A Quantum Route to Hamilton-Jacobi Theory: Considerations on the Quantum-to-Classical Transition

Collaborators:
J. F. Cariñena, A. Ibert, G. Morandi
N. Mukunda, G. Sudarshan, G. Esposito, J. Clemente-Gallardo
X. Gràcia, E. Martínez, M. Muñoz, N. Román
J. Grabowski, M. Kus
Lie-Scheffers System: A Geometric Approach

J.F. Cariñena, J. Grabowski, G. Marmo
Two quotations from Dirac

The Method of Classical Analogy

“... The value of classical analogy in the developments of quantum mechanics depends on the fact that classical mechanics provides a valid description of dynamical systems...”

“... Classical mechanics must therefore be a limiting case of quantum mechanics...”
Two quotations from Dirac

The Hamiltonian form of field dynamics

“..In classical dynamics, one has usually supposed that, when one has solved the equation of motion, one has done everything worth doing...”

Talking about the family of solutions of Hamilton equations which fill a Lagrangian submanifold transversal to the fibers of the cotangent bundle, Dirac says:

“... The family does not have any importance from the point of view of Newtonian mechanics; but it is a family which corresponds to one state of motion in the quantum theory, so presumable the family has some deep significance in nature, not yet properly understood...”
General settings

• \( \dot{q} = \frac{\partial H}{\partial p} \), \( \dot{p} = -\frac{\partial H}{\partial q} \) \hspace{1cm} \text{Hamilton equations}

• \( H \left( q, \frac{\partial S}{\partial q} \right) = E = \frac{\partial S}{\partial t} \) \hspace{1cm} \text{Hamilton-Jacobi}

• \( H(\vec{q}, -i\hbar\vec{\nabla}) = i\hbar \frac{\partial}{\partial t} \) \hspace{1cm} \text{Schrödinger}
Linear versus non-linear

Lie-Scheffers ideology: superposition rules

\[ \Gamma = a^j(t)X_j, \quad [X_j, X_k] = c^l_{jk}X_l \]

\[ \mathcal{H}, \quad \mathcal{R}(\mathcal{H}), \quad U(\mathcal{H}) \]

\[ TU(\mathcal{H}) \equiv [U(\mathcal{H})]^\mathbb{C}, \quad Tu(\mathcal{H}) \iff T^*u(\mathcal{H}) \iff T^*u^*(\mathcal{H}) \]

\( Tu(\mathcal{H}) \) is a \( C^* \)-algebra

GNS-construction: from \( Tu(\mathcal{H}) \) to \( \mathcal{H} \).

With any \( C^* \)-algebra on any manifold \( M \), we construct Hilbert spaces.
Second order dynamics on $U(\mathcal{H})$

Quantum systems on $\mathcal{R}(\mathcal{H})$

\[
\frac{d}{dt} \left( \frac{\vert \psi \rangle \langle \psi \vert}{\langle \psi \vert \psi \rangle} \right) = \frac{A \vert \psi \rangle \langle \psi \vert}{i\hbar \langle \psi \vert \psi \rangle} - \frac{1}{i\hbar} \frac{\vert \psi \rangle \langle \psi \vert}{\langle \psi \vert \psi \rangle} A
\]

$\mathcal{R}(\mathcal{H})$ homogeneous space of $U(\mathcal{H})$

$T\mathcal{R}(\mathcal{H})$ homogeneous space of $TU(\mathcal{H}) \simeq GL(\mathcal{H})$

Quantum systems on $\mathcal{R}(\mathcal{H}) \equiv$

complete solution of H.J. associated with

\[
\mathcal{L} = \frac{1}{2} \text{tr}(\rho_\psi (U^{-1} \dot{U})^2) = \frac{1}{2} \frac{\langle \psi \vert (U^{-1} \dot{U})^2 \vert \psi \rangle}{\langle \psi \vert \psi \rangle}
\]
Dirac interaction picture

On the group

\[ \mathcal{L} = \frac{1}{2} \text{tr}(U^{-1} \dot{U})^2 \rightarrow \text{E.L. equations } \frac{d}{dt}(U^{-1} \dot{U}) = 0 \]

Family of generalized solutions of the H.J. on the group

\[ \theta_\mathcal{L} = \text{tr}(U^{-1} \dot{U})(U^{-1} dU) \]
\[ \omega_\mathcal{L} = \text{tr } d \left[ (U^{-1} \dot{U}) \wedge (U^{-1} dU) \right], \quad d(U^{-1} U) = 0 \]
\[ U^{-1} \dot{U} = iA, \quad iX(U^{-1} dU) = iA, \quad A \text{ Hermitian} \]
\[ X^*(\theta_\mathcal{L}) = i \text{tr}(AU^{-1} dU) \]
Unfolding nonlinear classical dynamics

Quantization: A procedure to associate linear equations with non-linear ones.

More specifically: Write the equations of motion as a Lie-Scheffers system.

\[ i\hbar \frac{d}{dt} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \xi = \frac{z_1}{z_2} \Rightarrow \]

\[ \Rightarrow i\hbar \frac{d}{dt} \xi = H_{12} + (H_{11} - H_{22})\xi - H_{21}\xi^2 \]

From an equation on the group we obtain a linear equation on a vector space for any representation of the group.
More on Schrödinger equation and Hamilton-Jacobi equation

\[
i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \Delta \psi + V \psi, \quad \psi = Ae^{iS/\hbar}
\]

We obtain

\[
\frac{\partial A}{\partial t} = -\frac{1}{2m} (A\Delta S + 2 \text{grad}A \cdot \text{grad}S)
\]

\[
\frac{\partial S}{\partial t} = - \left[ \frac{(\text{grad}S)^2}{2m} + V(\vec{r}) - \frac{\hbar^2}{2m} \frac{\Delta A}{A} \right]
\]

Setting \( \vec{u} = \frac{1}{m} \text{grad}S \), \( A^2 = \psi^*\psi = \rho \), \( -\frac{\hbar^2}{2m} \frac{\Delta A}{A} = - \left( \frac{\hbar}{2} \right)^2 \frac{1}{m} \frac{\Delta \rho}{\rho} \)

\[
\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{u}) = 0
\]

\[
\int_{\mathbb{R}^3} \rho \, d^3x = 1
\]

Hamilton-Jacobi \( \Leftrightarrow \frac{\hbar^2}{2m} \frac{\Delta A}{A} = 0 \)
Field theoretical aspect

\[ L = \int_V \rho \left( \frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + V \right) d^3x \, dt, \quad \delta S|_{\partial V} = 0, \quad \delta \rho|_{\partial V} = 0 \]

\[ L = \int_V \rho \left( \frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + \left( \frac{\hbar}{2} \right)^2 \frac{1}{2m \rho^2} \nabla \rho \cdot \nabla \rho + V \right) d^3x \, dt \]

Fixed end-point variation with respect to \( S \) leads to the continuity equation

\[ \frac{\partial \rho}{\partial t} + \nabla \left( \rho \frac{1}{m} \nabla S \right) = 0 \]

Fixed end-point variation with respect to \( \rho \) leads respectively to

\[ \frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + V = 0 \]

\[ \frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + V + \left( \frac{\hbar}{2} \right)^2 \left( \frac{1}{2m \rho^2} \nabla \rho \cdot \nabla \rho - \frac{2}{\rho} \nabla^2 \rho \right) = 0 \]
The fate of the continuity equation

\[
\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{u}) = 0
\]

Geometric interpretation:

\[
dS : Q \times \Lambda \rightarrow T^* Q
\]

\[
dS_*(\theta_0) = \frac{\partial^2 S}{\partial \lambda_j \partial q_k} d\lambda_j \wedge dq_k
\]

A symplectic structure.

Liouville theorem on \( Q \times \Lambda \times \mathbb{R} \) gives the continuity equations.
Generalized Coherent States

Embedding “classical manifolds” into $\mathcal{R}(\mathcal{H})$.

$M$ a manifold with a volume form $\Omega$.

$$m \mapsto |\psi(m)\rangle \in \mathcal{H} \quad \Rightarrow \quad \frac{|\psi(m)\rangle\langle\psi(m)|}{\langle\psi(m)|\psi(m)\rangle}$$

Pull-back of Hermitian tensor fields

$$\langle d\psi(m)|d\psi(m)\rangle \to \text{Riemannian + skew-symmetric (2,0) tensors on } M$$

Fisher-Rao quantum information metric.

Transformations $T : |\psi(m)\rangle \to |\psi(m')\rangle$ defines

$$\phi_T : M \to M, \quad m \mapsto m'$$
Generalized Coherent States

Examples: Quantizers and Dequantizers

\[ W : m \mapsto W(m) \in u(\mathcal{H}) \quad |0\rangle \in \mathcal{H} \text{ a fiducial state} \]
\[ |\psi(m)\rangle = W(m)|0\rangle \]

\[ \mathcal{O}_P(\mathcal{H}) \to \mathcal{F}(M) \]
\[ A \mapsto \text{tr}(A W(m)) = f_A(m), \quad (f_A * f_B)(m) = \text{tr}(A B W(m)) \]

Remark: When \( M \) is a group and \( W \) a representation, the \(*\)-product becomes the convolution product on the group algebra.
If $D : M \rightarrow u(\mathcal{H})$ is another association such that

$$\text{tr } D(m) W(m') = \delta(m, m'),$$

we define a “quantizer” map

$$A = \int_M f_A(m) D(m) d\mu_\Omega$$

$$M = T^*\mathbb{R}^n \quad (q, p) \mapsto D(q, p) \quad \text{Weyl system, projective unitary representation}$$

$$(f_A * f_B)(q, p) \quad \text{Moyal product}$$

Locality versus non-locality of the product.
“Quantum Hamilton-Jacobi”

Replace $Q \times \Lambda$ with operators

$$\hat{p} = \frac{\partial}{\partial \hat{q}} S(\hat{q}, \hat{\lambda}, t), \quad \hat{P} = -\frac{\partial}{\partial \hat{\lambda}} S(\hat{q}, \hat{\lambda}, t)$$

$$H \left( \hat{q}, \frac{\partial S}{\partial \hat{q}}, t \right) + \frac{\partial}{\partial t} S(\hat{q}, \hat{\lambda}, t) = 0$$

“Well ordering”

$$S(\hat{q}, \hat{\lambda}, t) = \sum_{\alpha} f_{\alpha}(\hat{q}, t) g_{\alpha}(\hat{\lambda}, t)$$

Main ingredient

$$\langle q | S(\hat{q}, \hat{\lambda}, t) | \lambda \rangle = S(q, \lambda, t) \langle q | \lambda \rangle$$

$$\Rightarrow \frac{1}{2m} \left[ \left( \frac{\partial}{\partial q} S'(q, \lambda, t) \right)^2 - i\hbar \frac{\partial^2}{\partial q^2} S(q, \lambda, t) \right] + V(q) + \frac{\partial}{\partial t} S(q, \lambda, t) = 0$$

$$\psi(q, \lambda, t) = e^{i\frac{\hbar}{2m} S(q, \lambda, t)}$$
Summarizing

Nonlinear equations \( \rightarrow \) linear equations
(classical) \( \rightarrow \) (quantum)

Lie Scheffers: \( \Gamma = a^i X_j \)

\( g^{-1} \dot{g} = a^j X_j \)

\( U^{-1}(g) \dot{U}(g) = a^j A_j, \quad U(g) \in u(\mathcal{H}) \)