A counterexample related to a Host-Kra-Maass' problem

Abstract

In 2014, Host, Kra and Maass proved that for a minimal s-step nilsystem but not $(s-1)$-step nilsystem (X,T), there exist positive constants $c(\epsilon), c'(\epsilon)$ and $p \geq s-1$ such that the topological complexity $r(n,\epsilon)$ of (X,T) satisties

$$c(\epsilon)n^p \leq r(n,\epsilon) \leq c'(\epsilon)n^p \quad \text{for every} \quad n \geq 1.$$

Moreover, $c(\epsilon) \to +\infty$ as $\epsilon \to 0$.

A natural question attracting their attention is that what systems have the same topological complexity as nilsystems.

Question. Characterize the minimal TDS (X,T) satisfying the following property: for every $\epsilon > 0$ small enough, there exist constants $c_1(\epsilon)$, $c_2(\epsilon) > 0$ depending only on ϵ such that

$$c_1(\epsilon)n \leq r(n,\epsilon) \leq c_2(\epsilon)n$$

for every $n \geq 1$ and $c_1(\epsilon) \to \infty$ as $\epsilon \to 0$. If in addition, (X,T) is distal, then is it a 2-step nilsystem?

In this talk, we compute the topological complexity of a special class of skew products on the 2-torus (i.e. group extensions over irrational rotations on the torus). Employing a characterization of 2-step nilsytems, we construct a minimal (distal) group extension over an irrational rotation which has the same topological complexity as above but is not a 2-step nilsystem. This answers the latter part of the above question negatively.