FUNDAMENTA
MATHEMATICAE
145 (1994)

Density of periodic sources in the boundary of a basin of attraction for iteration of holomorphic maps: geometric coding trees technique

by

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Abstract. We prove that if A is a basin of immediate attraction to a periodic attracting or parabolic point for a rational map f on the Riemann sphere, then the periodic points in the boundary of A are dense in this boundary. To prove this in the non-simply connected or parabolic situations we prove a more abstract, geometric coding trees version.

Introduction. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of the Riemann sphere $\overline{\mathbb{C}}$. Let $J(f)$ denote its Julia set. We say a periodic point p of period m is *attracting* (a *sink*) if $|(f^m)'(p)| < 1$, *repelling* (a *source*) if $|(f^m)'(p)| > 1$ and *parabolic* if $(f^m)'(p)$ is a root of unity. We say that $A = A_p$ is an *immediate basin of attraction* to a sink or a parabolic point p if A is a component of $\overline{\mathbb{C}} \setminus J(f)$ such that $f^{nm}|_A \to p$ as $n \to \infty$ and $p \in A_p$ for p attracting, and $p \in \partial A$ for p parabolic.

We shall prove the following fact, which answers a question posed by G. Levin:

Theorem A. *If A is a basin of immediate attraction for a periodic attracting or parabolic point for a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ then the periodic points contained in ∂A are dense in ∂A.*

The classical Fatou–Julia theorem says that the periodic sources are dense in $J(f)$. However, these periodic sources could only converge to ∂A, without being in ∂A.

1991 *Mathematics Subject Classification*: Primary 58F23.

The first author would like to thank the Institute of Mathematical Sciences of SUNY, Stony Brook, and the Institute of Mathematics of Yale University for their hospitality. The work on this paper was begun during his stays at these institutions in 1991/92. Both authors acknowledge the support of Polish KBN grants 210469101 "Iteracje i Fraktale" and 210909101 "Układy Dynamiczne".

The density of periodic points in Theorem A immediately implies the density of periodic sources because for every rational map there are only finitely many periodic points which are not sources and the Julia set has no isolated points.

We remark that just the existence of some periodic points in ∂A was proved by Fatou [F, p. 81] and Pommerenke [Po]. In fact, every periodic branch of a geometric coding tree in A (see Section 2 for the definition) converges to a periodic point.

The idea of proving Theorem A using Pesin's theory and Katok's proof of the density of periodic points [K], showing that $f^{-n}(B(x,\varepsilon)) \subset B(x,\varepsilon)$ for some branches of f^{-n}, is also too crude. The problem is that the resulting fixed point for f^n in $B(x,\varepsilon)$ could be outside ∂A. However, this gives a hint for a correct proof. We shall consider points in ∂A together with "tails", some curves in A along which these points are accessible. (We say $x \in \partial A$ is *accessible* from A if there exists a continuous curve $\gamma : [0,1] \to \overline{\mathbb{C}}$ such that $\gamma([0,1)) \subset A$ and $\gamma(1) = x$. We then also say that x is accessible along γ.)

Thus we shall in fact prove something stronger than Theorem A:

COMPLEMENT TO THEOREM A. *The periodic points in ∂A accessible from A along f-invariant finite length curves are dense in ∂A.*

If f is a polynomial (or a polynomial-like map) then it follows automatically that these periodic points are accessible along external rays. See [LP] for the proof and for the definition of external rays in the case where A is not simply connected.

It is an open problem whether all periodic sources in ∂A are accessible from A (see [P3] for a discussion of this and related problems). It was proved that this is so in case f is a polynomial and A is the basin of attraction to ∞ in [EL], [D], and later in [Pe], [P4] in more general situations: for f any rational function and A a completely invariant (i.e. $f^{-1}(A) = A$) basin of attraction to a sink or a parabolic point.

The paper is organized as follows: In Section 1 we shall prove Theorem A directly for A simply connected and p attracting. In Section 2 we shall introduce a more general point of view: geometric coding trees, studied and exploited already in [P1], [P2], [PUZ] and [PS], and formulate and prove Theorems B and C in the trees setting, which easily yields Theorem A.

1. Theorem A for A simply connected and p attracting. First let us state a lemma which belongs to Pesin's theory.

LEMMA 1. *Let $(X, \mathcal{F}, \nu)$ be a measure space with a measurable automorphism $T : X \to X$. Let μ be a probability ergodic f-invariant measure on a compact set Y in the Riemann sphere, for f a holomorphic mapping from a neighbourhood of Y to $\overline{\mathbb{C}}$ keeping Y invariant, with positive Lyapunov* *exponent, i.e.* $\chi_\mu(f) := \int \log|f'|\,d\mu > 0$. *Let* $h : X \to Y$ *be a measurable mapping such that* $h_*(\nu) = \mu$ *and* $h \circ T = f \circ h$ *a.e. Then for* ν*-almost every* $x \in X$ *there exists* $r = r(x) > 0$ *and univalent branches* F_n *of* f^{-n} *on* $B(h(x), r)$ *for* $n = 1, 2, \ldots$ *for which* $F_n(h(x)) = h(T^{-n}(x))$. *Moreover*, *for every* λ *with* $\exp(-\chi_\mu(f)) < \lambda < 1$ (*not depending on* x) *and a constant* $C = C(x) > 0$,

$$|F_n'(h(x))| < C\lambda^n \quad \textit{and} \quad \frac{|F_n'(h(x))|}{|F_n'(z)|} < C$$

for every $z \in B(h(x), r)$, $n > 0$ (*distances and derivatives in the Riemann spherical metric on* $\overline{\mathbb{C}}$).

Moreover, r *and* C *are measurable functions of* x.

Let $R : \mathbb{D} \to A_p$ be a Riemann mapping such that $R(0) = p$. Define $g := R^{-1} \circ f \circ R$ on $\mathbb{D}$. We know that g extends holomorphically to a neighbourhood of $\operatorname{cl} A$ and is expanding on ∂A (see [P2]). (In fact, g is a finite Blaschke product, because we assume in this section that f is defined on the whole A; see [P1]. However, we only need the assumption that f is defined on a neighbourhood of ∂A as in [P2].)

For every $\zeta \in \partial\mathbb{D}$, every $0 < \alpha < \pi/2$ and every $\varrho > 0$ consider the cone

$$\mathcal{C}_{\alpha,\varrho}(\zeta) := \{z \in \mathbb{D} : |\operatorname{Arg}\zeta - \operatorname{Arg}(\zeta - z)| < \alpha,\ |\zeta - z| < \varrho\}.$$

In the sequel we shall need the following simple

LEMMA 2. *There exist* $\varrho_0 > 0$, $C > 0$ *and* $0 < \alpha_0 < \pi/2$ *such that for every* $\varrho \le \varrho_0$, $n \ge 0$, $\zeta \in \partial\mathbb{D}$ *and every branch* G_n *of* g^{-n} *on the disc* $B(\zeta, \varrho_0)$ *the following inclusion holds*:

$$G_n(\{z \in \mathbb{D} : z = t\zeta,\ 1 - t < \varrho\}) \subset \mathcal{C}_{\alpha_0, C\varrho}(G_n(\zeta)).$$

Remark. Considering an iterate of f and g we can assume that $C = 1$, because above we can write in fact $\mathcal{C}_{\alpha_0, C\xi^n\varrho}$ for some $0 < \xi < 1$.

Proof of Theorem A for a simply connected basin of a sink. Keep the notation of this section: A the basin of attraction to a fixed point, a sink p, $R : \mathbb{D} \to A$ a Riemann mapping and g the pull-back of f extended beyond $\partial\mathbb{D}$, just a finite Blaschke product.

Consider $\mu := \overline{R}_*(l)$, where $\overline{R}$ denotes the radial limit of R and l is the normalized length measure on $\partial\mathbb{D}$. In fact, μ is the harmonic measure on ∂A viewed from p. This measure is ergodic f-invariant and $\chi_\mu(f) = \chi_l(g) > 0$ (see [P1, P2]). Also $\operatorname{supp}\mu = \partial A$.

Indeed, for every $\varepsilon > 0$, $x \in \partial A$ and $x_n \in A$ such that $x_n \to x$ we have for harmonic measures: $\omega(x_n, B(x, \varepsilon)) \to 1 \ne 0$. But the measures $\omega(p, \cdot)$ and $\omega(x_n, \cdot)$ are equivalent, hence $\omega(p, B(x, \varepsilon)) > 0$.

We shall not use the assumption that μ is a harmonic measure any more; we only use the above-mentioned properties.

From the existence of the nontangential limit $\overline{R}$ of R a.e. [Du] it follows easily that for all $\varepsilon > 0$, $0 < \alpha < \pi/2$ and $\varrho > 0$ there exists $K_\varepsilon \subset \partial\mathbb{D}$ such that $l(K_\varepsilon) \geq 1 - \varepsilon$ and for all $\zeta \in K_\varepsilon$,

$$R(z) \to \overline{R}(\zeta) \quad \text{uniformly as } z \to \zeta,\ z \in \mathcal{C}_{\alpha,\varrho}(\zeta).$$

Namely, for every $\delta_1 > 0$ there exists $\delta_2 > 0$ such that for every $\zeta \in K_\varepsilon$ if $z \in \mathcal{C}_{\alpha,\delta_2}(\zeta)$ then $\mathrm{dist}(R(z), \overline{R}(\zeta)) < \delta_1$, distance in the Riemann spherical metric on $\overline{\mathbb{C}}$.

Consider the inverse limit (the natural extension in Rokhlin's terminology [Ro]) $(\widetilde{\partial\mathbb{D}}, \widetilde{\mathcal{B}}, \widetilde{l}, \widetilde{g})$ of $(\partial\mathbb{D}, \mathcal{B}, l, g)$. Here $\mathcal{B}$ stands for the Borel σ-algebra. Recall [Ro] that the natural extension can be defined as the space of all g-trajectories $(\zeta_n), n = \ldots, -1, 0, 1, \ldots$ (or equivalently backward trajectories: $n = \ldots, -1, 0$) with the shift map $\widetilde{g}((\zeta_n)) = (\zeta_{n+1})$. Define $\pi_j((\zeta_n)) = \zeta_j$. Then the σ-algebra $\widetilde{\mathcal{B}}$ is defined to be generated by all sets $\pi_j^{-1}(A)$ for $A \in \mathcal{B}$. The measure $\widetilde{l}$ satisfies $\widetilde{l}(\pi_j^{-1}(A)) = l(A)$. (In our case $\widetilde{\mathcal{B}}$ is just the Borel σ-algebra in the topological inverse limit $\widetilde{\partial\mathbb{D}}$.)

By Lemma 1 applied to $(\widetilde{\partial\mathbb{D}}, \widetilde{\mathcal{B}}, \widetilde{l})$, to the automorphism $\widetilde{g}$, the map $h = \overline{R} \circ \pi_0$, $Y = \partial A$ and f our rational map, there exist constants $C, r > 0$, this time not depending on x, and a measurable set $\widetilde{K} \subset \widetilde{\partial\mathbb{D}}$ such that $\widetilde{l}(\widetilde{K}) \geq 1 - 2\varepsilon$, $\widetilde{K} \subset \pi_0^{-1}(K_\varepsilon)$ and for every g-trajectory $(\zeta_n) \in \widetilde{K}$ the assertion of Lemma 1 holds with the constants C and r.

Let $t = t(r) > 0$ be such that for every $\zeta \in K_\varepsilon$ and $z \in \mathcal{C}_{\alpha,t}(\zeta)$ we have

$$\mathrm{dist}(R(z), \overline{R}(\zeta)) < r/3. \tag{1}$$

We additionally assume that $t < \varrho_0$ from Lemma 2. Also the α is α_0 from Lemma 2.

By the Poincaré Recurrence Theorem for $\widetilde{g}$, for a.e. trajectory $(\zeta_n) \in \widetilde{K}$ there exists a sequence $n_j \to \infty$ as $j \to \infty$ such that

$$\zeta_{-n_j} = \pi_0 \widetilde{g}^{-n_j}((\zeta_n)) \to \zeta_0 \tag{2}$$

and $\widetilde{g}^{-n_j}((\zeta_n)) \in \widetilde{K}$, hence

$$\zeta_{-n_j} \in K_\varepsilon. \tag{3}$$

Indeed, we can take a sequence of finite partitions $\mathcal{A}_j$ of $\pi_0(\widetilde{K})$ such that the maximal diameters of sets of $\mathcal{A}_j$ converge to 0 as $j \to \infty$. Almost every $(\zeta_n) \in \widetilde{K}$ is in $\bigcap_j \pi_0^{-1}(A_j)$ where $A_j \in \mathcal{A}_j$ and there exists n_j such that $\widetilde{g}^{-n_j}((\zeta_n)) \in \pi_0^{-1}(A_j)$.

For a.e. $(\zeta_n) \in \widetilde{K}$ fix $N = N((\zeta_n))$ such that

$$\zeta_{-N} \in B(\zeta_0, t(r)\sin\alpha), \tag{4}$$

N arbitrarily large.

Denote by G_N the branch of g^{-N} such that $G_N(\zeta_0) = \zeta_{-N}$. By Lemma 2, $G_N(\tau\zeta_0) \in \mathcal{C}_{\alpha,t}(\zeta_N)$ for every $1 - t < \tau < 1$.

By (4) there exists $1-t < \tau_0 < 1$ such that $\tau_0\zeta_0 \in \mathcal{C}_{\alpha,t}(\zeta_{-N})$ (see Fig. 1).

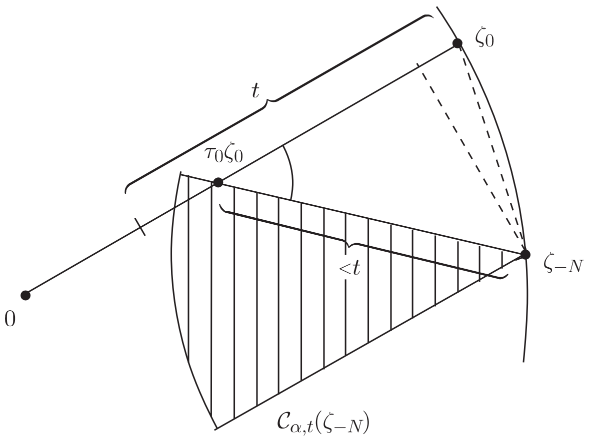

Fig. 1

By (3) we can apply (1) to $\zeta = \zeta_{-N}$. Thus by (1) applied to $z = \tau_0\zeta_0$, $\zeta = \zeta_0$ and $\zeta = \zeta_{-N}$ we obtain

$$\operatorname{dist}(\overline{R}(\zeta_{-N}), \overline{R}(\zeta_0)) < \tfrac{2}{3}r\,.$$

So, if N has been taken large enough, we obtain by Lemma 1 for the branch F_N of f^{-N} appearing in the statement of Lemma 1,

$$F_N(B(\overline{R}(\zeta_0), r)) \subset B(\overline{R}(\zeta_{-N}), r/3) \subset B(\overline{R}(\zeta_0), r) \tag{5}$$

(see Fig. 2).

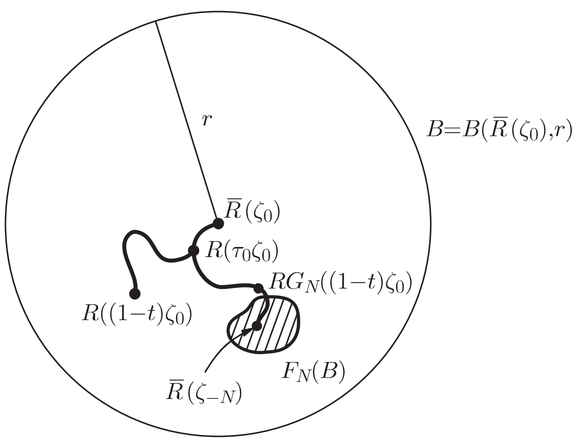

Fig. 2

Moreover, F_N is a contraction, i.e. $|(F_N|_{B(\bar{R}(\zeta_0),r)})'| < C\lambda^N < 1$.

The interval I joining $\tau_0\zeta_0$ and $G_N((1-t)\zeta_0)$ is in $\mathcal{C}_{\alpha,t}(\zeta_{-N})$, hence

$$R(I) \subset B(\overline{R}(\zeta_{-N}), r/3) \subset B(\overline{R}(\zeta_0), r).$$

By the definitions of F_N, G_N we have $\overline{R} \circ G_N = F_N \circ \overline{R}$ at ζ_0. To prove this equality on $[(1-t)\zeta_0, \zeta_0]$ we must know that for f^{-N} we really have the branch F_N. But this is indeed the case because the maps involved are continuous on the domains under consideration and $[(1-t)\zeta_0, \zeta_0]$ is connected. So

$$F_N(R(1-t)\zeta_0) = RG_N((1-t)\zeta_0). \tag{6}$$

Let γ be the concatenation of the curves $R([(1-t)\zeta_0, \tau\zeta_0])$ and $R(I)$. By (6) it joins $R((1-t)\zeta_0)$ to $F_N(R((1-t)\zeta_0))$ and lies entirely in $B(\overline{R}(\zeta_0), r)$. Let Γ be the concatenation of the curves $\gamma, F_N(\gamma), F_N^2(\gamma), \ldots$ Then one end, say a, of Γ is in ∂A and Γ is periodic of period N (Γ makes sense by (5)). Moreover,

$$\operatorname{length} \Gamma \le \sum_{n\ge 0} C\lambda^n \operatorname{length} \gamma < \infty.$$

We have $\operatorname{dist}(a, \overline{R}(\zeta_0)) < r$. Because $\operatorname{supp} \mu = \partial A$ and ε and r can be taken arbitrarily close to 0, this proves the density of the periodic points in ∂A.

2. Geometric coding trees: completion of the proof of Theorem A. We shall prove a more abstract and general version of Theorem A here. This will allow us to deduce Theorem A immediately in the parabolic and non-simply connected cases.

Let U be an open connected subset of the Riemann sphere $\overline{\mathbb{C}}$. Consider any holomorphic mapping $f : U \to \overline{\mathbb{C}}$ such that $f(U) \supset U$ and $f : U \to f(U)$ is a proper map. Define $\operatorname{Crit} f = \{z : f'(z) = 0\}$, the set of critical points for f. Suppose that $\operatorname{Crit} f$ is finite. Consider any $z \in f(U)$. Let $z^1, \ldots, z^d$ be some of the f-preimages of z in U where $d \ge 2$. Consider smooth curves $\gamma^j : [0,1] \to f(U)$, $j = 1, \ldots, d$, joining z to z^j respectively (i.e. $\gamma^j(0) = z$, $\gamma^j(1) = z^j$), such that in $\bigcup_{j=1}^d \gamma^j$ there are no critical values for iterations of f, i.e. $\gamma^j \cap f^n(\operatorname{Crit} f) = \emptyset$ for every j and $n > 0$.

Let $\Sigma^d := \{1, \ldots, d\}^{\mathbb{Z}^+}$ denote the one-sided shift space and σ the shift to the left, i.e. $\sigma((\alpha_n)) = (\alpha_{n+1})$. For every sequence $\alpha = (\alpha_n)_{n=0}^\infty \in \Sigma^d$ we define $\gamma_0(\alpha) := \gamma^{\alpha_0}$. Suppose that for some $n \ge 0$, for every $0 \le m \le n$, and all $\alpha \in \Sigma^d$, the curves $\gamma_m(\alpha)$ are already defined. Suppose that for $1 \le m \le n$ we have $f \circ \gamma_m(\alpha) = \gamma_{m-1}(\sigma(\alpha))$, and $\gamma_m(\alpha)(0) = \gamma_{m-1}(\alpha)(1)$.

Define the curves $\gamma_{n+1}(\alpha)$ so that the previous equalities hold by taking suitable f-preimages of the curves γ_n. For every $\alpha \in \Sigma^d$ and $n \ge 0$ set $z_n(\alpha) := \gamma_n(\alpha)(1)$.

For every $n \geq 0$ denote by $\Sigma_n = \Sigma_n^d$ the space of all sequences of elements of $\{1,\ldots,d\}$ of length $n+1$. Let π_n denote the projection $\pi_n : \Sigma^d \to \Sigma_n$ defined by $\pi_n(\alpha) = (\alpha_0,\ldots,\alpha_n)$. As $z_n(\alpha)$ and $\gamma_n(\alpha)$ only depend on $(\alpha_0,\ldots,\alpha_n)$, we can consider z_n and γ_n as functions on Σ_n.

The graph $\mathcal{T}(z,\gamma^1,\ldots,\gamma^d)$ with the vertices z and $z_n(\alpha)$ and edges $\gamma_n(\alpha)$ is called a *geometric coding tree* with root at z. For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \geq 0$ is called a *geometric branch* and denoted by $b(\alpha)$. The branch $b(\alpha)$ is called *convergent* if the sequence $\gamma_n(\alpha)$ converges to a point in $\operatorname{cl} U$. We define the *coding map* $z_\infty : \mathcal{D}(z_\infty) \to \operatorname{cl} U$ by $z_\infty(\alpha) := \lim_{n\to\infty} z_n(\alpha)$ on the domain $\mathcal{D} = \mathcal{D}(z_\infty)$ of all α's for which $b(\alpha)$ is convergent.

(This convergence is called in [PS] strong convergence. In the previous papers [P1], [P2], [PUZ] we mainly considered convergence in the sense that $z_n(\alpha)$ converges to a point, but here we shall need the convergence of the edges γ_n.)

In the sequel we also need the following notation: for each geometric branch $b(\alpha)$ denote by $b_m(\alpha)$ the part of $b(\alpha)$ starting from $z_m(\alpha)$, i.e. consisting of the vertices $z_k(\alpha)$, $k \geq m$, and of the edges $\gamma_k(\alpha)$, $k > m$.

The basic theorem concerning convergence of geometric coding trees is the following

Convergence Theorem. 1. *Every branch except branches in a set of Hausdorff dimension* 0 *in the standard metric on* Σ^d, *is convergent* (*i.e.* $\mathrm{HD}(\Sigma^d \setminus \mathcal{D}) = 0$). *In particular*, *for every Gibbs measure* ν_φ *for a Hölder continuous function* $\varphi : \Sigma^d \to \mathbb{R}$, $\nu_\varphi(\Sigma^d \setminus \mathcal{D}) = 0$, *so the measure* $(z_\infty)_*(\nu_\varphi)$ *makes sense.*

2. *For every* $z \in \operatorname{cl} U$, $\mathrm{HD}(z_\infty^{-1}(\{z\})) = 0$. *Hence for every* ν_φ *the entropies satisfy* $\mathrm{h}_{\nu_\varphi}(\sigma) = \mathrm{h}_{(z_\infty)_*(\nu_\varphi)}(\bar f) > 0$ (*provided we assume that there exists a continuous extension* $\bar f$ *of* f *to* $\operatorname{cl} U$).

The proof of this theorem can be found in [P1] and [P2] under some stronger assumptions (slow convergence of $f^n(\operatorname{Crit} f)$ to γ^i as $n \to \infty$). To obtain the above version one should also rely on [PS] (where even $f^n(\operatorname{Crit} f) \cap \gamma^i \neq \emptyset$ is allowed).

Recently (see [P4]), a complementary fact was proved. In case f is a rational map on the Riemann sphere, U is a completely invariant basin of attraction to a sink or a parabolic periodic point and the condition (i) (see statement of Theorem C below) is satisfied, this fact can be formulated as follows:

3. *Every* f*-invariant probability ergodic measure* μ, *of positive Lyapunov exponent*, *supported by* ∂U *is the* $(z_\infty)_*$*-image of a probability* σ*-invariant measure on* Σ^d.

Suppose, in Theorems B, C which follow, that the map f extends holomorphically to a neighbourhood of the closure of the limit set Λ of a tree, $\Lambda = z_\infty(\mathcal{D}(z_\infty))$. Then Λ is called a *quasi-repeller* (see [PUZ]).

THEOREM B. *Let Λ be a quasi-repeller for a geometric coding tree $\mathcal{T}(z,\gamma^1,\ldots,\gamma^d)$ for a holomorphic map $f : U \to \overline{\mathbb{C}}$. Then for every Gibbs measure ν for a Hölder continuous function φ on Σ^d, the periodic points in Λ for the extension of f to Λ are dense in* $\operatorname{supp}(z_\infty)_*(\nu)$.

This is all we can prove in the general case. In the next theorem we shall introduce additional assumptions.

Define

$$\widehat{\Lambda} := \{\text{all limit points of the sequences } z_n(\alpha^n),\ \alpha^n \in \Sigma^d,\ n \to \infty\}.$$

THEOREM C. *Suppose we have a tree as in Theorem* B *which additionally satisfies the following conditions for every $j = 1,\ldots,d$:*

(i) $\gamma^j \cap \operatorname{cl}\bigcup_{n>0} f^n(\operatorname{Crit} f) = \emptyset$,

(ii) *there exists a neighbourhood $U^j \subset f(U)$ of γ^j such that*

$$\operatorname{Vol} f^{-n}(U^j) \to 0$$

where Vol *denotes the standard Riemann measure on $\overline{\mathbb{C}}$.*

Then the periodic points in Λ for $\overline{f}$ are dense in $\widehat{\Lambda}$.

Theorem C immediately follows from Theorem B if we prove the following:

LEMMA 3. *Under the assumptions of Theorem* C (*except that we need not assume that f extends to $\overline{f}$*), *for every Gibbs measure ν on Σ^d we have* $\operatorname{supp}(z_\infty)_*(\nu) = \widehat{\Lambda}$.

P r o o f. The proof is a minor modification of the proof of the Convergence Theorem, part 1, but for completeness we give it here.

Let U^j and U'^j be open connected simply connected neighbourhoods of γ^j for $j = 1,\ldots,d$ respectively, such that $\operatorname{cl} U'^j \subset U^j$, $U^j \cap \operatorname{cl}\bigcup_{n>0} f^n(\operatorname{Crit} f) = \emptyset$ and (ii) holds.

By (ii), $\varepsilon(n) := \operatorname{Vol} f^{-n}(\bigcup_{j=1}^d U^j) \to 0$ as $n \to \infty$.

Define $\varepsilon'(n) = \sup_{k\ge n} \varepsilon(n)$. We have $\varepsilon'(n) \to 0$.

Denote the components of $f^{-n}(U^j)$ and of $f^{-n}(U'^j)$ containing $\gamma_n(\alpha)$, where $\alpha_n = j$, by $U_n(\alpha), U'_n(\alpha)$ respectively. Similarly to $z_n(\alpha)$ and $\gamma_n(\alpha)$ each such component only depends on the first $n+1$ numbers in α so in our notation we can replace α by $\pi_n(\alpha) = (\alpha_0,\ldots,\alpha_n) \in \Sigma_n$.

Fix arbitrary $n \ge 0$, $\alpha \in \Sigma_n$ and $\delta > 0$. For every $m > n$ define

$$\mathcal{B}(\alpha, m) = \{(j_0,\ldots,j_m) \in \Sigma_m :\ j_k = \alpha_k \text{ for } k = 0,\ldots,n\}$$

and

$$\mathcal{B}_\delta(\alpha, m) = \{(j_0, \ldots, j_m) \in \mathcal{B}(\alpha, m) : \mathrm{Vol}\, U_m(j_0, \ldots, j_m) \\ \le \varepsilon(m) \exp(-(m-n)\delta)\}\,.$$

Set also $\mathcal{B}(\alpha) = \pi_n^{-1}(\{\alpha\}) \subset \Sigma^d$.

Since for every j_m all $U_m(j_0, \ldots, j_m)$ are pairwise disjoint,

$$\sharp\mathcal{B}(\alpha) - \sharp\mathcal{B}_\delta(\alpha, m) \le d \exp((m-n)\delta)\,. \tag{7}$$

By the Koebe distortion theorem for the branches f^{-m} leading from U^j to $U_m(\beta)$ for $\beta \in \Sigma^d$, $\beta_m = j$ we have

$$\mathrm{diam}\, \gamma_m(\beta) \le \mathrm{diam}\, U'_m(\beta) \le \mathrm{Const}(\mathrm{Vol}\, U'_m(\beta))^{1/2} \le \mathrm{Const}(\mathrm{Vol}\, U_m(\beta))^{1/2}\,.$$

Thus if $\beta \in \mathcal{B}(\alpha)$ and $\pi_m(\beta) \in \mathcal{B}_\delta(\alpha, m)$ for every $m > m_0 \ge n$ then

$$\mathrm{length}\, b_{m_0}(\beta) \le \mathrm{Const} \sum_{m > m_0} \varepsilon(m)^{1/2} \exp(-(m-n)\delta/2)\,.$$

Now we shall rely on the following property of ν true for the Gibbs measure for every Hölder continuous function φ on Σ^d:

There exists $\theta > 0$ depending only on φ such that for every pair of integers $k < m$ and every $\beta \in \Sigma^d$,

$$\frac{\nu(\pi_m^{-1}(\pi_m(\beta)))}{\nu(\pi_k^{-1}(\pi_k(\beta)))} < \exp(-(m-k)\theta)\,.$$

So with the use of (7) we obtain

$$\frac{\nu(\mathcal{B}(\alpha) \setminus \bigcap_{m > m_0} \pi_m^{-1}(\mathcal{B}_\delta(\alpha, m)))}{\nu(\mathcal{B}(\alpha))} \le \sum_{m > m_0} d \exp((m-n)\delta) \exp(-(m-n)\theta)\,.$$

We consider $\delta < \theta$.

As the conclusion we obtain the following

Claim. *For every $r > 0$ and $0 < \lambda < 1$, if n is large enough then for every $\alpha \in \Sigma_n^d$ there is $\mathcal{B}' \subset \mathcal{B}(\alpha)$ such that*

$$\frac{\nu(\mathcal{B}')}{\nu(\mathcal{B}(\alpha))} > \lambda$$

and for every $\beta \in \mathcal{B}'$,

$$\mathrm{length}\, b_n(\beta) < r\,.$$

Indeed, it is sufficient to take $\mathcal{B}' = \bigcap_{m > m_0} \pi_m^{-1}(\mathcal{B}_\delta(\alpha, m))$, where m_0 is the smallest integer $\ge n$ such that $\sum_{m > m_0} d \exp((m-n)(\delta - \theta)) \le 1 - \lambda$. (Of course the constant $m_0 - n$ does not depend on n, α.) Then for every $\beta \in \mathcal{B}'$,

$$\mathrm{length}\, b_n(\beta) < (m_0 - n)\varepsilon'(n) + \mathrm{Const}\, \varepsilon'(m_0)^{1/2} \sum_{m > m_0} \exp(-(m-n)\delta/2) < r$$

if n is large enough.

The above claim immediately proves our Lemma 3. ■

R e m a r k 4. Lemma 3 proves in particular (under the assumptions (i) and (ii) but without assuming that f extends to $\bar{f}$) that $\operatorname{cl} \Lambda = \widehat{\Lambda}$.

R e m a r k 5. Observe that Lemma 3 without any additional assumptions about the tree, like (i), (ii), is false. For example take $z = p$ to be the sink, $z^1 = p$, $z^j \neq p$ for $j = 2, \ldots, d$ and $\gamma^1 \equiv p$. Then $p \in \Lambda$ but $p \notin \operatorname{supp}(z_\infty)_*(\nu)$ for every Gibbs ν.

Observe that if (i) and (ii) are skipped in the assumptions of Theorem C then its assertion on the density of Λ or the density of periodic points in $\widehat{\Lambda}$ is also false. We can take z in a Siegel disc S but z different from the periodic point in S, $z^1 \in S$, $z^j \notin S$ for $j = 2, \ldots, d$.

Here Λ is not even dense in the set Λ' intermediary between Λ and $\widehat{\Lambda}$,

$$\Lambda' := \bigcup_{\alpha \in \Sigma^d} \Lambda(\alpha)$$

$$\text{where} \quad \Lambda(\alpha) := \{\text{the set of limit points of } z_n(\alpha), n \to \infty\}$$

(because Λ' contains a "circle" in the Siegel disc).

$\widehat{\Lambda}$ corresponds to the union of impressions of all prime ends and Λ' corresponds to the union of all sets of principal points. See [P3] for this analogy.

We do not know whether Lemma 3 or Theorem C are true without the assumption (i), only with the assumption (ii).

Now we shall prove Theorem B:

P r o o f o f T h e o r e m B. We repeat the same scheme as in the proof of Theorem A, the case discussed in Section 1. Now $(\partial\mathbb{D}, g, l)$ is replaced by (Σ^d, σ, ν). Its natural extension is denoted by $(\widetilde{\Sigma}^d, \widetilde{\sigma}, \widetilde{\nu})$ (in fact $\widetilde{\Sigma}^d = \{1, \ldots, d\}^{\mathbb{Z}}$). As in Section 1 we find a set $\widetilde{K}$ with $\widetilde{\nu}(\widetilde{K}) > 1 - 2\varepsilon$ so that all points of $\widetilde{K}$ satisfy the assumptions of Lemma 1 with constants C, r. The map $\overline{R}$ is replaced by z_∞ and Y is $\operatorname{cl} \Lambda$ now.

Condition (1) makes sense along branches (which play the role of cones), i.e. it takes the form: there exists $M = M(r)$ arbitrarily large such that for every $\alpha \in \widetilde{K}$,

$$b_M(\alpha) \subset B(z_\infty(\alpha), r/3)\,. \tag{8}$$

The crucial property we need in order to refer to Lemma 1 is $\chi_{(z_\infty)_*(\nu)}(\bar{f}) > 0$. It holds because by the Convergence Theorem, part 2, we know that $\mathrm{h}_\nu(\sigma) = \mathrm{h}_{(z_\infty)_*(\nu)}(\bar{f}) > 0$ and by [R], $\chi_{(z_\infty)_*(\nu)}(\bar{f}) \geq \frac{1}{2}\mathrm{h}_{(z_\infty)_*(\nu)}(\bar{f}) > 0$.

As in Section 1, for every $\alpha = (\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots) \in \widetilde{K}$ there exists N arbitrarily large such that $\beta = \pi_0 \widetilde{\sigma}^{-N}(\alpha) \in \widetilde{K}$ is close to α. In particular,

$$\beta = (\alpha_0, \alpha_1, \ldots, \alpha_M, w, \alpha_0, \alpha_1, \ldots)$$

where w stands for a sequence of $N-M-1$ symbols from $\{1,\ldots,d\}$ and $N>M$.

By (8) we have

$$b_M(\alpha)\subset B(z_\infty(\alpha),r/3)\quad\text{and}\quad b_M(\beta)\subset B(z_\infty(\beta),r/3)\,.$$

We also have

$$z_M(\alpha)=z_M(\beta)\,.$$

So $\gamma:=\bigcup_{n=M+1}^{N+M}\gamma_n(\beta)\subset B(z_\infty(\alpha),r)$. Since $F_N(z_\infty(\alpha))=z_\infty(\beta)$ we have, as in Section 1, (6), $F_N(z_M(\alpha))=z_{M+N}(\beta)$, i.e. F_N maps one end of γ to the other. We also have, similarly to (5), $F_N(B(z_\infty(\alpha),r))\subset B(z_\infty(\alpha),r)$ and F_N is a contraction.

One end of the curve Γ built from $\gamma,F_N(\gamma),F_N^2(\gamma),\ldots$ is periodic of period N, is in $B(z_\infty(\alpha),r)$ and is the limit of the branch of the periodic point

$$(\alpha_0,\ldots,\alpha_M,w,\alpha_0,\ldots,\alpha_M,w,\ldots)\in\Sigma^d\,.$$

Theorem B is proved. ∎

Proof of Theorem A (conclusion). Write

$$\mathrm{Crit}^+:=\bigcup_{n>0}f^n(\mathrm{Crit}\,f|_A)\,.$$

Let p denote the sink in A or the parabolic point in ∂A attracting A.

Take an arbitrary point $z\in A\setminus\mathrm{Crit}^+$, $z\neq p$. Take an arbitrary geometric coding tree $\mathcal{T}(z,\gamma^1,\ldots,\gamma^d)$ in $A\setminus(\mathrm{Crit}^+\cup\{p\})$, where $d=\deg f|_A$.

Observe that (i) is satisfied because $\mathrm{cl}\,\mathrm{Crit}^+=\{p\}\cup\mathrm{Crit}^+$.

Condition (ii) also holds because taking $U^j\subset A$ we obtain $f^{-n}(U^j)\to\partial A$, hence there exists $N>0$ such that for every $n\ge N$ we have

$$f^{-n}(U^j)\cap U^j=\emptyset\,.$$

Indeed, if we had $\mathrm{Vol}\,f^{-n_t}(U^j)>\varepsilon>0$ for a sequence $n_t\to\infty$ we could assume that $n_{t+1}-n_t\ge N$. We would have $f^{-n_t}(U^j)\cap f^{-n_s}(U^j)=\emptyset$ for every $t\neq s$, hence $\mathrm{Vol}\bigcup_t f^{-n_t}(U^j)=\sum_t\mathrm{Vol}\,f^{-n_t}(U^j)\ge\sum_t\varepsilon=\infty$, a contradiction.

Thus we deduce from Theorem C that the periodic points in Λ are dense in $\widehat{\Lambda}$. The only thing to be checked is

$$\widehat{\Lambda}=\partial A\,. \tag{9}$$

(If A is completely invariant then $Z=\bigcup_{n\ge0}f^{-n}(z)$ is a subset of A. It is dense in the Julia set, in particular in ∂A. In general, however, $Z\not\subset A$ so the existence of a sequence in Z converging to a point in ∂A does not automatically imply the existence of such a sequence in $Z\cap A$.)

It is not hard to find a compact set $P \subset A$ such that $P \cap (\mathrm{Crit}^+ \cup \{p\}) = \emptyset$ and such that for every $\zeta_0 \in \partial A \setminus \{p\}$ and every $\zeta \in A$ close enough to ζ_0, there exists $n > 0$ such that $f^n(\zeta) \in P$. The closer ζ to ζ_0, the larger n.

Cover P by a finite number of topological discs $D_\tau \subset A$. There exist topological discs D'_τ whose union also covers P such that $\mathrm{cl}\, D'_\tau \subset D_\tau$. Join each disc D_τ to z by a curve δ_τ without selfintersections disjoint from Crit^+ and p. Then for every τ there exists a topological disc $V_\tau \subset A$ which is a neighbourhood of $D_\tau \cup \delta_\tau$ also disjoint from Crit^+ and p.

For every $\varepsilon > 0$ there exists $n_0 > 0$ such that for every $n > n_0$ and every branch F_n of $(f|_A)^{-n}$ on V_τ,

$$\operatorname{diam} F_n(D'_\tau \cup \delta_\tau) < \varepsilon$$

by the same reason by which $\mathrm{Vol}\, f^{-n}(U^j) \to 0$ and next (by the Koebe distortion theorem, see proof of Lemma 3), $\operatorname{diam} f^{-n}(U'^j) \to 0$.

So fix an arbitrary $\zeta_0 \in \partial A \setminus \{p\}$ and take $\zeta \in A$ close to ζ_0. Find N and τ such that $f^N(\zeta) \in D'_\tau$. We can assume $N > n_0$. Let F_N be the branch of f^{-N} on V_τ such that $F_N(f^N(\zeta)) = \zeta$. Then $\mathrm{dist}(\zeta, F_N(z)) < \varepsilon$. But $F_N(z)$ is a vertex of our tree. Letting $\varepsilon \to 0$ we obtain (9). ■

Remark 6. One can apply Theorem C to f a rational mapping on the Riemann sphere and $d = \deg f$ under the assumptions that for the Julia set $J(f)$ we have $\mathrm{Vol}\, J(f) = 0$ and the set $\mathrm{cl}\,\mathrm{Crit}^+$ does not dissect $\overline{\mathbb{C}}$. Indeed, in this case we take z in an immediate basin of a sink or a parabolic point and curves γ^j disjoint from $\mathrm{cl}\,\mathrm{Crit}^+$. Then the assumptions (i), (ii) are satisfied, so the periodic points in Λ are dense in $\widehat{\Lambda}$. A basic property of $J(f)$ says that $\bigcup_{n>0} f^{-n}(z)$ is dense in $J(f)$, i.e. $\widehat{\Lambda} = J(f)$, hence the periodic points in Λ are dense in $J(f)$.

In this case, however, we can immediately deduce the density of the periodic sources belonging to Λ in $J(f)$ from the fact that the periodic sources are dense in $J(f)$ and from the theorem saying that every periodic source q is the limit of a branch $b(\alpha)$, $\alpha \in \Sigma^d$, converging to it. So q belongs to Λ automatically. For details see [P4].

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Received 30 April 1993;
in revised form 28 September 1993