Koopman Representations for Positive Definite Functions and Van Der Corput Sets

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Overview

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Positive Definite Sequences

Definition

Let G be a locally compact second countable (l.c.s.c.) group. A function $f : G \to \mathbb{C}$ is **positive definite** if for any $c_1, \cdots, c_n \in \mathbb{C}$ and $g_1, \cdots, g_n \in G$, we have $\sum_{i,j=1}^{n} c_i \overline{c_j} f(g_i^{-1} g_j) \geq 0$. We denote the set of all continuous positive definite functions on G by $\mathbf{P}(G)$.

Theorem (Gelfand and Raikov, [Tem92, Theorem 5.B])

If $\phi \in \mathbf{P}(G)$ then there exists a unitary representation U of G in a Hilbert space $\mathcal{H}$ and a cyclic vector $f \in \mathcal{H}$ such that $\phi(g) = \langle U_g f, f \rangle$.

A consequence of the Gaussian Measure Space Construction is that in the preceding theorem, U can be taken to be the Koopman Representation of a measure preserving action of G.

Special Representation of PD Sequences

Theorem (F., 2023)

If G is l.c.s.c. and $\phi \in \mathbf{P}(G)$, then there exists an ergodic m.p.s. $(X, \mathscr{B}, \mu, \{T\}_{g\in G})$ and $f \in L^2(X,\mu)$ such that $\phi(g) = \langle T_g f, f\rangle$. Furthermore, if ϕ is real-valued, then f can also be taken to be real-valued.

If G is abelian and $\phi \in \mathbf{P}(G)$ satisfies $\phi(0) = 1$, then there exists a probability measure ν on $\widehat{G}$ for which $\phi(g) = \hat{\nu}(g)$.

Theorem (F., 2023)

Let G be a countably infinite abelian group and ν a probability measure on $\widehat{G}$. There exists a m.p.s. $(X, \mathscr{B}, \mu, \{T_g\}_{g\in G})$ and a $f \in L^2(X,\mu)$ satisfying $|f| = 1$, $\int_X f d\mu = \nu(\{0\})$, and $\hat{\nu}(g) = \langle T_g f, f\rangle$. Furthermore, if ϕ is real-valued, then there exists a real-valued $f' \in L^2(X,\mu)$ satisfying $|f'| \le \sqrt{2}$, $\int_X f' d\mu = \frac{1}{\sqrt{2}}\nu(\{0\})$, and $\hat{\nu}(g) = \langle T_g f', f'\rangle$.

An Example

A set $E \subseteq [0,1]$ is **symmetric** if $E = 1 - E$, and it is a **Kronecker set** if for any continuous map $\phi : E \to \mathbb{S}^1$, there exists a sequence of integers $(n_s)_{s=1}^{\infty}$ for which

$$\lim_{s\to\infty} \sup_{x\in E} |\phi(x) - \exp(2\pi i n_s x)| = 0. \tag{1}$$

[CFS82, Appendix 4] has a construction of a perfect Kronecker set.

Theorem (Foiaș-Strătilă,[FS68], [CFS82, Theorem 14.4.2′])

If $E \subseteq [0,1]$ is a symmetric Kronecker set, ν a continuous measure supported on E, and $(X, \mathscr{B}, \mu, \{T^n\}_{n\in\mathbb{Z}})$ is an ergodic m.p.s. with some $f \in L^2(X,\mu)$ for which $\hat{\nu}(n) = \langle T^n f, f\rangle$, then f has a Gaussian distribution.

It follows that for ν as above, we cannot have the system $(X, \mathscr{B}, \mu, \{T^n\}_{n\in\mathbb{Z}})$ be ergodic and $|f| = 1$ at the same time.

Questions

Let G be a l.c.s.c. group and $\mathcal{X} := (X, \mathscr{B}, \mu, \{T_g\}_{g \in G})$ a m.p.s.

Question

If $\phi \in \mathbf{P}(G)$ with $\phi(0) = 1$, does there exists a m.p.s. $\mathcal{X}$ and a $f \in L^2(X, \mu)$ with $|f| = 1$ for which $\phi(g) = \langle T_g f, f \rangle$?

Question (A)

If $\phi \in \mathbf{P}(G)$ is real-valued and satisfies $\phi \geq 0$, then what special m.p.s. $\mathcal{X}$ and $f \in L^2(X, \mu)$ can we pick so that $\phi(g) = \langle T_g f, f \rangle$?

Question (B)

Can we describe the $\phi \in \mathbf{P}(G)$ for which there exists a m.p.s. $\mathcal{X}$ and a $f \in L^2(X, \mu)$ with $f \geq 0$ such that $\phi(g) = \langle T_g f, f \rangle$?

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The Gaussian Measure Space Construction (GMSC)

If G is a l.c.s.c. group and $\phi \in \mathbf{P}$ takes values in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then let $X = \mathbb{K}^G$, let $\mathscr{B}$ denote the product σ-algebra, and let $T_g(x_n)_{n\in G} = (x_{ng})_{n\in G}$. There exists a T-invariant probability measure $\mu = \mu_\phi$ on $(X, \mathscr{B})$ and $\pi_e \in L^2_{\mathbb{K}}(X, \mu)$ for which $\phi(g) = \langle \pi_g, \pi_e \rangle = \langle T_g \pi_e, \pi_e \rangle$, where $\pi_g : X \to \mathbb{K}$ is the projection onto the gth coordinate. In this construction, π_e, and hence all π_g, will have a Gaussian distribution, and $(X, \mathscr{B}, \mu, \{T_g\}_{g\in G})$ is a **Gaussian dynamical system**.

Properties of the GMSC

Theorem ([Gla03, Theorem 3.59])

Let $\mathcal{X} := (X, \mathscr{B}, \mu, \{T_g\}_{g\in G})$ be a Gaussian dynamical system. The following conditions are equivalent:

1. *The system $\mathcal{X}$ is ergodic.*
2. *The system $\mathcal{X}$ is weakly mixing.*
3. *The system $\mathcal{X}$ is weakly mixing on the first chaos, which is the smallest T-invariant subspace of $L^2(X, \mu)$ containing π_e.*

If $G = \mathbb{Z}$ and $\phi \in \mathbf{P}(\mathbb{Z})$ is given by $\phi(n) = e^{2\pi i n\sqrt{2}}$, then the GMSC gives us a m.p.s. $\mathcal{X} := (X, \mathscr{B}, \mu, \{T^n\}_{n\in\mathbb{Z}})$ and a $f \in L^2(X, \mu)$ for which $\langle T^n f, f\rangle = e^{2\pi i n\sqrt{2}}$. Since f is an eigenvector of T for the eigenvalue $e^{2\pi i\sqrt{2}}$, we see that $\mathcal{X}$ is not weakly mixing, so it will not be ergodic either.

Ergodic Representations on Groups

Theorem (Tucker-Drob, 2023)

Let $\phi \in \mathbf{P}(G)$ take values in $\mathbb{K}$. Suppose that there exists a unitary representation U of G on $\mathcal{H}$ that decomposes into a direct sum of finite dimensional representations, and there exists $f \in \mathcal{H}$ for which $\phi(g) = \langle U_g f, f \rangle$. There exists an ergodic m.p.s. $(K, \mathscr{B}, \lambda_K, \{T_g\}_{g\in G})$ and $F \in L^2_{\mathbb{K}}(K, \mu_K)$ for which $\phi(g) = \langle T_g F, F \rangle$.

Proof Sketch: In this case we see that $K := \overline{U(G)}$ is a compact group, and that $\phi : G \to \mathbb{K}$ uniquely extends to some $\tilde{\phi} \in \mathbf{P}(K)$ taking values in $\mathbb{K}$. Since K is compact and $\tilde{\phi}$ is continuous, we see that $\tilde{\phi} \in L^2(K, \lambda_K)$. Letting L denote the left regular representation of K, it is a classical result that there exists some $F \in L^2(K, \lambda_K)$ for which $\tilde{\phi}(k) = \langle L_k F, F \rangle$, so it suffices to restrict L to a representation of G.

Proof Sketch

Theorem (F., 2023)

If G is l.c.s.c. and $\phi \in \mathbf{P}(G)$, then there exists an ergodic m.p.s. $(X, \mathscr{B}, \mu, \{T\}_{g\in G})$ and $f \in L^2(X,\mu)$ such that $\phi(g) = \langle T_g f, f\rangle$. Furthermore, if ϕ is real-valued, then f can also be taken to be real-valued.

Proof Sketch: Let U be a representation of G on $\mathcal{H}$ and f' is cyclic vector for which $\phi(g) = \langle U_g f', f'\rangle$. Let $\mathcal{H} = \mathcal{H}_w \oplus \mathcal{H}_c$, with the restriction of U to $\mathcal{H}_w$ being weakly mixing, and the restriction to $\mathcal{H}_c$ being a group rotation. Let $f' = f'_w + f'_c$ with $f'_w \in \mathcal{H}_w$ and $f'_c \in \mathcal{H}_c$. Using the GMSC and the ergodic representation on groups, we get for $k \in \{w, c\}$ a weakly mixing for w and ergodic for c m.p.s. $\mathcal{X}_k := (X_k, \mathscr{B}_k, \mu_k, \{T_{k,g}\}_{g\in G})$ and some $f_k \in L^2(X_w, \mu_w)$ for which $\langle U_g f'_k, f'_k\rangle = \langle T_{k,g} f_k, f_k\rangle$. We now take $\mathcal{X} = \mathcal{X}_w \times \mathcal{X}_c$, and $f \in L^2(X,\mu)$ given by $f(x,y) = f_w(x) + f_c(y)$.

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Representations using Cesàro Averages 1/2

Theorem (Ruzsa, 1984)

If ν is a probability measure on $\mathbb{T}$, then there exists a sequence of complex numbers $(c_n)_{n=1}^{\infty}$ of modulus 1 for which

$$\hat{\nu}(h) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_{n+h}\overline{c_n} \text{ and } \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n = \nu(\{0\}). \quad (2)$$

This result had appeared implicitly in the work of Ruzsa [Ruz84], and explicitly in the work of Bergelson and Lesigne [BL08, Page 29] when it was generalized from $\mathbb{Z}$ to $\mathbb{Z}^d$.

Representations using Cesàro Averages 2/2

Definition

Let G be a countable abelian group. Let $S(G) \subseteq \mathbb{S}^1$ denote the smallest set containing the image of each character $\chi \in \widehat{G}$.

We see that $S(\mathbb{Z}^d) = \mathbb{S}^1$, $S(\oplus_{n=1}^{\infty}(\mathbb{Z}/m\mathbb{Z}))$ consists of the mth roots of unity, and $S(\oplus_{n=1}^{\infty}(\mathbb{Z}/n\mathbb{Z}))$ consists of all roots of unity.

Theorem (F., 2023)

Let G be a countable abelian group and $(F_n)_{n=1}^{\infty}$ a Følner sequence. There exists a Følner subsequence $(F_n')_{n=1}^{\infty}$ such that for any probability measure ν on $\widehat{G}$, there exists a sequence $(c_g)_{g\in G} \subseteq S(G)$ for which

$$\hat{\nu}(h) = \lim_{N\to\infty} \frac{1}{|F_N'|} \sum_{g\in F_N'} c_{g+h}\overline{c_g} \text{ and } \nu(\{\chi\}) = \lim_{N\to\infty} \frac{1}{|F_N'|} \sum_{g\in F_N'} c_g\overline{\chi(g)}.$$

Proof Sketch

Theorem (F., 2023)

Let G be a countable abelian group and $(F_n)_{n=1}^\infty$ a Følner sequence. There exists a Følner subsequence $(F'_n)_{n=1}^\infty$ such that for any probability measure ν on $\widehat{G}$, there exists a sequence $(c_g)_{g\in G} \subseteq S(G)$ for which

$$\hat{\nu}(h) = \lim_{N\to\infty} \frac{1}{|F'_N|} \sum_{g\in F'_N} c_{g+h}\overline{c_g} \text{ and } \nu(\{\chi\}) = \lim_{N\to\infty} \frac{1}{|F'_N|} \sum_{g\in F'_N} c_g\overline{\chi(g)}.$$

A Proof Sketch: Pick a sequence of monotilings $(\mathcal{T}_k)_{k=1}^\infty$ of G whose tile shapes S_k are $(K_k, \frac{1}{k})$-invariant Følner blocks (see [CC19]). Pick $(F'_k)_{k=1}^\infty$ such that $|F'_k| > k^3|F'_{k-1}| \cdot |S_k|$ and F'_k is sufficiently translation invariant. If $S_k + d$ is some tile of $\mathcal{T}_k$ contained in $F'_k \setminus F'_{k-1}$ and $g + d \in S_k + d$, we let $c_{g+d} = \hat{g}(X_d)$, where $(X_d)_{d\in G}$ are i.i.d. random variables taking values in $\widehat{G}$ with distribution ν.

Creating a m.p.s.

Theorem (F., 2023)

Let G be a countably infinite abelian group and ν a probability measure on $\widehat{G}$. There exists a m.p.s. $(X, \mathscr{B}, \mu, \{T_g\}_{g\in G})$ and a $f \in L^2(X,\mu)$ satisfying $|f| = 1$, $\int_X f d\mu = \nu(\{0\})$, and $\hat{\nu}(g) = \langle T_g f, f\rangle$.

Proof Sketch: Let $(c_g)_{g\in G} \subseteq \mathbb{S}^1$ and $(F_n)_{n=1}^{\infty}$ be such that

$$\hat{\nu}(h) = \lim_{N\to\infty} \frac{1}{|F_N'|} \sum_{g\in F_N'} c_{g+h}\overline{c_g} \text{ and } \nu(\{0\}) = \lim_{N\to\infty} \frac{1}{|F_N'|} \sum_{g\in F_N'} c_g.$$

Let βG denote the Stone-Čech compactification of G, let $\tilde{c} : \beta G \to \mathbb{S}^1$ be the unique continuous extension of $c : G \to \mathbb{S}^1$, and let $T : \beta G \to \beta G$ be given by $T_g(p) = g \cdot p$. The Følner sequence $(F_n)_{n=1}^{\infty}$ can be used to create a mean m on G, which lifts to a measure $\tilde{m}$ on βG, so we take our system to be a standard factor of $(\beta G, \mathscr{B}, \tilde{m}, \{T_g\}_{g\in G})$ and note that $\hat{\nu}(g) = \langle T_g \tilde{c}, \tilde{c}\rangle$.

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Van der Corput sets

Definition ([NRS12])

A set $V \subseteq \mathbb{N}$ is a **van der Corput (vdC) set** if for any unitary operator $U : \mathcal{H} \to \mathcal{H}$ and any $f \in \mathcal{H}$ satisfying $\langle U^v f, f \rangle = 0$ for all $v \in V$, we have $Pf = 0$, where P is the projection onto the subspace of U-fixed points.

Theorem (F., 2023)

The following are equivalent to V being a vdC-set:

1. *For any ergodic m.p.s. $\mathcal{X}$ and any $f \in L^2(X, \mu)$ satisfying $\langle T^v f, f \rangle = 0$ for all $v \in V$, we have $\int_X f d\mu = 0$.*
2. *For any m.p.s. $\mathcal{X}$ and any $f \in L^2(X, \mu)$ satisfying $|f| = 1$ and $\langle T^v f, f \rangle = 0$ for all $v \in V$, we have $\int_X f d\mu = 0$.*
3. *(i) with f real-valued, or (ii) with f real-valued and bounded.*

See [KL18] for a longer list of characterizations of vdC-sets.

Sets of Recurrence

Definition

A set $R \subseteq \mathbb{N}$ is a **set of (measurable) recurrence** if for any m.p.s. $\mathcal{X} := (X, \mathscr{B}, \mu, \{T^n\}_{n \in \mathbb{Z}})$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists some $r \in R$ for which $\mu(A \cap T^r A) > 0$.

Theorem

The following are equivalent to R being a set of recurrence:

1. *For any m.p.s. $\mathcal{X}$, and any $f \in L^2(X, \mu) \setminus \{0\}$ with $f \geq 0$, there exists $r \in R$ for which $\langle T^r f, f \rangle > 0$.*
2. *R is a set of recurrence for all ergodic system.*
3. *(i) holds for all ergodic system.*
4. *For any sequence $(c_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$, we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_{n+r} c_n = 0 \text{ for all } r \in R \Rightarrow \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_n = 0.$$

Ruzsa's Problem

Kamae and Mendes France [KMF78] showed that every vdC-set is a set of recurrence. Ruzsa [Ruz84] asked if every set of recurrence is also a vdC-set? Bourgain [Bou87] constructed the first example of a set of recurrence that is not a vdC-set. Bourgain's construction was then refined by Mountakis [Mou23]. Furthering our understanding of the difference between vdC-sets and sets of recurrence is one purpose of the Questions (A) and (B).

Question (A)

If $\phi \in \mathbf{P}(G)$ is real-valued and satisfies $\phi \geq 0$, then what special m.p.s. $\mathcal{X}$ and $f \in L^2(X, \mu)$ can we pick so that $\phi(g) = \langle T_g f, f \rangle$?

Question (B)

Can we describe the $\phi \in \mathbf{P}(G)$ for which there exists a m.p.s. $\mathcal{X}$ and a $f \in L^2(X, \mu)$ with $f \geq 0$ such that $\phi(g) = \langle T_g f, f \rangle$?

Quasi-van der Corput sets

Let us provisionally define $Q \subseteq \mathbb{N}$ to be a **quasi-vdC-set** if for any m.p.s. $(X, \mathscr{B}, \mu, \{T^n\}_{n\in\mathbb{Z}})$ and any measurable $f : X \to \{-1, 1\}$, we have that

$$\langle T^q f, f\rangle = 0 \ \forall \ q \in Q \Rightarrow \int_X f d\mu = 0. \tag{3}$$

Theorem

Q is a quasi-vdC set if and only if for any m.p.s. $\mathcal{X}$ and any $A \in \mathscr{B}$ satisfying $\mu(A) = \frac{1}{4} + \mu(A \cap T^q A)$ for all $q \in Q$, we have $\mu(A) = \frac{1}{2}$. Hence quasi-vdC$\Rightarrow$recurrence.

Question ([KL18])

Is every quasi-vdC set a vdC-set? Is every set of recurrence a quasi-vdC set?

Nice sets

Definition

$R \subseteq \mathbb{N}$ is a **set of nice recurrence** if for any m.p.s. $\mathcal{X}$, any $A \in \mathscr{B}$, and any $\epsilon > 0$ there exists $r \in R$ for which $\mu(A \cap T^r A) > \mu(A)^2 - \epsilon$.

Definition

$V \subseteq \mathbb{N}$ is a **nice vdC-set** if for any $(c_n)_{n=1}^{\infty} \subseteq \mathbb{S}^1$ we have

$$\sup_{v \in V} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} c_{n+v} \overline{c_n} \right| \geq \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} c_n \right|^2$$

Definition

$F \subseteq \mathbb{N}$ is a **nice FC$^+$-set** if for any probability measure ν on $\mathbb{T}$ we have $\sup_{f \in F} |\hat{\nu}(f)| \geq \nu(\{0\})$.

Relations between Nice sets

Theorem (Bergelson and Lesigne, [BL08, Section 3])

(i) *Every nice FC^+-set is a nice vdC-set and a set of nice recurrence.*

(ii) *If V is a nice vdC-set and ν is a probability measure on $\mathbb{T}$, then $\sup_{v\in V}|\hat{\nu}(v)| \geq \nu(\{0\})^2$.*

Theorem (F., [Far22, Chapter 5])

(i) *Every nice vdC-set is a set of nice recurrence.*

(ii) *F is a nice FC^+-set if and only if for any $(c_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ satisfying $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}|c_n|^2 < \infty$, we have*

$$\sup_{v\in V} \limsup_{N\to\infty} \frac{1}{N}\left|\sum_{n=1}^{N} c_{n+v}\overline{c_n}\right| \geq \limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} c_n\right|^2.$$

More Quesitons

Question (Bergelson and Lesigne, [BL08, Section 3])

Is every set of nice recurrence also a nice vdC-set? Is every nice vdC-set also a nice FC^+-set?

Question

Is there any relationship between the class of sets of nice recurrence and the class of vdC-sets?

Question

Suppose that $E \subseteq [0,1]$ is a symmetric Kronecker set and ν is a probability measure supported on E. What can we say about $Z_\nu := \{n \in \mathbb{N} \mid \hat{\nu}(n) = 0\}$?

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