

Estimates of the topological entropy from below for continuous self-maps on some compact manifolds

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Abstract

Extending our results of [17], we confirm that Entropy Conjecture holds for every continuous self-map of a compact $K(\pi, 1)$ manifold with the fundamental group π torsion free and virtually nilpotent, in particular for every continuous map of an infra-nilmanifold. In fact we prove a stronger version, a lower estimate of the topological entropy of a map by the logarithm of the spectral radius of an associated "linearization matrix" with integer entries.

From this, referring to known estimates of Mahler measure of polynomials, we deduce some absolute lower bounds for the entropy.

1 Introduction

Let M be a compact manifold and d be a metric on M consistent with the topology. Let $f : M \rightarrow M$ a continuous self-map of M . The topological entropy $h_{\text{top}}(f)$, denoted shortly by $\mathbf{h}(f)$, is defined as $\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} 1/n \log \sup \#Q$, with the supremum taken over all Q being (ϵ, n) -separated. Q is called (ϵ, n) -separated, if for every two distinct points $x, y \in Q$, $\max_{j=0, \dots, n} d(f^j(x), f^j(y)) \geq \epsilon$. In fact $\mathbf{h}(f)$ does not depend on the metric (cf. [28]).

Entropy Conjecture, denoted shortly as EC, says that the topological entropy of f is larger or equal to the logarithm of the spectral radius of the linear operator induced by f on the linear spaces of cohomology of M with real coefficients. It was posed by M. Shub in seventies who asked what suppositions on f and M imply EC.

A group π is called *virtually nilpotent* if it contains a finite index nilpotent subgroup. We prove the following

Theorem A *Assume that a compact manifold M is a $K(\pi, 1)$ -space with the fundamental group π being torsion free and virtually nilpotent. Then EC holds for every continuous self-map f of M . In particular EC holds for every continuous self-map of any compact infra-nilmanifold.*

- We can assume that a nilpotent finite index subgroup in a virtually nilpotent group π is a normal subgroup.

Indeed, for any pair of groups $K \subset L$, where K has a finite index in L , one has a homomorphism $\rho : L \rightarrow \text{Sym}(L/K)$ into the symmetry group of the quotient space. Then $\ker \rho$ is a normal subgroup of L , it has finite index, and is contained in K .

- One can replace the assumption that π is virtually nilpotent by the assumption that π has polynomial growth, [9].

- Theorem A is a step towards proving a conjecture by A. Katok [11] saying that EC holds for every continuous map if the universal cover of M is homeomorphic to an Euclidean space \mathbb{R}^d .

- One can even ask whether EC holds for every continuous self-map of a $K(\pi, 1)$ compact manifold (or a finite CW-complex).

1.1. Affine maps

We refer to the following theorem by A. Malcev and L. Auslander about the *existence of a model* [7, p.76]:

Assume that π is finitely generated torsion free virtually nilpotent group. Then it contains a finite index maximal nilpotent normal subgroup Γ . This subgroup can be embedded as a lattice, i.e. a discrete co-compact subgroup, in a connected, simply connected nilpotent Lie group G . The embedding can be extended to an embedding of π in the group $\text{Aff}(G)$ of affine mappings of G , so that $\pi \cap G = \Gamma$. More precisely, if $\mathbf{C} \subset \text{Aut}(G)$ denotes the maximal compact subgroup of the group of automorphisms of G , then $\pi \subset G \rtimes \mathbf{C} \subset G \rtimes \text{Aut}(G)$; this embedding of π is called an *almost Bieberbach group*.

It follows then from the definition that π acts on G properly discontinuously. (First note that if $\alpha \in \pi$ has a fixed point $z \in G$, then $\alpha^\ell(z) = z$ and $\alpha^\ell \in \Gamma$ for $\ell = \#(\pi/\Gamma)$. Then $\alpha^\ell = e$ the unity of π , hence $\alpha = e$ by the assumption that π is torsion free.)

The quotient manifold $\mathbf{IN} = G/\pi$ is called an *infra-nilmanifold*. It is regularly finitely covered by the nilmanifold $\mathbf{N} = G/\Gamma$, with the deck transformation group equal to $H = \pi/\Gamma$.

Note that every compact manifold finitely covered by a nilmanifold, in particular every infra-nilmanifold, satisfies the assumptions of Theorem A. Indeed, G is homeomorphic to \mathbb{R}^d , where $d = \dim M$ (cf. [22]). If π were not torsion free, a cyclic subgroup generated by an element $\{g\} \simeq \mathbb{Z}_p$ of a prime order would act freely on \mathbb{R}^d . The latter is impossible, as follows from the Smith theory (cf. [1]).

The image of the embedding of π into $G \rtimes \mathbf{C} \subset \text{Aut}(G)$ will be denoted by $\pi_{\mathbf{IN}}$. It is a deck transformation group of the cover $p_{\mathbf{IN}} : G \rightarrow G/\pi_{\mathbf{IN}}$ and, distinguishing an arbitrary $z \in G$, it can be identified in a standard way with the fundamental group $\pi_1(\mathbf{IN}, p_{\mathbf{IN}}(z))$ of \mathbf{IN} .

Consider now any M being $K(\pi, 1)$ as in Theorem A. The group π acts properly discontinuously on the universal cover space \tilde{M} and it can be identified with π_M , which

is the deck transformation group of the universal cover $p_M : \tilde{M} \rightarrow M$.

(Similarly to $\mathbf{N} \rightarrow \mathbf{IN}$, we have a regular finite cover $\tilde{M}/\Gamma \rightarrow M = \tilde{M}/\pi$, with the deck transformation group H .)

Every continuous map $f : M \rightarrow M$ induces an endomorphism $F = F_f$ of π_M , unique up to an inner automorphism. To define this use $f_{\#} : \pi_1(M, z) \rightarrow \pi_1(M, f(z))$ between the fundamental groups and standard identifications of these groups with π_M . For more details see Section 3.

When we identify π with $\pi_{\mathbf{IN}}$, we can consider F as an endomorphism of $\pi = \pi_{\mathbf{IN}}$.

By K. B. Lee, [13, Th.1.1 and Cor 1.2], there exists an affine self-map $\Phi = \Phi_f = (b, B)$ of G , with $b \in G$, $B \in \text{End}(G)$, such that for all $x \in G$, $\alpha \in \pi_{\mathbf{IN}}$ we have

$$\Phi(\alpha(x)) = F(\alpha)(\Phi(x)). \quad (1)$$

Hence, in view of (1), there exists a factor $\phi = \phi_f$ of Φ on \mathbf{IN} under the action of $\pi_{\mathbf{IN}}$. In general one calls factors of affine Φ on G satisfying (1), affine maps on \mathbf{IN} . In particular, ϕ_f is an affine map on \mathbf{IN} .

Since both M and \mathbf{IN} are $K(\pi, 1)$ -spaces, they are homotopically equivalent, [26]. Let $h : M \rightarrow \mathbf{IN}$ be a homotopy equivalence that implements an identification between π_M and $\pi_{\mathbf{IN}}$. Then $\phi \circ h$ is homotopic to $h \circ f$, see [26, Section 8.1] and Section 3. Note, that if $M = \mathbf{IN}$, i.e. f is a self-map of an infra-nilmanifold \mathbf{IN} , then ϕ is homotopic to f , [13].

Let $\Phi = \Phi_f = (b, B)$ be an affine self-map of G associate to $f : M \rightarrow M$ (in fact to the homomorphism F). The differential at the unity, $D(B)(e)$, is an endomorphism of \mathcal{G} , the Lie algebra of G . Denote $D(B)(e)$ by D_f . Let $\sigma(D_f) = \{\lambda_1, \dots, \lambda_d\}$ be the set of all its eigenvalues counted with multiplicities. Consider $\text{sp}(\wedge^* D_f)$, the spectral radius of the full exterior power $\wedge^* D_f = \bigoplus_0^d \wedge^k D_f$ of D_f . Of course it is equal to $\prod_{j:|\lambda_j|>1} |\lambda_j|$, where $\lambda_j \in \sigma(D_f)$, provided at least one λ_j has absolute value larger than 1. Otherwise it is equal to 1 (in $\wedge^0 D_f$).

By [16], G containing Γ as a lattice, is unique. By [13] (see also Section 2) the endomorphism B is unique up to an inner automorphism of G , so its choice does not influence the spectra.

1.2. Linearization matrices

One can assign to an endomorphism F_f of the deck transformation group π (or $f_{\#}$ for π identified with the fundamental group) not only a linear map $D_f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\mathcal{G} \cong \mathbb{R}^d$, but also an integer $d \times d$ matrix $A_{[f]}$ called the linearization of the homotopy class $[f]$. Note that the inner conjugacy classes of endomorphisms of π are in one-one correspondence with homotopy classes $[f]$ of self-maps of a $K(\pi, 1)$ -space.

In general an endomorphism $F : \pi \rightarrow \pi$ does not preserve the nilpotent subgroup $\Gamma \triangleleft \pi$. But Γ contains a subgroup $\Gamma' \triangleleft \pi$ such that Γ' is nilpotent, has finite index in π and is invariant for F . (Γ' will be defined in Proposition 5). See also [14]. Since Γ' has finite index in Γ it is also a lattice in G . By (1) the endomorphism $B : G \rightarrow G$ is an extension of $F : \Gamma' \rightarrow \Gamma'$, see the end of Section 2.

Let G be a nilpotent connected simple-connected Lie group and Γ a lattice in G . The descending central series of commutators $G_0 = G$, $G_{i+1} = [G, G_i]$ of G is finite with the last group trivial $G_k = \{e\}$. For each i define $\Gamma_i = G_i \cap \Gamma$. Each Γ_i is a lattice in G_i , as follows from Malcev theorem [16, Theorem 1]. Consequently, if $B : G \rightarrow G$ is a homomorphism of G preserving a lattice Γ it preserves each subgroup Γ_i . Thus it induces an endomorphism B_i on each factor group Γ_i/Γ_{i+1} , $0 \leq i \leq k$. Clearly Γ_i/Γ_{i+1} is abelian and torsion free of dimension d_i . Therefore the action of B_i on $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}^{d_i}$ yields an integer $d_i \times d_i$ matrix A_i which is uniquely defined up to a choice of basis, i.e. up to a conjugation by a unimodular matrix.

Let us apply this construction to Γ' , using the fact it is invariant under B . We conclude with $A_{[f]} := \bigoplus_{i=0}^k A_i$, which defines an integer $d \times d$ matrix, because $\bigoplus_{i=0}^k d_i = d$. (See [12] and [10] for some extensions to solvmanifolds.)

It is easy to show that $\sigma(A_{[f]}) = \sigma(DB(e))$, thus consequently $\sigma(\wedge^* A_{[f]}) = \sigma(\wedge^* D_f)$, which implies $\text{sp}(\wedge^* A_{[f]}) = \text{sp}(\wedge^* D_f)$ (cf. [17], [10]).

In fact the matrix $A_{[f]}$ can be defined directly, using $F = f_{\#} : \pi \rightarrow \pi$, without constructing G . One can use the series of *isolators* $\sqrt[\ell]{\Gamma'_i} = \{x \in \Gamma' : (\exists \ell > 0) x^\ell \in \Gamma'_i\}$, for Γ'_i being commutators in the descending central series for Γ' , i.e. $\Gamma'_{i+1} = [\Gamma', \Gamma'_i]$, see [2, Section 3] and for example [19, Lemma 11.1.8]. In fact $\sqrt[\ell]{\Gamma'_i} = G_i \cap \Gamma'$ defined above, see [3, Lemma 1.2.6].

1.3. Conclusion

We are in the position to formulate a sharper version of Theorem A, namely

Theorem B *For every continuous self-map f of a compact manifold M which is a $K(\pi, 1)$ -space with the group π torsion free and virtually nilpotent,*

$$\mathbf{h}(f) \geq \log \text{sp}(\wedge^* D_f) = \log \text{sp}(\wedge^* A_{[f]}).$$

In the case M is an infra-nilmanifold the equality holds for every affine map $\phi : M \rightarrow M$, a factor of an affine Φ satisfying (1), in particular for ϕ_f . In consequence for every continuous self-map f of M we have $\mathbf{h}(f) \geq \mathbf{h}(\phi_f)$, i.e. ϕ_f minimizes entropy in the homotopy class of f .

Maybe considering of M is not needed and it is sufficient to consider only **IN**. Since the topological entropy is an invariant of conjugation by a homeomorphism, this would follow from Borel conjecture, which states that the fundamental group π of a manifold which is a $K(\pi, 1)$ -space determines M up to homeomorphism, This has been confirmed by Farrell and Jones [5] for a class of groups that contains the almost-Bieberbach groups, except in dimension 3.

Formulating Theorems A and B we have followed a suggestion by M. Shub [24], to assume a discrete group point of view. Having given an endomorphism $F = f_{\#} : \pi \rightarrow \pi$ of a finitely generated torsion free virtually nilpotent group, we associate to it a linear operator D_f , or an integer $d \times d$ matrix $A_{[f]}$. As suggested in [24] the logarithm of spectral radius of $\text{sp}(\wedge^* D_f)$, or $\text{sp}(\wedge^* A_{[f]})$, is "a kind of volume growth" of $f_{\#}$. In Theorem A

it is replaced by the spectral radius of the map induced on the real cohomologies of the group π (that is the real cohomologies of M).

We shall present two proofs of Theorems A and B.

The first one, in Section 2, concludes Theorems A and B from analogous theorems in [17], with $f : \mathbf{N} \rightarrow \mathbf{N}$ a continuous map of a compact nilmanifold. However this proof of Theorems A and B does not work in dimension 3.

The second proof, in Section 3, holds for f on M and uses only a homotopy equivalence between M and \mathbf{IN} . It directly repeats the arguments of [17].

An important observation is that $A_{[f]}$ is an integer matrix. This allows, in Section 4, to prove absolute estimates from below for $\text{sp}(\wedge^* A_{[f]})$, where f is an expanding map of a compact manifold (without boundary) or an Anosov diffeomorphism of a compact infra-nilmanifold. The latter uses number theory results estimating the Mahler measure of an integer polynomial.

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2 Entropy Conjecture on infra-nilmanifolds

The proof of Theorems A and B we present in this section will hold for M finitely covered by a nilmanifold and follows from two standard facts and the main theorem of [17] in which the topological entropy of a continuous map of nilmanifold is estimated by the corresponding quantities. We begin with the following

Proposition 1 *Suppose that we have finite cover (\hat{M}, p, M) of compact metric spaces, i.e. \hat{M} is the total space of the cover, M the base space, and $p : \hat{M} \rightarrow M$ the covering space. Let a pair (\hat{f}, f) , $\hat{f} : \hat{M} \rightarrow \hat{M}$, $f : M \rightarrow M$, be a map of a this covering, i.e. $p\hat{f} = fp$. Then*

$$\mathbf{h}(f) = \mathbf{h}(\hat{f}).$$

PROOF: It is elementary and is given in [6, Section I]. Briefly: The p -preimage of an (n, ϵ) - f -separated set in M is (n, ϵ) - \hat{f} -separated in a metric \hat{d} on \hat{M} which is the lift of a metric d on M chosen to define the entropy. Hence $\mathbf{h}(f) \leq \mathbf{h}(\hat{f})$. (In fact only the continuity of p was substantial in this proof).

Conversely, let Q be an (n, ϵ) - \hat{f} -separated set in \hat{M} consisting of points in a ball $B(z, \epsilon/2)$. Let $\delta > 0$ be a constant such that p is injective on every ball in \hat{M} of radius δ . We prove that the set $p(Q)$ is (n, ϵ) - f -separated. Indeed, take $\epsilon < \delta$ and suppose that for $x, y \in Q$ we have $d(f^j(p(x)), f^j(p(y))) < \epsilon$ for all $j = 0, 1, \dots, n$. Let $j_0 \geq 0$ be the smallest $j \leq n$ such that $d(\hat{f}^{j_0}(x), \hat{f}^{j_0}(y)) \geq \epsilon$. Then $d(\hat{f}^{j_0}(x), \hat{f}^{j_0}(y)) \geq \delta - \epsilon$ (i.e. projections by p are close to each other but the points are in different components of preimages of a

small ball under the cover map). This is not possible for $j = 0$ by $Q \subset B(z, \epsilon/2)$. If it happens for another j it means that the \hat{f} image of two points within the distance $< \epsilon$ have distance $\geq \delta - \epsilon$, which for ϵ small enough contradicts the uniform continuity of \hat{f} . \square

The same trick as above has been used in [17, Proof of Theorem 2.2], to which we refer here in Section 3, in the setting of the projection p from the universal cover of M to M .

Proposition 2 *Let (\hat{M}, p, M) be a regular finite cover over a finite CW-complex, e.g. over a compact manifold with the deck transformation group H . Let next $f : M \rightarrow M$ be a continuous map and $\hat{f} : \hat{M} \rightarrow \hat{M}$ its lift to \hat{M} .*

Then $\sigma(H^(f)) \subset \sigma(H^*(\hat{f}))$ and consequently $\text{sp}(H^*(f)) \leq \text{sp}(H^*(\hat{f}))$.*

PROOF: The action of H on \hat{M} defines an action on the real cohomology space for any $g \in H$ given by $g \mapsto H^*(g) : H^*(\hat{M}) \rightarrow H^*(\hat{M})$. Since M is the orbit space of a free action of a finite group H on \hat{M} , we have $H^*(M; \mathbb{R}) \simeq H^*(\hat{M}; \mathbb{R})^H$, when the latter is the fixed point of action of H on the real cohomology spaces, i.e. the image of a linear projection $\frac{1}{|G|} \sum_{g \in H} H^*(g)$ (cf. [1] Section III.2). Moreover this isomorphism is given by the map $H^*(p) : H^*(M; \mathbb{R}) \rightarrow H^*(\hat{M}; \mathbb{R})$ induced by the covering projection p . Since $p\hat{f} = fp$, we have $H^*(\hat{f})H^*(p) = H^*(p)H^*(f)$. This means that the linear subspace $H^*(M; \mathbb{R}) = H^*(\hat{M}; \mathbb{R})^H = \text{im } H^*(p)$ is preserved by $H^*(\hat{f})$. Consequently $H^*(f)$ can be identified with a restriction of a linear map $H^*(\hat{f})$ to an invariant linear subspace, which shows that $\sigma(H^*(f)) \subset \sigma(H^*(\hat{f}))$, and consequently proves the statement. (Note that \hat{f} is not an H -equivariant map in general.) \square

An immediate corollary is

Corollary 3 *Let (\hat{M}, p, M) be a regular finite cover over a finite CW-complex, e.g. over a compact manifold. Let $f : M \rightarrow M$ be a continuous map which has a lift \hat{f} to \hat{M} and EC holds for \hat{f} , then EC holds for f .*

A problem in the general situation is the existence of the lift \hat{f} . Fortunately one can go around it if \hat{M} is a nilmanifold, as follows

Definition 4 *Let $\Gamma \triangleleft \pi$ be a normal nilpotent subgroup of finite index in π and let s be an endomorphism of π . We say that a group $\Gamma' \subset \Gamma$ is s -admissible if*

- 1) $s(\Gamma') \subset \Gamma'$,
- 2) $\Gamma' \subset \Gamma$ is normal in π and $[\Gamma : \Gamma'] < \infty$, i.e. Γ' has finite index in Γ .

Proposition 5 *For a nilpotent group Γ normal and of finite index in a group π there exists a group $\Gamma' \subset \Gamma$, $\Gamma' \triangleleft \pi$, admissible for every endomorphism s of π . (Sometimes such Γ' is called a fully characteristic subgroup).*

PROOF: Repeat verbatim the argument of Lemma 3.1 of [14] and define

$$\Gamma' = \text{group generated by } \{\gamma^k : \gamma \in \pi\}, \text{ where } k \text{ is the order of } H = \pi/\Gamma.$$

This is clearly a normal subgroup of π preserved by every endomorphism of π .

Next we define a group

$$\Gamma(k) := \text{group generated by } \{x^k\} : x \in \Gamma.$$

Of course $\Gamma(k) \subset \Gamma'$. It is enough to show that $\Gamma(k)$ is of finite index in Γ . Apply an argument used in [14]: Since Γ is nilpotent, it is polycyclic, [22]. But for any polycyclic group Γ the corresponding group $\Gamma(k)$ has a finite index, cf. [22, Lemma 4.1]. In particular adapting the argument of Lemma 4.1 of [22] one shows the assertion for a nilpotent group, by an induction over the length of nilpotency. \square

Note that the number k used to define the group $\Gamma(k)$ is not unique, e.g. we can take any its multiple getting a smaller group with the required property. To get a larger group Γ' than that of [14] we can use k equal to the $\text{LCM}\{\#h : h \in H\}$, where $\#h$ is the order of h , instead of $k = \#H$, the order of H .

Corollary 6 *For any compact manifold M finitely covered by a nilmanifold \mathbf{N} there exists a regular finite cover $(\hat{\mathbf{N}}, p, M)$ of M by a nilmanifold $\hat{\mathbf{N}}$ such that every continuous map $f : M \rightarrow M$ has a lift $\hat{f} : \hat{\mathbf{N}} \rightarrow \hat{\mathbf{N}}$, i.e. $p\hat{f} = fp$.*

PROOF: The assertion follows from Proposition 5 for π the fundamental group of M , $\mathbf{N} = G/\Gamma$ for Γ a subgroup of π and $s = f_{\#}$. We can assume that Γ is normal in π , see Introduction. We define $\hat{\mathbf{N}} := G/\Gamma'$. A lift \hat{f} exists, since the homomorphism $f_{\#} : \pi \rightarrow \pi$ preserves Γ' identified with $p_{\#}\pi_1(\hat{\mathbf{N}})$, see [26]. \square

Together with Corollary 3 and EC for nilmanifolds, [17], this proves EC, i.e. Theorem A, for all continuous self maps of M finitely covered by nilmanifolds, in particular for all M being infranilmanifolds.

PROOF OF THEOREM B (for M finitely covered by a nilmanifold G/Γ):

Consider $\Gamma' \subset \Gamma$ invariant for every endomorphism of π , defined above. Note that the equality (1) for $\alpha = g \in \Gamma'$ takes the form

$$B(g(x))b = F(g)B(x)b.$$

As the left hand side expression is equal to $B(g)B(x)b$, we get $F(g) = B(g)$, i.e. $F|_{\Gamma'} = f_{\#}|_{\Gamma'} = B|_{\Gamma'}$. (In fact B in (b, B) was found in [13] as an extension of $f_{\#}|_{\Gamma'}$ to G .)

This proof is similar to [13, Proposition 1.4]. Note that by the cause of F also B has been found up to an inner automorphism.

This allows to define $A_{[f]}$ as in Subsection 1.2.

By Corollary 6 f lifts to $\hat{f} : \hat{\mathbf{N}} \rightarrow \hat{\mathbf{N}}$. We have $\mathbf{h}(\hat{f}) \geq \log \text{sp}(\wedge^* A_{[\hat{f}]})$ by [17]. This implies $\mathbf{h}(f) \geq \log \text{sp}(\wedge^* A_{[f]})$. Indeed, $A_{[f]} = A_{[\hat{f}]}$ since both matrices are defined with respect to the same group Γ' , see Subsection 1.2, and $\mathbf{h}(\hat{f}) = \mathbf{h}(f)$ by Proposition 1.

Now $\mathbf{h}(f) \geq \log \text{sp}(\wedge^* DB(e))$ follows from the equality $\text{sp} D_f = \text{sp} A_{[f]}$, discussed already in Subsection 1.2.

Finally, if $M = G/\pi$ is an infra-nilmanifold, then the factor $\hat{\phi}$ of (b, B) to G/Γ' is a lift of the affine map ϕ (homotopic to f , provided f was given a priori). The asserted equality $\mathbf{h}(\phi) = \log \text{sp}(\wedge^* A_{[\phi]})$ follows from $\mathbf{h}(\hat{\phi}) = \log \text{sp}(\wedge^* A_{[\hat{\phi}]})$ proved in [17] and from Proposition 1 for $\hat{\phi}$ and ϕ (in place of \hat{f} and f). As in the case of f we have $A_{[\phi]} = A_{[\hat{\phi}]}$. \square

3 Another proof of EC

Now we provide another proof of Theorems A and B without additional assumptions by showing that a modification of the proof for nilmanifolds given in [17] works.

Let us remind the notation. M is a compact manifold, being $K(\pi, 1)$ for a virtually nilpotent torsion free group π . G is a connected simply connected nilpotent Lie group and $\mathbf{IN} = G/\pi$ where π is embedded in $\text{Aff}(G)$ as $\pi_{\mathbf{IN}}$, acting discontinuously on G so that \mathbf{IN} is an infra-nilmanifold. See Existence of a Model Theorem in Introduction. We have the universal covers $p_M : \tilde{M} \rightarrow M$ and $p_{\mathbf{IN}} : G \rightarrow \mathbf{IN}$.

Note that we use the right action, thus $\mathbf{IN} = G/\pi$, instead for $\pi \backslash G$ used in [13]. Then the action of an affine map (d, D) , $d \in G$, $D \in \text{Endo}(G)$, is given as $(d, D)x = (Dx)d$.

We assume that all metrics under consideration are induced by Riemannian metrics. We need the following

Lemma 7 [Lemma on the equivalence of metrics on G] *Any right invariant metric ρ on G is equivalent to τ_G (i.e. the mutual ratios are bounded) being a lift of an arbitrary metric $\tau_{\mathbf{IN}}$ on an infra-nilmanifold $\mathbf{IN} = G/\pi$.*

PROOF: By compactness the lift τ_{Γ} of $\tau_{\mathbf{IN}}$ to G/Γ , where $\Gamma = G \cap \pi$, is equivalent to ρ_{Γ} , the projection of ρ to G/Γ . Therefore the lifts to G are also equivalent. \square

Let us stop for a while on the homotopy equivalence between M and \mathbf{IN} making some explanations from Introduction more precise.

Lemma 8 *There exists a homotopy equivalence $h : M \rightarrow \mathbf{IN}$. Moreover for every continuous $f : M \rightarrow M$ there exists an affine mapping $\phi : \mathbf{IN} \rightarrow \mathbf{IN}$ such that $\phi \circ h \simeq h \circ f$, where \simeq means: homotopic.*

PROOF: The action π_M of π on \tilde{M} can be identified, chosen an arbitrary distinguished point $\tilde{z}_0 \in \tilde{M}$, with the fundamental group $\pi_1(M, p_M(\tilde{z}_0))$ by projecting by p_M of curves

joining $\pi_M(\tilde{z}_0)$ to \tilde{z}_0 . Similarly $\pi_{\mathbf{IN}}$ can be identified with $\pi_1(\mathbf{IN}, p_{\mathbf{IN}}(\tilde{w}_0))$ for an arbitrary distinguished point $\tilde{w}_0 \in G$.

Let $h : M \rightarrow \mathbf{IN}$ be a homotopy equivalence; its existence follows from [26, Section 8.1]. Choose its lift $\tilde{h} : \tilde{M} \rightarrow G$ and distinguish \tilde{z}_0, \tilde{w}_0 such that $\tilde{h}(\tilde{z}_0) = \tilde{w}_0$. Denote $z_0 = p_M(\tilde{z}_0)$ and $w_0 = p_{\mathbf{IN}}(\tilde{w}_0)$. We have $h_{\#} : \pi_1(M, z_0) \rightarrow \pi_1(\mathbf{IN}, w_0)$ which yields, with respect to distinguished \tilde{z}_0, \tilde{w}_0 , an isomorphism between the deck transformation groups $H : \pi_M \rightarrow \pi_{\mathbf{IN}}$.

Let $\tilde{f} : \tilde{M} \rightarrow \tilde{M}$ be a lift of f . Define $F_M : \pi_M \rightarrow \pi_M$, with respect to \tilde{z}_0 and $\tilde{f}(\tilde{z}_0)$ (similarly to the way we defined H). Finally define $F = F_{\mathbf{IN}} := H \circ F_M \circ H^{-1}$ and affine maps Φ and ϕ as in Introduction, relying on [13].

By construction we get $F_{\mathbf{IN}} \circ H = H \circ F_M$. Denote $x_1 = \Phi(\tilde{h}(\tilde{z}_0))$ and $x_2 = \tilde{h}(\tilde{f}(\tilde{z}_0))$. If $x_1 = x_2$ then $\phi_{\#}h_{\#} = h_{\#}f_{\#}$ on $\pi_1(M, z_0)$, hence $\phi \circ h \simeq h \circ f$, by [26, Section 8.1, Theorem 11]. Otherwise one can consider $k : \mathbf{IN} \rightarrow \mathbf{IN}$ homotopic to identity and its lift \tilde{k} , such that $\tilde{k}(x_1) = x_2$. It gives (with respect to distinguished x_1, x_2) identity on $\pi_{\mathbf{IN}}$. Hence $k_{\#}\phi_{\#}h_{\#} = h_{\#}f_{\#}$ on $\pi_1(M, z_0)$, hence again we deduce $\phi \circ h \simeq h \circ f$. \square

PROOF (OF THEOREMS A AND B): Consider metrics $\tau_{\tilde{M}}, \tau_G$ on \tilde{M}, G respectively, being lifts to the universal covers $p_M : \tilde{M} \rightarrow M$ and $p_{\mathbf{IN}} : G \rightarrow \mathbf{IN}$ of arbitrary metrics $\tau_M, \tau_{\mathbf{IN}}$ on M, \mathbf{IN} .

Let $f : M \rightarrow M$ be a continuous map and \tilde{f} its lift to \tilde{M} .

Let $h : M \rightarrow \mathbf{IN}$ be a homotopy equivalence such that $h \circ f \simeq \phi_f \circ h$, for h and $\phi = \phi_f$ as in Lemma 8. Let $\tilde{h} : \tilde{M} \rightarrow G$ be a lift of h . Then the distance in τ_G between $\tilde{h} \circ \tilde{f}$ and $\Phi \circ \tilde{h}$ is bounded, by a constant $\xi_1 > 0$ (since the lifts are joined by a lift of a homotopy, up to a deck transformation, that is up to an isometry).

Let $x_n, n = 0, 1, 2, \dots$ be an \tilde{f} trajectory. Then $w_n = \tilde{h}(x_n)$ is a $\xi_1 - \Phi$ -trajectory in the metric τ_G , hence, by Lemma 7, a $\xi_2 - \Phi$ -trajectory in ρ , the right invariant metric on G , for a constant $\xi_2 > 0$.

Finally for $\Phi = (b, B)$ the sequence w_n is a $\xi_3 - B'$ -trajectory for $B' = b^{-1}Bb$, i.e. $B'(x) = b^{-1} \cdot B(x) \cdot b$, for a constant $\xi_3 > 0$.

Indeed, $\rho(\Phi(w), B'(w)) = \rho(B(w)b, B'(w)) = \rho(bB'(w), B'(w)) = \rho(b, e)$, the latter equality by the right invariance of ρ . Hence w_n is a $\xi_2 + \rho(b, e)$ trajectory for B' .

Note that the spectra of the derivatives (linearizations) of $DB(e)$ and $DB'(e)$ coincide as these operators are conjugate.

Now we define a mapping $\Theta : \tilde{M} \rightarrow G^u$, the unstable subgroup for B' , by proceeding as in [17]: Let $x = x_0 \in \tilde{M}$. Let $(x_n, n = 0, 1, \dots)$ be its \tilde{f} -trajectory and $w_n = \tilde{h}(x_n)$. For each n define $w_n \mapsto \pi^u(w_n)$, the "projection" to G^u , i.e. we write $w_n = g^{cs} \cdot g^u$ where $g^{cs} \in G^{cs}$ the central stable subgroup and $\pi^u(w_n) := g^u \in G^u$. Finally $\Theta(x_n)$, in particular $\Theta(x)$, is defined as the unique B' -trajectory in G^u subexponentially "shadowing" $\pi^u(w_n)$. Here we need to consider the whole sequence (w_n) , rather than each w_n separately. As in [17, Proposition 2.10] we prove that Θ maps \tilde{M} , and even $\tilde{h}^{-1}(G^u)$, onto G^u . This is a crucial point which uses the fact that \tilde{h} is onto, since $|\deg h| = 1$. Compare Remark 4.8

in [17]. We get also by construction $\Theta \circ \tilde{f} = B' \circ \Theta$.

The proof concludes as follows. For an arbitrary $\epsilon > 0$, for $((1 + \epsilon)^j, n) - B'$ -separated points in G^u , $j = 0, \dots, n$, (contained in a small disc), i.e. such that for some j their j -th images under B' are within the distance at least $(1 + \epsilon)^j$, we choose points x in their $\Theta|_{\tilde{h}^{-1}(G^u)}$ -preimages (also in a small disc). If for two distinct points x, y chosen in this way $p_M(x), p_M(y)$ are $(\epsilon, n) - f$ -close (i.e. not separated), then so are x, y (cf. Proof of Proposition 1). Hence $\tilde{h}(x)$ and $\tilde{h}(y)$ are (ξ_4, n) -close (with respect to Φ , hence B') in ρ for a constant ξ_4 . Hence, the Θ -images of x, y are $((1 + \epsilon)^j, n) - B'$ -close, a contradiction (for details see [17]). \square

Note that we did not use the admissible group constructed in Proposition 5.

Remark 9 The statement of Theorem A, in a weaker form for flat manifolds, was posed as a question by A. Szczepański in [27]. Earlier, a very special case of Entropy Conjecture for an affine map of a compact affine manifold was proved by D. Fried and M. Shub in [8].

4 Absolute estimates of entropy

The famous Lehmer's conjecture in number theory states that there exists a constant C , called Lehmer constant, such that for every integer polynomial $w(x) = a_0x^d + a_1x^{d-1} + \dots + a_d$, not being a product of cyclotomic polynomials (all zeros being roots of 1) or x^k , for the Mahler $M(w)$ measure of w we have

$$M(w) := |a_0| \prod_{\lambda_i} \max(1, |\lambda_i|) \geq C,$$

where the product is taken over all zeros of $w(x)$.

There are estimates of the Mahler measure which depend on the degree of an irreducible polynomial (the degree of an algebraic number). Using an estimate given by Voutier in 1996 (cf. [29]),

$$M(w) > \tau(d) := 1 + \frac{1}{4} \left(\frac{\log \log d}{\log d} \right)^3,$$

which is the best known valid for every $d > 1$, not only asymptotically, we get the following

Theorem 10 *Let $f : M \rightarrow M$ be a continuous map of a compact infra-nilmanifold of dimension d . Then*

- a) either $\mathbf{h}(\phi_f) = 0$,
- b) or $\mathbf{h}(f) > \log \tau(d)$.

PROOF: Let $w(x) = w_1(x) \cdot w_2(x) \cdots w_k(x)$, $d_j = \deg w_j$, $d_1 \leq d_2 \leq \cdots \leq d_k$, be a decomposition of the characteristic polynomial of the linearization matrix A_f into irreducible terms. If $\mathbf{h}(\phi_f) > 0$ then by Theorem B at least one eigenvalue of A_f has the absolute value larger than 1. Hence, by Theorem B, using the property the sequence $\tau(n)$ is decreasing with respect to n ,

$$\mathbf{h}(f) \geq \log \prod_{1 \leq j \leq k} M(w_j(x)) \geq \log M(w_k(x)) \geq \log \tau(d).$$

□

For other estimates of Mahler measure see for example [4].

In particular from Smyth's theorem [25] (which is a partial answer to the Lehmer conjecture) it follows

Theorem 11 *Let $f : M \rightarrow M$ be a continuous map of a compact infra-nilmanifold of dimension d . If the characteristic polynomial of A_f is non-reciprocal, i.e. the set of zeros is not invariant under the symmetry $\lambda \mapsto \lambda^{-1}$, and if $\mathbf{h}(\phi_f) > 0$, then*

$$\mathbf{h}(f) \geq \prod_{\lambda_j \in \text{roots } w_j(x)} \max(1, |\lambda_j^j|) > \tau_0,$$

where τ_0 is the real root of polynomial $\tau^3 - \tau - 1$. □

One can check that the latter τ_0 is greater than 1.32471795. In particular, τ_0 does not depend neither on $w(x)$ nor on its degree d .

Note that $\mathbf{h}(f) \geq \mathbf{h}(\phi_f)$, $f \sim \phi_f$, $A_f = A_{[f]}$ assert that Theorem 11 is a statement about a homotopy property of f . A special case is when A_f is a hyperbolic matrix invertible over integers, i.e. ϕ_f is an Anosov automorphism, and d , the dimension of M , is odd. Then obviously the characteristic polynomial of A_f is non-reciprocal, hence Theorem 11 applies and we obtain

$$\mathbf{h}(f) \geq 1.32471795.$$

This is in fact an easy case whose proof does not need the use of Theorem B. Namely one can refer to Franks' theorem [7, Theorem 2.2], saying that such a map f is semiconjugate to ϕ_f , i.e. there exists a continuous map $\theta : M \rightarrow M$ such that $\theta \circ f = \phi_f \circ \theta$. This θ is found to be homotopic to identity, hence "onto". Therefore $\mathbf{h}(f) \geq \mathbf{h}(\phi_f)$, see Proposition 1. It is easy to check that if f is an Anosov diffeomorphism then A_f is a hyperbolic invertible matrix.

Other remarks

- The "projection – shadowing" construction of Θ in the proof of Theorem B in Section 3 and in [17] can be considered as a strengthening of Franks' theorem to the case central direction exists.

- It is sufficient to assume A_f is a hyperbolic endomorphism, i.e. without eigenvalues of absolute value 1, and without zero eigenvalues, to apply Franks' theorem. Then ϕ_f is an Anosov endomorphism and the semiconjugacy holds between the inverse limits, cf. [23] and [20].

- In the expanding case, i.e. if all the eigenvalues of A_f have absolute values larger than 1, the product is at least 2. Therefore $\mathbf{h}(f) \geq \log 2$. In this case, instead of Theorem B, one can refer to Shub's theorem [23] saying that f is semiconjugate to ϕ_f .

- Finally, if f itself is metric expanding on a compact orientable manifold (i.e. it expands all the distances between points close to each other, at least by a constant factor larger than 1) or at least if f is forward expansive, i.e. $\exists \delta > 0$ such that $\forall x \neq y \exists n \geq 0$ with $d(f^n(x), f^n(y)) \geq \delta$ (as this implies expanding in an appropriate metric, see [21, Section 3.6]), then for its degree $d(f)$ one has immediately $\mathbf{h}(f) \geq \log |d(f)| \geq \log 2$, see [28].

Note that f expanding (in a metric induced by a Riemannian metric) can happen only on infra-nilmanifolds, [9].

- In general $\mathbf{h}(f) \geq \log |d(f)|$ for all f being C^1 , see [18]. However the assumption that f is C^1 is essential in absence of the expanding property, namely there are easy examples of continuous, but not smooth maps f for which $\mathbf{h}(f) < \log |\deg(f)|$.

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