset B(z, e^{-2d(n+m)\delta})$ and it is easy to show that $\ell \le K_5 + K_6(n+m)\delta$ for some constants $K_5$, $K_6$. So in particular, one of the preimages $f^{-(n+m)}(z)$ is in $B(f^\ell(z), \Delta)$ and the corresponding pull-back $W \stackrel{\text{def}}{=} f^{-(n+m)}(B(z, e^{-2d(n+m)\delta}))$ satisfies $W \subset B(f^\ell(z), \Delta)$. It follows from (18) that $f^{-(n+m+\ell)} \colon W \to B(z, e^{-2(n+m)\delta})$ is univalent and hence that there is a repelling fixed point $p = f^{n+m+\ell}(p)$ in $B(z, e^{-(n+m+\ell)})$.

Using the above, the Lyapunov exponent of $p$ can be estimated by

$$\begin{aligned}\chi(p) &= \frac{1}{n+m+\ell}\log|(f^{n+m+\ell})'(p)|\\ &\le \frac{1}{n}\log\left|(f^n)'(w)\right| \frac{1}{1+(m+\ell)/n} + \frac{m+\ell}{n+m+\ell}\sup\log|f'|\\ &\le \left(\chi(x) + \frac{\delta}{2}\right)\frac{1}{1+(m+\ell)/n} + \frac{m+\ell}{n+m+\ell}\sup\log|f'|.\end{aligned}$$

Now recall that the hyperbolic time $n$ can be chosen arbitrarily large and that $\delta$ was chosen arbitrarily small. This proves the first claim of the lemma.

If there are no critical points in $J$ then $\log|f'|$ (and hence $\chi$) is bounded from below. The remaining claim of the lemma follows from [7, Lemmas 2.1 and 2.3]. □

## 4. Upper bounds for the dimension

In this section we will derive upper bounds for the Hausdorff dimension of the level sets. We first start with a particular case.

### 4.1. **No critical points in $J$.**

We study the particular case that there is no critical point in the Julia set $J$ (though, parabolic points in $J$ are allowed).

We start by taking a more general point of view and investigate those points that have some least degree of hyperbolicity. In terms of Lyapunov exponents, this concerns points $x$ with $\overline{\chi}(x) > 0$. We begin our analysis by presenting a simple lemma, that will be useful shortly after.

**Lemma 7.** *Let $\{a_n\}_n$ be a sequence of real numbers such that $\{a_{n+1}-a_n\}_n$ converges to zero but $\{a_n\}_n$ does not have a limit. Then for any natural number $r$ and for any number $q \in [\liminf_{n\to\infty} a_n, \limsup_{n\to\infty} a_n]$ there exists a subsequence $n_k \nearrow \infty$ such that*

$$\lim_{k\to\infty} a_{n_k} = q$$

*and for every $k$ we have*

$$a_{n_k} < a_{n_k+r}.$$

*Proof.* We will restrict our hypothesis to the case that $r=1$. The general case then follows from considering the subsequence $\{a_{rn}\}_n$.

First note that every number $q \in [\liminf a_n, \limsup a_n]$ is the limit of some subsequence of $\{a_n\}_n$. Hence, if $q \neq \liminf a_n$ then for every $\varepsilon > 0$ there must exist infinitely many numbers $m = m(\varepsilon)$ such that

$$a_m < q - \varepsilon < a_{m+1}.$$

Similarly, if $q \neq \limsup a_n$ then for every $\varepsilon > 0$ there must exist infinitely many numbers $m = m(\varepsilon)$ such that

$$a_m < q + \varepsilon < a_{m+1}.$$

Since we assume that the sequence $\{a_n\}_n$ does not have a limit, $q$ must satisfy one of the abovementioned properties. Hence, if we choose a decreasing family $\{\varepsilon_k\}_k$ and for each $\varepsilon_k$ one of the corresponding numbers $n_k = m(\varepsilon_k)$, we obtain

$$|a_{n_k} - q| \le \varepsilon_k + |a_{n_k} - a_{n_k-1}| \to 0.$$

The second part of the assertion is immediately satisfied. □

Lemma 7 will help us to establish some bounded distortion properties. The following result implies in particular that every point $x$ with $\overline{\chi}(x) > 0$ is conical (recall the definition of a conical point in Section 2.5).

**Lemma 8.** *Assume that $J$ does not contain any critical points of $f$. Let $x \in J$ be a point with $\overline{\chi}(x) > 0$. Then there exists a number $K > 0$ such that for every $q \in [\underline{\chi}(x), \overline{\chi}(x)] \setminus \{0\}$ there exists a number $\delta > 0$ and a sequence $n_k \to \infty$ such that*

i) $\displaystyle\lim_{k\to\infty} \frac{1}{n_k} \log |(f^{n_k})'(x)| = q,$

ii) $\operatorname{Dist} f^{n_k}|_{f_x^{-n_k}(B(f^{n_k}(x),\delta))} < K.$

*Here $K$ is a universal constant, while $\delta$ depends on the number $q$ but not on the point $x$.*

*Proof.* As $J$ does not contain any critical point, the only accumulation points in $J$ of the orbit of some critical point can be parabolic points. Let $r$ be the least common multiplier of the periods of all parabolic points in $J$.

Given a number $q \in [\underline{\chi}(x), \overline{\chi}(x)] \setminus \{0\}$, there exists a number $\delta_0 > 0$ such that if $y \in J$ is $\delta_0$-close to some parabolic point then

$$\frac{1}{r} \log |(f^r)'(y)| \leq \frac{q}{2}.$$

Further, there exists a number $\delta_1 > 0$ such that if $z \in J$ is $\delta_0$-away from any parabolic point then no orbit of a critical point passes through $B(z, 2\delta_1)$.

To prove the claimed property, it will suffice to find a sequence $\{n_k\}_k$ for which i) is satisfied and for which $f^{n_k}(x)$ is in distance at least $\delta_0$ from any parabolic point. Indeed, in such a situation the backward branch of $f_x^{-n_k}(B(f^{n_k}(x), 2\delta_1))$ will not catch any critical point and the distortion estimations ii) will follow from the Koebe distortion lemma.

If $x$ is a Lyapunov regular point and $q = \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|$ is its Lyapunov exponent (which, by our assumptions, must be positive) then the claim i) is automatically satisfied. In this case we just need to choose $n_k$ such that $f^{n_k}(x)$ is far away from any parabolic point. Note that there must be infinitely many such times $n_k$ because otherwise the Lyapunov exponent at $x$ would be no greater than $q/2$.

If $x$ is not a Lyapunov regular point, then we apply Lemma 7 to the sequence

$$a_n = \frac{1}{n} \log |(f^n)'(x)|.$$

Notice that $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$ is satisfied, because there are no critical points in $J$ and hence $|f'|$ is uniformly bounded. Hence, from the first assertion of Lemma 7 we obtain a sequence $\{n_k\}_k$ that satisfies i). Notice that we have

$$a_{n_k+r} = \frac{n_k}{n_k + r} a_{n_k} + \frac{1}{n_k + r} \log |(f^r)'(f^{n_k}(x))| \leq \frac{n_k}{n_k + r} a_{n_k} + \frac{r}{n_k + r} \frac{q}{2}$$

whenever $f^{n_k}(x)$ is $\delta_0$-close to some parabolic point. This inequality cannot be true for big $n_k$ (when $a_{n_k}$ is already close to $q$) because of the second part of assertion of Lemma 7. This proves that for any time $n_k$ large enough the point $f^{n_k}(x)$ is $\delta_0$-away from any parabolic point. □

We are now prepared to prove an upper bound for the Hausdorff dimension of the level sets under consideration. To start with the most general approach that will be needed in the subsequent analysis, we first study a set of points $x$ for which $\overline{\chi}(x) > 0$ and for which additionally the Lyapunov exponent (possibly with respect to some subsequence of times) is guaranteed to be within a given interval $[\alpha, \beta]$. Let us first introduce some notation. Given $0 \leq \alpha \leq \beta$, $\beta > 0$, let

$$\widehat{\mathcal{L}}(\alpha, \beta) \stackrel{\text{def}}{=} \{x \in J \colon \underline{\chi}(x) \leq \beta, \, \overline{\chi}(x) \geq \alpha, \, \overline{\chi}(x) > 0\}. \tag{19}$$

**Proposition 2.** *Assume that $J$ does not contain any critical points of $f$. For every $\beta > 0$ and $0 \leq \alpha \leq \beta$ we have*

$$\dim_{\mathrm{H}} \widehat{\mathcal{L}}(\alpha, \beta) \leq \max_{\alpha \leq q \leq \beta} F(q).$$

*If $\alpha > \alpha^+$ or $\beta < \alpha^-$ then $\widehat{\mathfrak{L}}(\alpha,\beta)=\varnothing$.*

*Proof.* By Lemma 8, for every point $x \in \widehat{\mathfrak{L}}(\alpha,\beta)$ there exist a number $q = q(x) \in [\alpha,\beta]\setminus\{0\}$, a number $\delta > 0$, and a sequence $\{n_k\}_k$ of numbers such that

$$\lim_{k\to\infty}\frac{1}{n_k}\log|(f^{n_k})'(x)| = q. \tag{20}$$

and

$$2\delta\,|(f^{n_k})'(x)|^{-1}K^{-1} \le \operatorname{diam} f_x^{-n_k}(B(f^{n_k}(x),\delta)) \le 2\delta\,|(f^{n_k})'(x)|^{-1}K. \tag{21}$$

Recall that for every $d \in \mathbb{R}$ there exists a $\exp(P(\varphi_d)-\varphi_d)$-conformal measure $\nu_d$ that gives positive measure to any open set (see Section 2.4). Hence there exists $c = c(\delta) > 0$ such that for every $n_k$ we have $c \le \nu_d(B(f^{n_k}(x),\delta)) \le 1$. Using again the distortion estimates, we can conclude that

$$\begin{aligned} &cK^{-d}\,e^{-n_kP(\varphi_d)}|(f^{n_k})'(x)|^{-d} \\ &\qquad\le \nu_d(f_x^{-n_k}(B(f^{n_k}(x),\delta))) \le K^de^{-n_kP(\varphi_d)}|(f^{n_k})'(x)|^{-d}, \end{aligned} \tag{22}$$

which implies that

$$\lim_{k\to\infty}\frac{1}{n_k}\log\,\nu_d(f_x^{-n_k}(B(f^{n_k}(x),\delta))) = -P(\varphi_d) - d\,q$$

and in particular that this limit exists. Hence, there exists $N > 0$ such that for every $n_k \ge N$ we have

$$\begin{aligned} \nu_d\big(f_x^{-n_k}(B(f^{n_k}(x),\delta))\big) &\\ &\ge e^{-n_k(P(\varphi_d)+dq+d\delta)} \\ &\ge e^{-n_kP(\varphi_d)}|(f^{n_k})'(x)|^{-d} \\ &\ge K^{-2d}e^{-n_kP(\varphi_d)}\left(\operatorname{diam} f_x^{-n_k}(B(f^{n_k}(x),\delta))\right)^d\left(\frac{1}{2\delta}\right)^d. \end{aligned}$$

Here we used Lemma 8 to obtain the last inequality. Applying (20) and (21) we yield

$$\underline{d}_{\nu_d}(x) \le \frac{P(\varphi_d)}{q} + d. \tag{23}$$

For any $q_0 \in [\alpha,\beta]$, $q_0 > 0$, and $\varepsilon > 0$ there exist a small interval $(q_1,q_2)$, $q_1 > 0$, containing $q_0$ and a number $d \in \mathbb{R}$ such that for all $q \in (q_1,q_2)$

$$\frac{1}{q}P(\varphi_d) + d < \begin{cases} F(q)+\varepsilon & \text{if } F(q) \ne -\infty, \\ -100 & \text{if } F(q) = -\infty. \end{cases} \tag{24}$$

We can choose a countable family of intervals $\{(q_1^{(i)},q_2^{(i)})\}_i$ covering $[\alpha,\beta]\setminus\{0\}$ and a sequence of corresponding numbers $\{d_i\}_i$. Defining the measure

$$\nu \overset{\text{def}}{=} \sum_i 2^{-i}\nu_{d_i}$$

we obtain

$$\underline{d}_\nu(x) \le \inf_{i\ge 1} \underline{d}_{\nu_{d_i}}(x) \le \max_{\alpha\le q\le\beta} F(q) + \varepsilon,$$

where the second inequality follows from (24). Applying the Frostman lemma we obtain that

$$\dim_{\rm H} \widehat{\mathcal{L}}(\alpha,\beta) \le \max_{\alpha\le q\le\beta} F(q) + \varepsilon$$

Since $\varepsilon$ can be chosen arbitrarily small, this finishes the proof of the first claim. The second claim was already proved in Lemma 5. □

Note that for every $q\in[\alpha,\beta]$, $q>0$, we have $\mathcal{L}(\alpha,\beta)\subset\widehat{\mathcal{L}}(q,q)$, which readily proves the following result.

**Corollary 1.** *Under the hypotheses of Proposition 2, we have*

$$\dim_{\rm H} \mathcal{L}(\alpha,\beta) \le \min_{\alpha\le q\le\beta} F(q) = \min\{F(\alpha),F(\beta)\}.$$

*In particular, for every $\alpha>0$, we have*

$$\dim_{\rm H} \mathcal{L}(\alpha) \le F(\alpha).$$

4.2. **The general case.** We now consider the general case that there are critical points inside the Julia set.

We need two technical results from the literature. The first one is the following telescope lemma from [16]. Recall the definition of hyperbolic times given in Section 2.5.

**Lemma 9.** *Given $\varepsilon>0$ and $\sigma>0$, there exist constants $K_1>0$ and $R_1>0$ such that the following is true. Given $x\in J$ with upper Lyapunov exponent $\overline{\chi}(x)>\sigma$, for every number $r<R_1$, for every $n\ge 1$ being a hyperbolic time for $x$ with exponent $\sigma$, and for every $0\le k\le n-1$ we have*

$$\operatorname{diam} f_{f^k(x)}^{-n+k}(B(f^n(x),r)) \le r\,K_1 e^{-(n-k)(\sigma-\varepsilon)}.$$

To formulate our second preliminary technical result we need the following construction, see [9, 20, 21].

**Pullback construction:** Fix some $n>0$ and let $y\in J\setminus\bigcup_{i=1}^n f^i(\mathrm{Crit})$. Fix some $R>0$ and let $\{y_i\}_{i=1}^n$ be some backward trajectory of $y$, i.e. $y_0=y$ and $y_{i+1}\in f^{-1}(y_i)$ for every $i=1,\ldots,n-1$. Let $k_1$ be the smallest integer for which $f_{y_{k_1}}^{-k_1}(B(y,R))$ contains a critical point. For every $\ell\ge 1$ let then $k_{\ell+1}$ be the smallest integer greater than $k_\ell$ such that $f_{y_{k_{\ell+1}}}^{-(k_{\ell+1}-k_\ell)}(B(y_{k_\ell},R))$ contains a critical point and so on. In this way, for each backward branch $\{y_i\}_i$ we construct a sequence $\{k_\ell\}_\ell$ that must have a maximal element not greater than $n$. Let this element be $k$ and consider the set $Z$ of all pairs $(y_k,k)$ built from all the backward branches of $f$ that start from $y$. Let $N(y,n,R)\stackrel{\rm def}{=}\#Z$. We have the following estimate, see [21, Lemma 3.7] and [20, Appendix A].

**Lemma 10.** *Given $\varepsilon > 0$, there exist $K_2 > 0$ and $R_2 > 0$ such that for all $R \le R_2$ we have*

$$N(y, n, R) < K_2 e^{n\varepsilon}$$

*uniformly in $y$ and $n$.*

Recall the definition of conical points in Section 2.5. Using the above two lemmas we can now show the following result.

**Proposition 3.** *The set of points $x \in J$ that are not conical and satisfy $\overline{\chi}(x) > 0$ has Hausdorff dimension zero.*

*Proof.* Let us choose some numbers $\sigma > 0$ and $\varepsilon > 0$. Let

$$r \overset{\text{def}}{=} \frac{1}{2K_1 + 4} \min\{R_1, R_2\},$$

where $K_1$ and $R_1$ are constants given by Lemma 9 and where $R_2$ is given by Lemma 10.

We can choose a finite family of balls $\{B_i\}_{i=1}^L$ of radius $3r$ such that any ball of radius $2r$ intersecting $J$ must be contained in one of the balls $B_i$. In the case that we can prove existence of a sequence $\{n_i\}_i$ such that $f_{f^k(x)}^{-(n_i-k)}(B(f^{n_i}(x), 2r))$ does not contain critical points for any $0 \le k \le n_i - 1$, the Koebe distortion lemma will imply that $x$ is a conical point for $r$, $\{n_i\}_i$, and $U_i = f_x^{-n_i}(B(f^{n_i}(x), r))$.

Let $G(m, \sigma)$ be the set of points $x \in J$ with upper Lyapunov exponent greater than $\sigma$ for which with $r$ chosen above for all $n > m$ the backward branch of $f^{-n}$ from ball $B(f^n(x), 2r)$ onto a neighborhood of $x$ will necessarily meet a critical point, that is, for some $0 \le k \le n-1$ we have

$$f_{f^k(x)}^{-(n-k)}(B(f^n(x), 2r)) \cap \text{Crit} \neq \varnothing.$$

First we claim that $\dim_{\text{H}} G(m, \sigma) = 0$. Let us denote by $G(m, \sigma, n)$ the subset of $G(m, \sigma)$ for which $n > m$ is a hyperbolic time with exponent $\sigma$. Recall that $\overline{\chi}(x) > \sigma > 0$ implies that there exist infinitely many hyperbolic times for $x$ with exponent $\sigma$. Hence, we have

$$G(m, \sigma) = \bigcap_{m_0 \ge m} \bigcup_{n > m_0} G(m, \sigma, n). \tag{25}$$

Let $x \in G(m, \sigma, n)$. Let $B_j = B(y, 3r)$ be the ball that contains $B(f^n(x), 2r)$. We will apply the "pullback construction" for Lemma 10 to the point $y$, the numbers $n$, $R = \frac{1}{2} \min\{R_1, R_2\}$, and to the backward branch $f_x^{-n}$. And let $k = \max_\ell k_\ell$ and $y_k$ be given by the pullback construction (compare Figure 1).

We first note that Lemma 9 and $\rho(f^n(x), y) \le r$ imply that

$$\rho(f^{n-k}(x), y_k) \le r\, K_1.$$

This implies $B(f^{n-k}(x), 2r) \subset B(y_k, R)$ and hence $k \ge n - m$ because $x \in G(m, \sigma)$. Thus, with fixed $n$ and $B_j$, by Lemma 10 the point $x$ must belong to one of at most $K_2 e^{n\varepsilon} (\deg f)^m$ preimages of $B_j$. As $B(y, 3r) \subset B(x, 4r)$

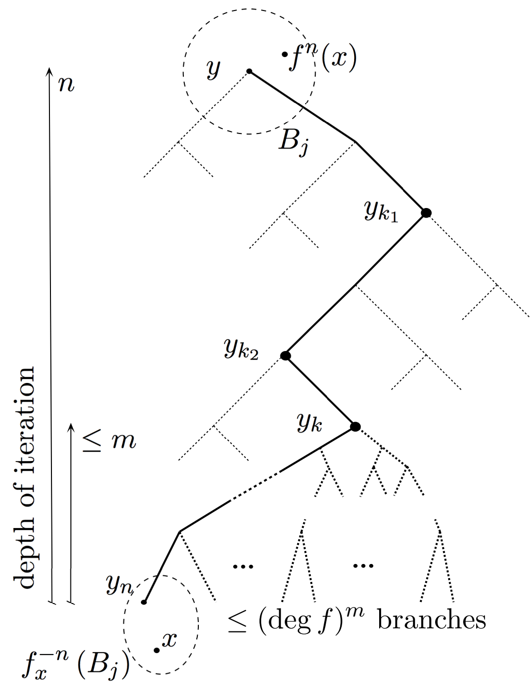


FIGURE 1. Pullback construction starting from the point $y$.

and $4r < R_1$, by Lemma 9 this pre-image of $B_j$ has diameter not greater than $8r\, K_1 e^{-n(\sigma-\varepsilon)}$.

Hence we showed that every point in $G(m,\sigma,n)$ belongs to the $n$th pre-image of some ball $B_j$ along a backward branch for that $k \geq m-n$ (where $k = \max_\ell k_\ell$ is as in the pullback construction). Thus, by Lemma 9 the set $G(m,\sigma,n)$ is contained in a union of at most $LK_2 e^{n\varepsilon}(\deg f)^m$ sets of diameter not greater than $8r\, K_1 e^{-n(\sigma-\varepsilon)}$. Using those sets to cover $G(m,\sigma,n)$ and applying (25), we obtain

$$\dim_{\mathrm{H}} G(m,\sigma) \leq \frac{\varepsilon}{\sigma-\varepsilon}.$$

As $\varepsilon$ can be chosen arbitrarily small, the claim follows.

Finally note that the set of points that we want to estimate in the proposition is contained in the union

$$\bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} G(m, \frac{1}{n}).$$

Since for any set in this union its Hausdorff dimension is zero, the assertion follows. □

Note that in the above proof we were able to show something more. Namely notice that the choice of $r$ depended on $\sigma$ alone (and not directly on the point $x$).

We are now prepared to prove the following estimate.

**Proposition 4.** *Let* $0 < \alpha \le \beta \le \alpha^+$. *We have*

$$\dim_{\rm H} \mathcal{L}(\alpha,\beta) \le \max\left\{0, \max_{\alpha\le q\le\beta} F(q)\right\}.$$

*Proof.* Let $\alpha$ and $\beta$ be like in the assumptions and let $x \in \mathcal{L}(\alpha,\beta)$. By Proposition 3 we can restrict our considerations to the case that $x$ is a conical point with corresponding number $r > 0$, sequence $\{n_k\}_k$, and family of neighborhoods $\{U_k\}_k$. Hence, there exist numbers $\delta > 0$ and $K > 1$ such that

$$2\delta\,|(f^{n_k})'(x)|^{-1}\,K^{-1} \le \operatorname{diam} f_x^{-n_k}(B(f^{n_k}(x),\delta)) \le 2\delta\,|(f^{n_k})'(x)|^{-1}\,K. \tag{26}$$

Choosing, if necessary, a subsequence of $\{n_k\}_k$, we can find $q = q(x) \in [\alpha,\beta]$ for which we have

$$\lim_{k\to\infty} \frac{1}{n_k} \log|(f^{n_k})'(x)| = q. \tag{27}$$

Recall that for any $d \in \mathbb{R}$ there exists a $\exp(P(\varphi_d) - \varphi_d)$-conformal measure $\nu_d$ that gives positive measure to any open set that intersects $J$ (see Section 2.4). Hence there exists a number $c_\delta > 0$ for which for every $n_k$ we have $c_\delta \le \nu_d(B(f^{n_k}(x),\delta)) \le 1$. Using again distortion estimates, we can conclude that

$$\begin{aligned} c\,e^{-n_k P(\varphi_d)} K^{-1} |(f^{n_k})'(x)|^{-d} \\ \le \nu_d(f_x^{-n_k}(B(f^{n_k}(x),\delta))) \le e^{-n_k P(\varphi_d)} K |(f^{n_k})'(x)|^{-d}. \end{aligned} \tag{28}$$

The rest of the proof is similar to the proof of Proposition 2. First we obtain that for every $d \in \mathbb{R}$ we have

$$\underline{d}_{\nu_d}(x) \le \frac{P(\varphi_d)}{q} + d.$$

Choosing the right number $d$, we then conclude that $\underline{d}_{\nu_d}(x) \le F(q)$. We finish the proof by constructing a measure such that for an arbitrarily small chosen number $\varepsilon$ and some number $q \in [\alpha,\beta]$ the lower pointwise dimension of that measure at every $x \in \mathcal{L}(\alpha,\beta)$ is not greater than $F(q) + \varepsilon$. □

**Remark 3.** Notice that in the general case, in which we do have critical points in $J$, for a point $x \in J$ with $\underline{\chi}(x) < \overline{\chi}(x)$ we cannot apply the same techniques as in Section 4.1. In particular, in the above proof for each point $x \in \mathcal{L}(\alpha,\beta)$ we only know that there exists some number $q = q(x) \in [\underline{\chi}(x), \overline{\chi}(x)]$ to which the Lyapunov exponents over some subsequence of times $\{n_k\}_k$ will converge, while in Proposition 2 we were able to take an arbitrary number $q$ in that interval. Hence, to show the following result we can only consider the set $\mathcal{L}(\alpha) = \mathcal{L}(\alpha,\alpha)$ and not $\widehat{\mathcal{L}}(q,q)$ for an arbitrary number $q \in [\alpha,\beta]$, and the following corollary is weaker than Corollary 1. However, the implications for the regular part of the spectrum remain the same.

Proposition 4 and $\mathcal{L}(\alpha) = \mathcal{L}(\alpha, \alpha)$ readily imply the following result.

**Corollary 2.** *For* $\alpha \in [\alpha^-, \alpha^+] \setminus \{0\}$ *we have*

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha) \leq F(\alpha).$$

## 5. Lower bounds for the dimension

In this section we will derive lower bounds for the Hausdorff dimension. We will either assume that $f$ is a non-exceptional map or that $f$ is exceptional but $\Sigma \cap J = \varnothing$. Recall that under those assumptions Proposition 1 is valid, so we can approximate the pressure with respect to $f|_J$ with pressures that are defined with respect to a sequence of Cantor repellers $f^{a_m}|_{\Lambda_m}$ that were constructed in Section 2.3.

One more case we would like to exclude is $\alpha^- = \alpha^+ = \alpha$. It is not very interesting because in this case any measure supported on any hyperbolic set $\Lambda_m \subset J$ has Lyapunov exponent $\alpha$. Hence we automatically have

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha) \geq \sup_{m \geq 1} F_m(\alpha)$$

and the supremum on the right hand side is in this case equal to $F(\alpha)$. Therefore, in the following considerations we will assume that $\alpha^- < \alpha^+$.

### 5.1. The interior of the spectrum.

We use the sequence of Cantor repellers to obtain for any exponent from the interior of the spectrum a big uniformly expanding subset of points with Lyapunov exponent $\alpha$ that provides us with an estimate from below.

**Proposition 5.** *For* $\alpha \in (\alpha^-, \alpha^+)$ *we have* $\dim_{\mathrm{H}} \mathcal{L}(\alpha) \geq F(\alpha)$.

*Proof.* Let us consider the sequence of Cantor repellers $f^{a_m}|_{\Lambda_m}$ from Proposition 1. By (11), for each number $\alpha^+ > \alpha > \alpha^-$ there exists $m_0 \geq 1$ such that $\alpha_m^+ > \alpha > \alpha_m^-$ for every $m \geq m_0$. Obviously we have

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha) \geq \sup_{m \geq 1} \dim_{\mathrm{H}} \mathcal{L}(\alpha) \cap \Lambda_m.$$

Since $\Lambda_m$ is a uniformly expanding repeller with respect to $f^{a_m}$, for any exponent $\alpha \in (\alpha_m^-, \alpha_m^+)$ there exists a unique number $q = q(\alpha) \in \mathbb{R}$ such that $\alpha = -\frac{1}{a_m} \frac{d}{ds} P_{f^{a_m}|\Lambda_m}(\varphi_s)|_{s=q(\alpha)}$ and an equilibrium state $\mu_q$ for the potential $\varphi_q$ (with respect to $f^{a_m}|_{\Lambda_m}$) such that the Lyapunov exponent of $\mu_q$ with respect to $f^{a_m}$ is equal to $a_m \alpha$ (compare the classical results in the introduction). For the measure

$$\nu_m = \sum_{i=0}^{a_m - 1} \mu_m \circ f^i$$

we have $\chi(\nu_m)=\alpha$. Hence, the variational principle implies

$$\begin{aligned}\max\Big\{h_\nu(f)\colon \nu\in\mathcal{M}_{\rm E}\Big(\bigcup_i f^i(\Lambda_m)\Big),\chi(\nu)=\alpha\Big\}\\ &\ge \frac{1}{a_m}P_{f^{a_m}|\Lambda_m}(S_{a_m}\varphi_q)+q\alpha\\ &\ge \inf_{d\in\mathbb{R}}\left(\frac{1}{a_m}P_{f^{a_m}|\Lambda_m}(S_{a_m}\varphi_d)+d\alpha\right)=F_m(\alpha).\end{aligned}$$

We obtain that $\dim_{\rm H}\nu=\frac{h_\nu(f)}{\chi(\nu)}$ whenever $\nu$ is an $f$-invariant ergodic Borel probability measure with positive Lyapunov exponent [11]. This implies that for every $m\ge m_0$

$$\dim_{\rm H}\mathcal{L}(\alpha)\cap\bigcup_i f^i(\Lambda_m)\ge\max\Big\{\frac{h_\nu(f)}{\alpha}\colon \nu\in\mathcal{M}_{\rm E}\Big(\bigcup_i f^i(\Lambda_m)\Big),\chi(\nu)=\alpha\Big\}.$$

From here the we can conclude that $\dim_{\rm H}\mathcal{L}(\alpha)\ge\sup_{m\ge1}F_m(\alpha)$. Together with Proposition 1 the statement is proved. □

5.2. **The boundary of the spectrum.** Unfortunately, the above approach does not suffice to analyze the level sets for exponents from the boundary of the spectrum. Our main goal in this section is to prove the following result. It will not only enable us to describe the boundary of the spectrum but also provide us with dimension lower bounds for level sets of irregular points.

**Theorem 3.** *Let $\{\Lambda_i\}_i$ be a sequence of subsets of $J$. We assume that each $\Lambda_i$ is a uniformly expanding repeller for some iteration $f^{a_i}$ and contains non-immediately postcritical points. Let $\{\phi_i\}_i$ be a sequence of Hölder continuous potentials and let $\{\mu_i\}_i$ be a sequence of equilibrium states for $\phi_i$ with respect to $f^{a_i}|_{\Lambda_i}$. Then*

$$\dim_{\rm H}\Big\{x\in J\colon \underline{\chi}(x)=\liminf_{i\to\infty}\chi(\mu_i),\,\overline{\chi}(x)=\limsup_{i\to\infty}\chi(\mu_i)\Big\}\ge\liminf_{i\to\infty}\dim_{\rm H}\mu_i$$

*and*

$$\dim_{\rm P}\Big\{x\in J\colon \underline{\chi}(x)=\liminf_{i\to\infty}\chi(\mu_i),\,\overline{\chi}(x)=\limsup_{i\to\infty}\chi(\mu_i)\Big\}\ge\limsup_{i\to\infty}\dim_{\rm H}\mu_i.$$

We derive the following estimates for level sets that include exponents at the boundary of the spectrum.

**Proposition 6.** *For $\alpha^-\le\alpha<\beta\le\alpha^+$ we have*

$$\dim_{\rm H}\widehat{\mathcal{L}}(\alpha,\beta)\ge\max_{\alpha\le q\le\beta}F(q).$$

*Proof.* The claimed estimate follows from Theorem 3.

We can also observe that for every $q\in(\alpha,\beta)$ we have $\widehat{\mathcal{L}}(\alpha,\beta)\supset\mathcal{L}(q,q)=\mathcal{L}(q)$. Hence we can apply Proposition 5 to derive $\dim_{\rm H}\mathcal{L}(q)\ge F(q)$ and prove the claimed estimate. □

**Proposition 7.** *For $\alpha^- \le \alpha \le \beta \le \alpha^+$ we have*

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) \ge \min\{F(\alpha), F(\beta)\}.$$

*Proof.* Since we assume $\alpha^- < \alpha^+$, there exists a sequence $\{\Lambda_i\}_i$ of uniformly expanding repellers and a sequence $\{\mu_i\}_i$ of equilibrium states for the potential $-\log|f'|$ with respect to $f|_{\Lambda_i}$ such that $\dim_{\mathrm{H}} \mu_i = F(\chi(\mu_i))$ and $\alpha = \lim_{i\to\infty} \chi(\mu_{2i})$ and $\beta = \lim_{i\to\infty} \chi(\mu_{2i+1})$. Theorem 3 then implies that

$$\mathcal{L}(\alpha, \beta) = \big\{x \in J \colon \underline{\chi}(x) = \alpha, \overline{\chi}(x) = \beta\big\} \ge \liminf_{i\to\infty} F(\chi(\mu_i)).$$

Since $\liminf_{i\to\infty} F(\chi(\mu_i)) \ge \min\{F(\alpha), F(\beta)\}$, this proves the claimed estimate. □

Recall that a point $x$ is said to be *recurrent* if $f^{n_i}(x) \to x$ for some sequence $n_i \nearrow \infty$.

**Corollary 3.** *Assume that $J$ does not contain any recurrent critical points of $f$, that $f$ is non-exceptional, and that $F(0) \ne -\infty$. Then we have*

$$\dim_{\mathrm{H}} \mathcal{L}(0) = \dim_{\mathrm{H}} J = F(0).$$

*Proof.* Proposition 7 implies that $\dim_{\mathrm{H}} \mathcal{L}(0) \ge F(0)$. As $F(0) \ne -\infty$, the pressure function $d \mapsto P(\varphi_d)$ is nonnegative and $F(0) = \inf\{d \colon P(\varphi_d) = 0\}$. Since $J$ does not contain any recurrent critical points of $f$, we can apply [24, Theorem 4.5] and conclude that $\inf\{d \colon P(\varphi_d) = 0\} = \dim_{\mathrm{H}} J$. □

To prove Theorem 3, we will construct a sufficiently large set of points that have precisely the given lower/upper Lyapunov exponent. Note that such a set will not "be seen" by any invariant measure in the non-trivial case that lower and upper exponents do not coincide. Also points with Lyapunov exponent zero, though Lyapunov regular, will not "be seen" by any interesting invariant measure. Our approach is to show that such a set is large with respect to some necessarily non-invariant probability measure. It is generalizing the construction of so-called w-measures introduced in [8] for which we strongly made use of the Markov structure that is not available to us here.

*A technical lemma.* In preparation for the proof of Theorem 3 we start with some technical result that will be usefull shortly after.

**Lemma 11.** *Let $g \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a conformal map and $\Lambda \subset \overline{\mathbb{C}}$ be a compact $g$-invariant hyperbolic topologically transitive set, $\mu$ a $g$-invariant ergodic measure on $\Lambda$ with Lyapunov exponent $\chi = \chi(\mu)$, entropy $h = h_\mu(g)$, and Hausdorff dimension $d = d(\mu)$. Let $V$ be an open set of positive measure $\mu$. If $\gamma$ is small enough then for any $\varepsilon > 0$ for $\mu$-almost every point $v \in \Lambda$ there exist a number $K > 0$ and a sequence $\{n_i\}_i$ such that for each $n_i$ there is a set $F_{n_i} \subset V \cap \Lambda$ such that for all $y_j \in F_{n_i}$*

i) *$g^{n_i}(y_j) = v$,*

ii) *$K^{-1} \exp(m(\chi - \varepsilon)) < |(g^m)'(y_j)| < K \exp(m(\chi + \varepsilon))$ for all $m \le n_i$,*

iii) *the branch* $g_{y_j}^{-n_i}$ *mapping* $v$ *onto* $y_j$ *extends to all* $B(v,\gamma)$ *and the distortion of the resulting map is bounded by* $K$*,*

iv) *we have*

$$\mu\left(\bigcup_{y_j\in F_{n_i}} g_{y_j}^{-n_i}(B(v,\gamma))\right)\geq K^{-1},$$

v) *for* $j\neq k$ *we have*

$$\rho\left(g_{y_j}^{-n_i}(B(v,\gamma)),g_{y_k}^{-n_i}(B(v,\gamma))\right)>K^{-1}\operatorname{diam} g_{y_j}^{-n_i}(B(v,\gamma)),$$

vi) *for any* $x\in V$ *and* $r>0$ *we have*

$$\mu\left(B(x,r)\cap\bigcup_{y_j\in F_{n_i}} g_{y_j}^{-n_i}(B(v,\gamma))\right)\leq Kr^{d-\varepsilon},$$

vii) *we have*

$$K^{-1}e^{-n(h+\varepsilon)}\leq\mu\left(g_{y_j}^{-n_i}(B(v,\gamma))\right)\leq e^{-n(h-\varepsilon)}.$$

*Proof.* As $g|_\Lambda$ is uniformly expanding, there exist numbers $\gamma>0$ and $c_1>0$ such that for every $n\geq 1$, $y\in\Lambda$, $v=f^n(y)\in\Lambda$, and a backward branch $g_y^{-n}$ mapping $v$ onto $y$ for every $x_1$, $x_2\in B(v,\gamma)$ we have

$$\frac{|(g_y^{-n})'(x_1)|}{|(g_y^{-n})'(x_2)|}\leq c_1, \tag{29}$$

and in particular the mapping $g_y^{-n}$ extends to all of $B(v,\gamma)$. Moreover, there is some constant $c_2>0$ not depending on $n$ or $y$ such that

$$\operatorname{diam} g_y^{-n}(B(v,\gamma))\leq c_2\gamma.$$

We assume that $\gamma$ is so small that for any two points $x$, $y$ with distance from $\Lambda$ and mutual distance $<c_2\gamma$, for any point $x'\in g^{-1}(x)$ there is at most one point $y'\in g^{-1}(y)$ such that $\rho(x',y')\leq c_2\gamma$. This is true for any small enough number $\gamma$ because $\Lambda$ is in positive distance from any critical point of $g$. Let $\widetilde{V}\subset V$ be such that $B(\widetilde{V},c_2\gamma)\subset V$ and that $\widetilde{V}$ is nonempty and of positive measure, which is possible whenever $\gamma$ is small enough.

Notice that $B(v,\frac{1}{c_2+1}\gamma)$ has positive $\mu$-measure for $\mu$-almost every $v$. Choose such point $v$ and let

$$\widetilde{U}\stackrel{\text{def}}{=}B\left(v,\frac{1}{c_2+1}\gamma\right),\quad U\stackrel{\text{def}}{=}B(v,\gamma).$$

Let

$$\delta\stackrel{\text{def}}{=}\frac{1}{2}\min\left\{\mu(\widetilde{U}),\mu(\widetilde{V})\right\}.$$

Let $\varepsilon>0$. There is a set of points $\Lambda'\subset\Lambda$ with $\mu(\Lambda')>1-\delta$ (actually, it can be chosen to have arbitrarily large measure) and a number $N>1$ such that: for every $n\geq N$ and for every $x\in\Lambda'$ we have

(C1) $\left|\frac{1}{n}\#\left\{k\in\{0,\dots,n-1\}\colon g^k(x)\in\widetilde U\cap\Lambda\right\}-\mu(\widetilde U)\right|\le\varepsilon$,

(C2) $\big|\log|(g^n)'(x)|-n\chi\big|\le n\varepsilon$,

(C3) $|\log \mathrm{Jac}_\mu g^n(x)-nh|\le n\varepsilon$,

(C4) for $r\le\operatorname{diam}\Lambda$ we have
$$\mu(B(x,r))<c_3r^{d-\varepsilon}.$$

Here (C1), (C2) simply follow from ergodicity, (C3) is a consequence of the Rokhlin formula and ergodicity, and (C4) follows from the definition of the dimension $d(\mu)$.

We choose now a family $\{E_n\}_{n\ge N}$ of subsets of $\Lambda'$ such that for every $n\ge N$ the set $E_n$ is a maximal $(n,\frac{c_2}{c_2+1}\gamma)$-separated subset of $\Lambda'$. Given $n\ge N$, let $V_n\stackrel{\text{def}}{=}E_n\cap\widetilde V$. For each $n\ge N$ let
$$F_n\stackrel{\text{def}}{=}\left\{x\in V_n\colon g_x^{-n}(U)\subset V\right\}.$$
For every $z_j\in F_n$ let $V_{n,j}\stackrel{\text{def}}{=}g_{z_j}^{-n}(\widetilde U)$. Obviously, all the sets $V_{n,j}$ are pairwise disjoint.

By (C1) the trajectory of every point from $\widetilde V\cap\Lambda'$ visits $\widetilde U$ at some time $n\ge N$ (in fact, at infinitely many times). Let $x$ be such a point and $n$ be such a time, that is, $g^n(x)\in\widetilde U$. Because $E_n$ is maximal, $x$ must be $(n,\frac{c_2}{c_2+1}\gamma)$-close to some point $z_j\in E_n$. Hence we have
$$g^n(z_j)\in B\left(g^n(x),\frac{c_2}{c_2+1}\gamma\right)\subset U$$
and
$$x\in V_{n,j}\subset g_{z_j}^{-n}(U)\subset B(x,c_2\gamma)\subset V.$$
This shows that $z_j\in F_n$. This together with (C1) implies that
$$\sum_{\ell=0}^{n-1}\mu\left(\bigcup_j V_{\ell,j}\right)\ge(\mu(\widetilde U)-\varepsilon)\,n\,\mu(\Lambda')$$
and hence, as it is satisfied for all $n>N$, we obtain
$$\sum_j\mu\left(V_{\ell,j}\right)\ge\frac12(\mu(\widetilde U)-\varepsilon)\mu(\Lambda')$$
for infinitely many $\ell\ge N$. This gives us a sequence of times $n_i\stackrel{\text{def}}{=}\ell$ and points $y_j=g_{z_j}^{-n_i}(v)$ for which the assertion of the lemma is true. Indeed, $g_{z_j}^{-n_i}(U)$ and $g_{z_k}^{-n_i}(U)$ for $j\ne k$ are disjoint because we are in a sufficiently large distance from Crit. We also know that those maps $g_{z_j}^{-n_i}|_U$ have uniformly bounded distortion. Recall that $U=B(v,\gamma)$, which implies that $g_{z_j}^{-n_i}(B(v,\gamma/2))$ and $g_{z_k}^{-n_i}(B(v,\gamma/2))$ for $j\ne k$ are not only disjoint but in distance that is comparable to the sum of their diameters and v) follows.

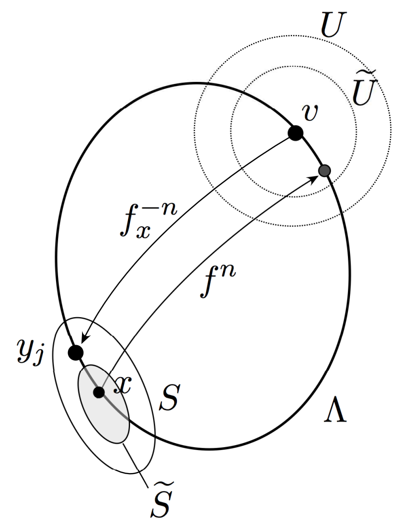


FIGURE 2. Finding $y_j$

Further properties i) and iii) are checked directly from the construction. Moreover, vi) follows from (C4), iv) follows from our choices of $\ell$, ii) from (C2) and vii) from iv) and (C3). □

*Construction of a Cantor set.* We now continue with some preliminary constructions that will be needed in the proof of Theorem 3.

As a first step, we are going to construct "bridges" between the repellers $\Lambda_i$ (compare Figure 3). This is very similar to the proof of Lemma 2 though here we will not necessarily require that the repellers are disjoint. To fix some notation, let $\{(B_i, b_i)\}_i$ be a collection of *bridges*, that is, let

$$B_i \overset{\text{def}}{=} B(z_i, r_i) \subset B(\Lambda_i, \gamma_i),$$

where the numbers $r_i$, $b_i$, $\gamma_i$ and the points $z_i$ are appropriately chosen such that $f^{b_i}|_{B_i}$ is a homeomorphism and that

$$\mu_{i+1}(f^{b_i}(B_i)) > 0, \quad f^{b_i}(z_i) \in \Lambda_{i+1},$$

and

$$\rho(f^k(B_i), \text{Crit}) \geq \delta_i \quad \text{for every } 1 \leq k \leq b_i.$$

The particular choice of the numbers and points will be specified in the following so that we are able to apply Lemma 11 to the each of the sets $\Lambda_i$ and the maps $g = f^{a_i}$.

Let us outline the following Cantor set construction. Lemma 11 enables us to select sufficiently many preimages for each of the disks $B_i$ (as we choose $B_i \subset B(v_i, \gamma_i)$ and then select preimages $g^{-n_i}(v_i)$ using the lemma). The construction of this Cantor set is easily described in terms of backward branches: at level $i$ we start with the disk $B_i$, apply Lemma 11 and find a large number of components of $f^{-a_i n_i}(B_i)$ in $f^{b_{i-1}}(B_{i-1})$, then we go backwards through the bridge obtaining the components in $B_{i-1}$. We repeat the procedure in $B_{i-1}$, ..., $B_1$.

Given the sequence of uniformly expanding repellers $f^{a_i}|_{\Lambda_i}$ with equilibrium states $\mu_i$ with Lyapunov exponent $\chi(\mu_i)$ and entropy $h_{\mu_i}(f^{a_i})$ (both with respect to the map $f^{a_i}$), let us denote

$$\chi_i \stackrel{\text{def}}{=} \frac{1}{a_i}\chi(\mu_i), \quad h_i \stackrel{\text{def}}{=} \frac{1}{a_i}h_{\mu_i}(f^{a_i}), \quad d_i \stackrel{\text{def}}{=} d(\mu_i) = \frac{h_i}{\chi_i},$$

and

$$s_i \stackrel{\text{def}}{=} |(f^{b_i})'(z_i)|, \quad t_i \stackrel{\text{def}}{=} \operatorname{Dist} f^{b_i}|_{B_i}.$$

Let

$$w_i \stackrel{\text{def}}{=} \inf_{x\colon \rho(x,\text{Crit})>\delta_i} |f'(x)|, \quad W \stackrel{\text{def}}{=} \sup_{x\in J} |f'(x)|.$$

The constants $\gamma_i$ can be chosen to be arbitrarily small (at the cost of decreasing $r_i$ and increasing $b_i$, thus changing $t_i$ and $s_i$ accordingly). In particular we can choose each $\gamma_i$ sufficiently small such that Lemma 11 applies to the map $g = f^{a_i}$ and the set $\Lambda_i$. Let

$$V_1 \stackrel{\text{def}}{=} B(\Lambda_1, \gamma_1) \quad \text{and} \quad V_{i+1} \stackrel{\text{def}}{=} f^{b_i}(B_i).$$

We choose a fast decreasing sequence $\{\varepsilon_i\}_i$. We will denote by $K_i$ the numbers $K_i = K(\Lambda_i, \mu_i, V_i, \varepsilon_i, v_i)$ as given by Lemma 11 (for $g = f^{a_i}$) where we choose each point $v_i \in \Lambda_i$ such that $B_i \subset B(v_i, \gamma_i)$. Let $A_1 \stackrel{\text{def}}{=} f_{y_j}^{-a_1 n_1}(B_1)$ for one point $y_j$ and $n_1$ as provided by Lemma 11. We have $\operatorname{Dist} f^{a_1 n_1}|_{A_1} \le K_1$ and for every $x \in A_1$ and $m \le n_1$ we have

$$K_1^{-1} e^{a_1 m(\chi_1 - \varepsilon_1)} < |(f^{a_1 m})'(x)| < K_1 e^{a_1 m(\chi_1 + \varepsilon_1)}$$

(note that $n_1$ can be chosen to be arbitrarily big, as guaranteed by Lemma 11).

We apply Lemma 11 now to $\Lambda_2$, $\mu_2$, $V_2$, $\varepsilon_2$, $v_2$, $g = f^{a_2}$, which provides us with a family $F_2 \in f^{-a_2 n_2}(v_2)$. Let $\widehat{A}_{2j} = f_{y_j}^{-a_2 n_2}(B(v_2, \gamma_2))$ for $y_j \in F_2$. Those sets are contained in the set $V_2 = f^{b_1}(B_1)$. We also have $\operatorname{Dist} f^{a_2 n_2}|_{\widehat{A}_{2j}} \le K_2$ and for every $x \in \widehat{A}_{2j}$ and $m \le n_2$ we have

$$K_2^{-1} e^{a_2 m(\chi_2 - \varepsilon_2)} < |(f^{a_2 m})'(x)| < K_2 e^{a_2 m(\chi_2 + \varepsilon_2)}.$$

Such sets will be used later to distribute the w-measure accordingly. It provides us also with a family of sets $\widetilde{A}_{2j} \subset \widehat{A}_{2j}$ such that $f^{a_2 n_2}(\widetilde{A}_{2j}) = B_2$. Such sets will be used later to define our Cantor set on which the w-measure is supported.

We repeat this procedure for every $i$, using Lemma 11 repeatedly: for every $i > 1$ we find a family $F_i \subset f^{-a_i n_i}(v_i)$ and the corresponding components $\widehat{A}_{ij}$ of $f^{-a_i n_i}(B(v_i, \gamma_i))$ contained in $V_i$ that satisfy

$$\rho(\widehat{A}_{ij}, \widehat{A}_{ik}) \ge K_i^{-1} \operatorname{diam} \widehat{A}_{ij}, \quad j \ne k,$$

and for every $x \in \widehat{A}_{ij}$ and $m \le n_i$

$$K_i^{-1} e^{a_i m(\chi_i - \varepsilon_i)} < |(f^{a_i m})'(x)| < K_i e^{a_i m(\chi_i + \varepsilon_i)}$$

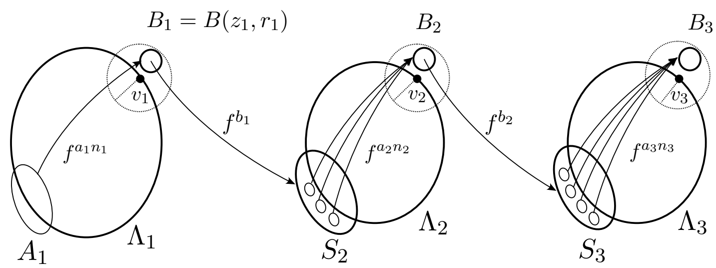


FIGURE 3. Connecting the hyperbolic sets by bridges.

and

$$\operatorname{Dist} f^{a_i n_i}|_{\widehat{A}_{ij}} \le K_i.$$

In addition, by Lemma 11 iv) we have

$$\mu_i \left( \bigcup_j \widehat{A}_{ij} \right) \ge K_i^{-1},$$

and by Lemma 11 vi) for any $x \in V_i$ and $r > 0$ we have

$$\mu_i \left( B(x,r) \cap \bigcup_j \widehat{A}_{ij} \right) \le K_i \, r^{d_i - \varepsilon_i}.$$

Note here that for every $i \ge 1$ we have

$$r_i \, s_i \, t_i^{-1} \le \operatorname{diam} V_{i+1} \le r_i \, s_i \, t_i.$$

Let

$$m_i \stackrel{\text{def}}{=} \sum_{k=1}^{i} (a_k n_k + b_k).$$

Let $A_{2\,j}$ be the component of $A_1 \cap f^{-m_1}(\widetilde{A}_{2j})$ for which $f^{a_1 n_1}(A_{2\,j}) \subset B_1$. Similarly, let $A_{i\,j_1 \ldots j_{i-1}}$ be the component of $A_{(i-1) j_1 \ldots j_{i-2}} \cap f^{-m_{i-1}}(\widetilde{A}_{i\,j_{i-1}})$ for which $f^{a_{i-1} n_{i-1} + m_{i-2}}(A_{i\,j_1 \ldots j_{i-1}}) \subset B_{i-1}$.
The sets

$$A_i \stackrel{\text{def}}{=} \bigcup_{j_1 \ldots j_{i-1}} A_{i\,j_1 \ldots j_{i-1}}$$

form a decreasing sequence of unions of topological balls. Moreover, the pair $\big(A_{(i-1) j_1 \ldots j_{i-2}}, \{A_{i\,j_1 \ldots j_{i-2}\,k}\}_k\big)$ is an image of $(V_i, \{\widetilde{A}_{i\,k}\}_k)$ under a branch of the map $f^{-m_{i-1}}$, the distortion of that branch is bounded by

$$\widetilde{K}_{i-1} \stackrel{\text{def}}{=} \prod_{k=1}^{i-1} K_k t_k,$$

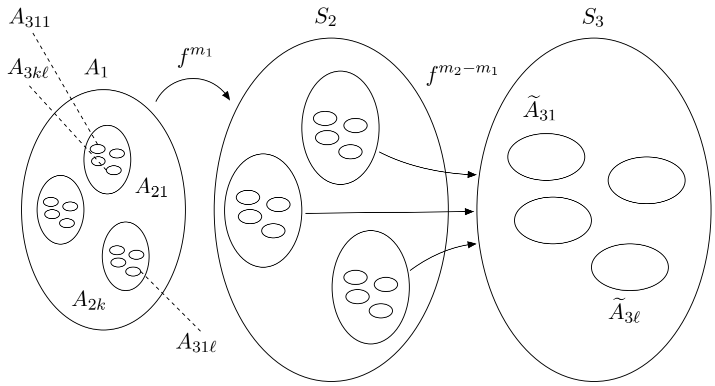


FIGURE 4. Local structure of the Cantor set $A$

and the absolute value of its derivative is between
(30)
$$L_{i-1} \stackrel{\text{def}}{=} \prod_{k=1}^{i-1} s_k^{-1} t_k^{-1} e^{-a_k n_k(\chi_k+\varepsilon_k)} \quad \text{and} \quad \widehat{L}_{i-1} \stackrel{\text{def}}{=} \prod_{k=1}^{i-1} s_k^{-1} t_k e^{-a_k n_k(\chi_k-\varepsilon_k)}$$

($\widetilde{K}_i, L_i, \widehat{L}_i$ depend on $i$ only). We can now define the Cantor set

$$A \stackrel{\text{def}}{=} \bigcap_{i\geq 1} A_i \tag{31}$$

(compare Figure 4). We will summarize its geometric and dynamic properties in the following lemma.

**Lemma 12.** *The above defined set $A$ possesses the following properties:*

i) *(Lyapunov exponents on the islands) for $x \in A_i$ and $k \leq n_i$ we have*
$$K_i^{-1} e^{a_i k(\chi_i-\varepsilon_i)} < |(f^{a_i k})'(f^{m_{i-1}}(x))| < K_i e^{a_i k(\chi_i+\varepsilon_i)},$$

ii) *(Lyapunov exponents on the bridges) for $x \in A_i$ and $k \leq b_i$ we have*
$$w_i^k < |(f^k)'(f^{m_{i-1}+a_i n_i}(x))| < W^k,$$

iii) *we have*
$$K_i^{-1} \widetilde{K}_{i-1}^{-1} L_{i-1} r_i \, e^{-a_i n_i(\chi_i+\varepsilon_i)} \leq \operatorname{diam} A_{ij_1\ldots j_{i-1}} \leq K_i \widetilde{K}_{i-1} \widehat{L}_{i-1} r_i \, e^{-a_i n_i(\chi_i-\varepsilon_i)},$$

iv) *there are at least* $\exp(a_i n_i(h_i-\varepsilon_i))$ *sets $A_{i\,j_1\ldots j_{i-1}}$ contained in every set $A_{(i-1)j_1\ldots j_{i-2}}$,*

v) *for any $k_{i-1} \neq j_{i-1}$ and any $i, j_1, \ldots, j_{i-2}$ we have*
$$\rho(A_{ij_1\ldots j_{i-1}}, A_{ij_1\ldots j_{i-2}k_{i-1}}) \geq \widetilde{K}_{i-1}^{-1} K_i^{-1} \operatorname{diam} A_{ij_1\ldots j_{i-1}}$$

vi) *we have*
$$\mu_i\left(\bigcup_j \widehat{A}_{ij}\right) \geq K_i^{-1},$$

vii) *for any $x \in V_i$ and $r > 0$ we have*

$$\mu_i\left(B(x,r) \cap \bigcup_j \widehat{A}_{ij}\right) \le K_i\, r^{d_i - \varepsilon_i},$$

viii) *we have*

$$K_i^{-1} e^{-a_i n_i (h_i + \varepsilon_i)} \le \mu_i\left(\widehat{A}_{ij}\right) \le e^{-a_i n_i (h_i - \varepsilon_i)}.$$

Additional assumptions on $\{n_i\}_i$ and properties guaranteed by Lemma 11 will enable us to estimate the Hausdorff and packing dimensions of the constructed set and describe the upper and lower Lyapunov exponents at each point. This will be done in the following.

*Construction of a w-measure on the Cantor set.* We continue to consider the Cantor set $A$ constructed in (31). Let $\mu$ be the probability measure which on each level $i$ is distributed on the cylinder sets $A_{i\, j_1 \ldots j_{i-1}}$ of level $i$ in the following way

$$\mu(A_{i\, j_1 \ldots j_{i-1}}) \stackrel{\text{def}}{=} \mu(A_{i-1\, j_1 \ldots j_{i-2}}) \frac{\mu_i(\widehat{A}_{i\, j_{i-1}})}{\sum_k \mu_i(\widehat{A}_{i\, k})}\,. \tag{32}$$

We extend the measure $\mu$ arbitrarily to the Borel $\sigma$-algebra of $A$. We call the probability measure a *w-measure* with respect to the sequence $\{f|_{\Lambda_i}, \phi_i, \mu_i\}_i$.

After these preparations, we are now able to prove Theorem 3. We will continue to use the notations done in the above construction of the set $A$.

*Proof of Theorem 3.* In the course of the following proof we will choose some sequence $\{\varepsilon_k\}_k$ and then construct a sequence of positive integers $\{n_i\}_i$. Here each of those numbers $n_i$ has to satisfy several conditions that depend on $\{\varepsilon_k\}_k$, the parameters of the hyperbolic sets $\{\Lambda_k\}_k$ and the measures $\{\mu_k\}_k$, and the previously chosen numbers $n_j$, $j = 1, 2, \ldots, i-1$. Naturally, it is always possible to satisfy all those conditions at the same time.

We will first check that the Cantor set defined in (31) satisfies

$$A \subset \mathcal{L}\Big(\liminf \chi(\mu_i), \limsup \chi(\mu_i)\Big)$$

(under some appropriate assumptions about $\{n_i\}_i$). Then we will estimate the Hausdorff and packing dimensions of $A$ using the w-measure $\mu$ defined in (32).

Let us first consider the Lyapunov exponent at a point in the set $A$. Let

$$\ell_n(x) \stackrel{\text{def}}{=} \frac{1}{n} \log |(f^n)'(x)|.$$

For $n \le a_1 n_1$ we have

$$\ell_n(x) \in \left(\chi_1 - \varepsilon_1 - \frac{1}{n}\log K_1 - O(\frac{a_1}{n}), \chi_1 + \varepsilon_1 + \frac{1}{n}\log K_1 + O(\frac{a_1}{n})\right).$$

We know that $f^{n_1+k}(x)$ stays in distance at least $\delta_1$ from the critical points for every $k \le b_1$. Hence, for $a_1 n_1 < n \le a_1 n_1 + b_1$ we have

$$\ell_n(x) = \frac{a_1 n_1}{n} \ell_{a_1 n_1}(x) + \frac{n - a_1 n_1}{n} O(|\log w_1|) \tag{33}$$

and for $n_1$ big enough (in comparison with $\log K_1$ and $b_1 |\log w_1|$) the right hand side of (33) is between $\chi_1 - 2\varepsilon_1$ and $\chi_1 + 2\varepsilon_1$. For $m_{i-1} < n \le m_{i-1} + a_i n_i$ we have

$$\ell_n(x) = \frac{m_{i-1}}{n} \ell_{m_{i-1}}(x) + \frac{1}{n} \log |(f^{n-m_{i-1}})'(f^{m_{i-1}}(x))| + O(\frac{a_i}{n})$$

while for $m_{i-1} + a_i n_i < n \le m_i$ we have

$$\ell_n(x) = \frac{m_{i-1}}{n} \ell_{m_{i-1}}(x) + \frac{1}{n} \log |(f^{a_i n_i})'(f^{m_{i-1}}(x))| + \frac{n - m_{i-1} - a_i n_i}{n} O(|\log w_i|).$$

Estimating the second summand using Lemma 12 i) and the third one using Lemma 12 ii) and assuming that $n_i$ is big enough (in comparison with $n_{i-1}$, $b_i |\log w_i|$, $a_{i+1}$, and $\log K_i$), we can first prove that

$$|\ell_{m_i}(x) - \chi_i| < 2\varepsilon_i$$

(by induction) and then prove that for all $m_i < n < m_{i+1}$ we have

$$\left| \ell_n(x) - \left( \frac{m_i}{n} \chi_i + \frac{n - m_i}{n} \chi_{i+1} \right) \right| < 2(\varepsilon_i + \varepsilon_{i+1}).$$

As the upper (the lower) Lyapunov exponent of $x$ equals the upper (the lower) limit of $\ell_n(x)$ as $n \to \infty$, we have shown that every point $x \in A$ satisfies

$$\underline{\chi}(x) = \liminf_{i\to\infty} \chi(\mu_i), \quad \overline{\chi}(x) = \limsup_{i\to\infty} \chi(\mu_i).$$

Let us now estimate the Hausdorff and packing dimensions of the set $A$. To do so, we will apply the Frostman lemma. Let us calculate the pointwise dimension of the measure $\mu$ defined in (32) at an arbitrary point $x \in A$. Notice that we can write

$$\{x\} = \bigcap_{i=1}^{\infty} A_{i\, j_1 \ldots j_{i-1}}$$

for some appropriate symbolic sequence $(j_1 j_2 \ldots)$. By Lemma 12 v), the ball $B(x,r)$ does not intersect any of the sets $A_{i\, k_1 \ldots k_{i-1}}$ if $k_1 \ldots k_{i-1} \ne j_1 \ldots j_{i-1}$whenever we have

$$r \le R_i(x) \stackrel{\text{def}}{=} \widetilde{K}_{i-1}^{-1} K_i^{-1} \operatorname{diam} A_{i\, j_1 \ldots j_{i-1}}.$$

Consider $R_{i+1}(x) < r \le R_i(x)$. We have

$$\begin{aligned} \mu(B(x,r)) &\le \sum_{A_{i+1\, j_1 \ldots j_{i-1} k} \cap B(x,r) \ne \varnothing} \mu(A_{i+1\, j_1 \ldots j_{i-1} k}) \\ &\le \sum_{A_{i+1\, j_1 \ldots j_{i-1} k} \subset B\left(x, r + \max_\ell \operatorname{diam} A_{i+1\, j_1 \ldots j_{i-1} \ell}\right)} \mu(A_{i+1\, j_1 \ldots j_{i-1} k}). \end{aligned}$$

Let $D_i \stackrel{\text{def}}{=} \max_\ell \operatorname{diam} A_{i+1\, j_1\dots j_{i-1}\ell}$. We continue with

$$\begin{aligned}\mu(B(x,r)) &\le \frac{\mu(A_{i\, j_1\dots j_{i-1}})}{\sum_\ell \mu_{i+1}(\widehat{A}_{i+1\,\ell})} \sum_{k:\ \widehat{A}_{i+1\,k}\subset f^{m_i}(B(x,r+D_i))} \mu_{i+1}(\widehat{A}_{i+1\,k})\\ &\le \frac{\mu(A_{i\, j_1\dots j_{i-1}})}{\sum_\ell \mu_{i+1}(\widehat{A}_{i+1\,\ell})} \mu_{i+1}\left(B\left(f^{m_i}(x), L_i^{-1}(r+D_i)\right)\right).\end{aligned}$$

Using Lemma 12 vi) and vii) we can estimate

$$\begin{aligned}\mu(B(x,r)) &\le \mu(A_{i\, j_1\dots j_{i-1}})K_{i+1}\, \mu_{i+1}\left(B\left(f^{m_i}(x), L_i^{-1}(r+D_i)\right)\right)\\ &\le \mu(A_{i\, j_1\dots j_{i-1}}) \frac{K_{i+1}^2}{L_i^{d_{i+1}-\varepsilon_{i+1}}} (r+D_i)^{d_{i+1}-\varepsilon_{i+1}}. \qquad (34)\end{aligned}$$

Let

$$\Xi \stackrel{\text{def}}{=} \mu(A_{i\, j_1\dots j_{i-1}}) \frac{K_{i+1}^2}{L_i^{d_{i+1}-\varepsilon_{i+1}}}. \qquad (35)$$

Using Lemma 12 vi) and viii), we can now estimate the first factor in (35) and we obtain

$$\begin{aligned}\log \mu(A_{i\, j_1\dots j_{i-1}}) &\le \sum_{k=2}^{i} \log \frac{\mu(A_{k\, j_1\dots j_{k-1}})}{\mu(A_{k-1\, j_1\dots j_{k-2}})}\\ &\le \sum_{k=2}^{i} \left(\log \mu_k\left(\widehat{A}_{k\, j_{k-1}}\right) + \log K_k\right)\\ &\le -\sum_{k=2}^{i} (a_k n_k (h_k - \varepsilon_k) + \log K_k) \le -a_i n_i (h_i - 2\varepsilon_i)\end{aligned}$$

provided that we assume that $n_i$ has been chosen big enough (in comparison with $a_k$, $n_k$, $\log K_k$, $k < i$). For the second factor in (35) we yield

$$-\log L_i = \sum_{k=1}^{i} \log s_k + \log t_k + a_k n_k (\chi_k + \varepsilon_k) \le a_i n_i (\chi_i + 2\varepsilon_i).$$

provided that we assume that $n_i$ has been chosen big enough (in comparison with $a_k$, $n_k$, $\log K_k$, and $s_k$, $k < i$). This implies that

$$\log \Xi = a_i n_i \chi_i\, (d_{i+1} - d_i) + n_i\, O(\varepsilon_i, \varepsilon_{i+1}).$$

Further, using (30) and Lemma 12 iii) we obtain

$$\frac{\operatorname{diam} A_{i+1\, j_1\dots j_{i-1}\ell}}{R_{i+1}} \le \widetilde{K}_i^3\, K_{i+1}^3 \prod_{k=1}^{i+1} e^{2n_k\varepsilon_k}$$

and hence

$$D_i \le e^{3n_{i+1}\varepsilon_{i+1}}\, R_{i+1}.$$

In addition, by Lemma 12 iii) we have

$$\log R_{i+1} = a_{i+1}n_{i+1}(\chi_{i+1} + O(\varepsilon_{i+1}))$$

and

$$\log D_i \leq -a_{i+1}n_{i+1}(\chi_{i+1} + O(\varepsilon_{i+1})).$$

assuming that $n_i$ has been chosen big enough. To estimate (34), we consider now the two cases: a) that $r \leq D_i$ and b) that $r > D_i$. In case a) we have that

$$\mu(B(x,r)) \leq \Xi\,(2D_i)^{d_{i+1}-\varepsilon_{i+1}}.$$

Note that the the right hand side of this estimate no longer depends on $r$ and hence,

(36)

$$\frac{\log \mu(B(x,r))}{\log r} \geq \frac{\log\left(\Xi \cdot (2D_i)^{d_{i+1}-\varepsilon_{i+1}}\right)}{\log R_{i+1}} = O\left(\frac{n_i}{n_{i+1}}\right) + d_{i+1} + O(\varepsilon_{i+1}).$$

In case b) we have

$$\mu(B(x,r)) \leq \Xi\,(2r)^{d_{i+1}-\varepsilon_{i+1}}$$

and hence

$$\frac{\log \mu(B(x,r))}{\log r} \geq \frac{\log\left(\Xi \cdot (2\,r)^{d_{i+1}-\varepsilon_{i+1}}\right)}{\log r}.$$

We again need to distinguish two cases: If $d_{i+1} > d_i$, then

$$\frac{\log \mu(B(x,r))}{\log r} \geq \frac{\log \mu(B(x,R_i))}{\log R_i} \geq d_i + O(\varepsilon_i, \varepsilon_{i+1}), \tag{37}$$

while in the case $d_i \geq d_{i+1}$ we have

$$\frac{\log \mu(B(x,r))}{\log r} \geq \frac{\log \mu(B(x,R_{i+1}))}{\log R_{i+1}} \geq d_{i+1} + O\left(\frac{n_i}{n_{i+1}}\right) + O(\varepsilon_i, \varepsilon_{i+1}). \tag{38}$$

The estimations (36), (37), and (38) prove that if the sequence $\{n_i\}_i$ increases fast enough then for every $x \in A$ we have

$$\underline{d}_\mu(x) \geq \liminf_{i\to\infty} d_i \quad \text{and} \quad \overline{d}_\mu(x) \geq \limsup_{i\to\infty} d_i.$$

Hence, applying the Frostman lemma, we obtain

$$\dim_{\mathrm{H}} A \geq \liminf_{i\to\infty} d_i \quad \text{and} \quad \dim_{\mathrm{P}} A \geq \limsup_{i\to\infty} d_i,$$

and hence the assertion of Theorem 3 follows. □

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