Warsaw, Jan 3, 1994

Dear Alex,

It seems to me I can prove that the upper exponent can be approximated on periodic orbits.

Theorem. Let f be a rational map of the Riemann sphere. Let

$$\chi =: \limsup_{n\to\infty} \sup_{x\in J(f)} \frac{1}{n} \log |(f^n)'(x)| > 0.$$

Then for every $\chi' < \chi$ there exist x, n such that $f^n(x) = x$ and $\frac{1}{n} \log |(f^n)'(x)| \geq \chi'$.

Proof. Let n_1 be an integer such that for every $n > n_1$ and every $x \in J(f)$

$$\frac{1}{n} \log |(f^n)'(x)| < \chi^+ \tag{0}$$

for an arbitrary fixed $\chi^+ > \chi, \chi^+ \approx \chi$.

Fix also an arbitrary $\chi^- < \chi, \chi^- \approx \chi$.

For $\log |(f^n)'(x)| \geq n(\frac{\chi+\chi^-}{2})$ we find $0 < n' \leq n$ such that $m\chi^- - \log |(f^m)'(x)|$ attains at n' its minimum.

Observe that $n' \to \infty$ as $n \to \infty$. In particular for n large we have

$$n' > n_1. \tag{1}$$

Denote by g_m the branch of f^{-m} from a neighbourhood of $y = f^{n'}(x) = y$ such that $g_m(y) = f^{n'-m}(x)$. We obtain

$$\log |g_m'(y)| \leq -m\chi^- \quad \text{for every} \quad 0 < m < n'. \tag{2}$$

We have

$$|f'(g_{m+1}(y))(f^m)'(g_m(y))| \geq \exp(m+1)\chi^-. \tag{3}$$

So for $m > n_1$

$$|f'(g_{m+1}(y))| \geq \exp(m+1)\chi^- \exp -m\chi^+,$$

from (0) applied to $x = g_m(y)$ and $n = m$.

In particular

$$\operatorname{dist}(g_{m+1}(y), \operatorname{Crit}(f)) \geq \frac{1}{2} \exp(m+1)(\chi^+ - \chi^-) \tag{4}$$

For $m < n_1$ using $\chi^- > 0$ we obtain from (3)

$$\operatorname{dist}(g_{m+1}(y), \operatorname{Crit}(f)) \geq (\sup |f'|)^{-n_1} \tag{5}$$

Due to (5) there exists $\varepsilon_0 > 0$ such that if y, n' satisfy (1),(2) then g_m exists on $B(y, \varepsilon_0)$ for every $m = 0, 1, ..., n_1$.

Now our aim is to find a constant ε_1 such that g_m exists on $B(y, \varepsilon_1)$ for every $m = 0, 1, ..., n'$.

We use induction. Suppose g_m is defined on $B(y, r_m)$ and distortion is bounded by

$$C \exp 4m(\chi^+ - \chi^-) \tag{6}$$

Then

$$\mathrm{diam} g_m(B(y, r_m)) < \mathrm{Const}(\exp 4m(\chi^+ - \chi^-))|g_m'(y)|,$$

$$\begin{aligned} \mathrm{diamComp} f^{-1} g_m(B(y, r_m)) &< \mathrm{Const}\nu \exp 4m(\chi^+ - \chi^-)|g_{m+1}'(y) \\ &\le \mathrm{Const}\nu \exp(4m(\chi^+ - \chi^-)) \exp(-(m+1)\chi^-) \end{aligned}$$

where Comp is the component containing $g_{m+1}(y)$ and ν is maximal multiplicity of f at critical points.

So $\mathrm{Comp} f^{-1} g_m(B(y, r_m)) \cap \mathrm{Crit}(f) = \emptyset$ by (4), (if $5(\chi^+ - \chi^-) < \chi^-$)

So the branch g_{m+1} on $B(y, r_m)$ exists (i.e. instead of $\mathrm{Comp} f^{-1} g_m ...$ we can write g_{m+1}). By Koebe distortion th. for $r_{m+1} = r_m(1 - \exp(-m(\chi^+ - \chi^-)))$ the property (6) is satisfied for $m + 1$.

We summarize: the branches $g_m, m = 0, 1, ..., n'$ are well defined on $B(y, \varepsilon_1)$ where

$$\varepsilon_1 = \varepsilon_0 \prod_m (1 - \exp(-m(\chi^+ - \chi^-))).$$

Thus we proved the following:

For every $0 < \chi^- < \chi$ there exists a constant $\varepsilon_1 > 0$ and pairs $x \in J, n > 0$ with n arbitrarily large such that there exist univalent branches g_m of f^{-m} on $B(y, \varepsilon_1)$, where $y = f^n(x)$, such that $g_m(y) = f^{n-m}(x)$ and $|g_m'(y)| \le \exp -m\chi^-$, for every $m = 0, 1, ...n$.

Now let n_0 be an integer such that for every $y \in J(f)$

$$f^{n_0}(B(y, \frac{1}{2}\varepsilon_1)) \supset (J(f))$$

Let $A =: \bigcup_{0<l\le n_0} f^l(\mathrm{Crit}(f))$.

Suppose first that there are no eventually periodic critical points in $J(f)$.

Then there exists $\delta_0 > 0$ such that for every $x \in J(f)$ there exists $0 \le k \le \sharp(A) + 1$ such that $\mathrm{dist}(f^k(x), A) > \delta_0$.

For n large enough $n - k$ is large hence $\mathrm{diam} g_{n-k}(B(y, \frac{2}{3}\varepsilon_1)) < \delta_0$. So we can make n_0 steps backward without meeting critical values of f so that the f^{-n_0} preimage of $g_{n-k}(y)$ lands in $B(y, \frac{1}{2}\varepsilon_1$. Denote the appropriate branch by $f_\nu^{-n_0}$.

For n large enough $|g'_{n-k}(y)|$ is so small that this implies

$$f_\nu^{-n_0}(g_{n-k}(B(y, \frac{2}{3}\varepsilon_1))) \subset B(y, \frac{2}{3}\varepsilon_1).$$

So there is an $n-k+n_0$ - periodic point in $\mathrm{cl}B(y, \frac{2}{3}\varepsilon_1)$ with exponent at least $(n-k+n_0)\chi^-$ (up to a constant). We used here the fact that $f_\nu^{-n_0}g_{n-k}$ is defined and univalent on $B(y, \varepsilon_1)$, Koebe distortion th. and an upper bound for $|f_\nu^{-n_0})'|$ on $g_{n-k}B(y, \frac{2}{3}\varepsilon_1)$ by Const$/\delta_0$. (Recall that $f_\nu^{-n_0}$ is defined and univalent on $B(g_{n-k}(y), \delta_0)$.)

We used here only the fact that $n-k$ is arbitrarily large. So in the case A contains periodic orbits, most of $x, f(x), ..., f^n(x)$ is close to a periodic orbit. So this periodic orbit is *a priori* that periodic orbit we look for.

Remark From the beginning one can find $B(y, \varepsilon)$ such that

$$B_1 =: \text{ cl Comp} f^{-(n+n_0)}(B(y, \varepsilon)) \subset B(y, \varepsilon).$$

and

$$\frac{\mathrm{diam} B_1}{\mathrm{diam} B(y, \varepsilon)} \leq \exp -n\chi^-.$$

Then $F =: f^{n+n_0}$ is polynomial-like on B_1 and has an F-fixed point z in the "local Julia set" $J(F)$. However there is no reason to expect that $|F'(z)|$ is large (of order at least $\exp n\chi^-$).

This approximation by periodic orbits is a "specification" in Bowen's terminology, but our Theorem proves *specification property which respects exponent.*