Oct 4, 2022

Baltic Set Theory Seminar, Fall 2022

A stationary-tower-free proof of Woodin's Sealing Theorem

(with G. Sargsyan and B. Wcisło)

Shoenfield's Absoluteness Theorem: Σ^1_2-facts are forcing absolute

Thm (Woodin): If A is a universally Baire set of reals and there is a class of Woodin cardinals, then the theory of $L(A, \mathbb{R})$ cannot be changed by forcing, i.e., for any set generic extensions $V[g] \subseteq V[g*h]$, there is an elem. emb.

$$j : L(A_g, \mathbb{R}_g) \longrightarrow L(A_{g*h}, \mathbb{R}_{g*h}).$$

$\mathbb{R}^{V[g]}$

The canonical interpretation of A in $V[g]$, i.e., if (T,S) witness that A is uB, then $A_g = (p[T])^{V_g}$.

Definition (Woodin):

$\boxed{\text{Sealing}}$ is the conjunction of the following statements:

(1) For every set gen. g over V, $L(\Gamma^\infty_g, \mathbb{R}_g) \models AD^+$ and

$$\mathcal{P}(\mathbb{R}_g) \cap L(\Gamma^\infty_g, \mathbb{R}_g) = \Gamma^\infty_g.$$

(2) For every set generic extensions $V[g] \subseteq V[g*h]$, there is an elem. emb

$\mathbb{R}^{V[g]}$

$$j : L(\Gamma^\infty_g, \mathbb{R}_g) \longrightarrow L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h})$$

the uB sets of reals in $V[g]$

s.t. for every $A \in \Gamma^\infty_g$, $j(A) = A_h$.

Question A: Is there a large cardinal that implies Sealing?

Observation: Mice cannot satisfy Sealing, clause (1).

Question B: Does Sealing hold after collapsing a large cardinal? Yes.

What we know about Sealing

- (Woodin) Sps. there is a proper class of Woodin cards and κ is a supercompact card. Then Sealing holds after collapsing 2^{2^κ} to be countable.

- (Sargsyan-Trang, 2019) If there is a Woodin limit of Woodin cards, then Sealing is consistent.

- (Sargsyan-Trang, 2019) Suppose there is a proper class of Woodin cards and a strong card, and self-iterability holds. Then Sealing holds after collapsing the successor of the least strong card to be countable.

- (M-Sargsyan-Wcisło, 2022) There is a new stationary-tower-free proof of Woodin's Sealing Theorem, build on (Sargsyan, Trang, 2019).

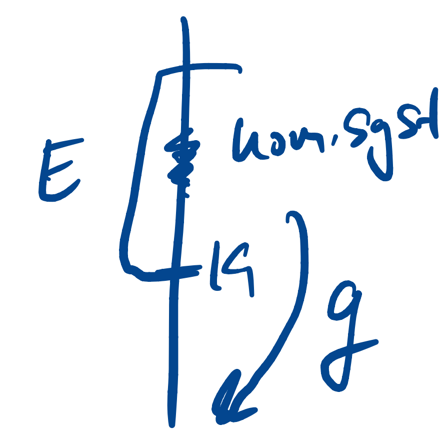
Outline of the proof

afterwards

- Analyze $L(\Gamma_g^\infty, \mathbb{R}_g)$ and obtain it as a derived model of a direct limit model

make reals generic + make uB set generic, in fact their representations

We need to ensure iterability of the models in question, for this we will use the s.c.!



now + next week

- This requires to preserve uB sets to ultrapowers via extenders

↳ isolate a property that allows us to do that

here the s.c. will also show up.

V, $V[g]$, $g \subseteq \mathrm{Col}(\omega, 2^{2^\kappa})$, κ s.c.

WTS: $V[g] \models$ Sealing.

$L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h})$

further gen. ext.

<u>Def</u>: A set $A \subseteq X^\omega$ is <u>γ-hom. Suslin</u> if there is a γ-complete homogeneity system $\bar{\mu} = (\mu_s \mid s \in X^{<\omega})$ and a tree T s.t. $A = p[T]$ and, for all $s \in X^{<\omega}$, $\mu_s(T_s) = 1$.

$T_s = \{t \mid s^\frown t \in T\}$

In particular, $A = \{x \in X^\omega \mid (\mu_{x \restriction n} \mid n < \omega)$ is well-founded$\}$.

We write TW_Y for the set of all towers of measures $\vec{\mu} = (\mu_i \mid i < \omega)$ s.t. $\mu_i \in Y$.

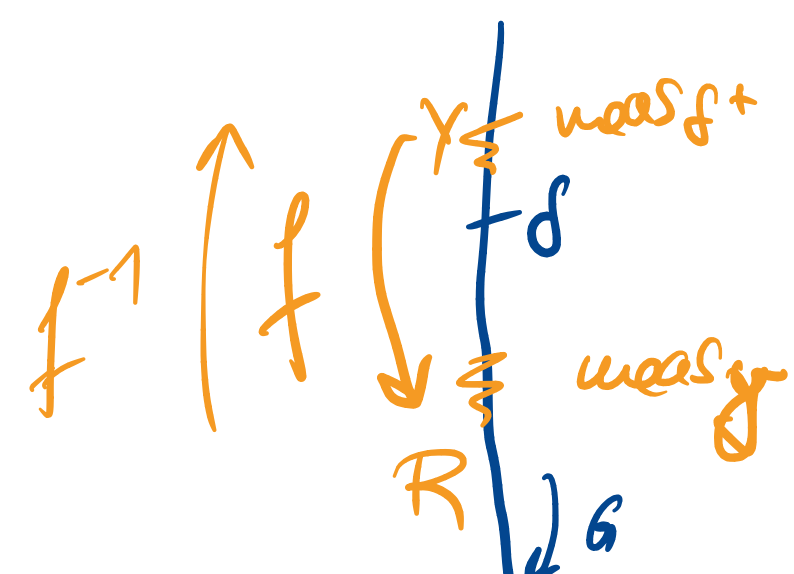
Flipping functions

Lemma (Flipping lemma):
Let δ be a Woodin card. and let $Y \subseteq \mathrm{meas}_{\delta^+}$ be s.t. $|Y| < \delta$.
Then for any $\gamma < \delta$, there is some $R \subseteq \mathrm{meas}_\gamma$ as well as a Lipschitz function $f : TW_Y \to TW_R$ s.t.

(1) f is 1-to-1, and

(2) for all $<\gamma$-generics G and all $\vec{\mu} \in (TW_Y)^{V[G]}$,

$$\vec{\mu} \text{ is well-founded} \iff f(\vec{\mu}) \text{ is ill-founded}$$



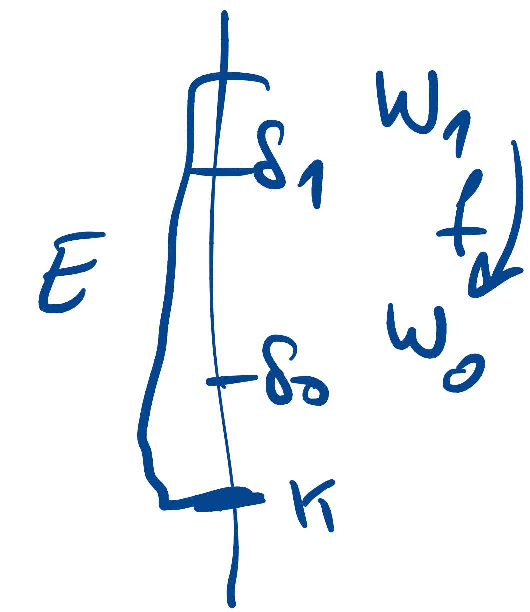
<u>Def</u>: Let κ be a card., let $\delta_0 < \delta_1$ be Woodin card. above κ.

Let $W_0 \leq \text{meas}_{\kappa^+}$ with $W_0 \in V_{\delta_0}$,

let $W_1 \leq \text{meas}_{\delta_0^+}$ with $W_1 \in V_{\delta_1}$, $|W_1| < \delta_0$,

let $f: TW_{W_1} \longrightarrow TW_{W_0}$ be a flipping function.

Let E be an extender with crit. pt. κ.



Then E is uB-preserving for (W_0, W_1, f) iff

(1) $lh(E) = str(E) > \delta_1$, and

(2) $\boxed{\text{in } Ult(V,E)}$ there are Lipschitz embeddings h_0, h_1 s.t.

$$\begin{tikzcd} TW_{W_1} \arrow[r, "h_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & \pi_E(TW_{W_1}) \arrow[d, "\pi_E(f)"] \\ TW_{W_0} \arrow[r, "h_0"] & \pi_E(TW_{W_0}) \end{tikzcd}$$

Keep in mind: how systems will live in some $V[g]$. But the measures come from V-measures.

Question: What is the large card. assumption necessary to get such an E?

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Def. We say a cardinal κ is <u>uB-preserving</u> if for all Woodin cards. $\delta_0 < \delta_1$ above κ and for all W_0, W_1, and f, as above, there is an extender E with c.p. κ s.t. E is uB-preserving for (W_0, W_1, f).

<u>Lemma</u>: Sps. there is a proper class of Woodin cardinals and let κ be a supercompact cardinal. Then κ is uB-preserving.

<u>proof</u>: Let $\kappa < \delta_0 < \delta_1$, W_0, W_1, f be as above.
Let $j : V \to M$ be an elem. emb. witnessing that κ is δ_1^+-s.c.
and let E be the (κ, δ_1^+)-extender derived from j.
Now $h_0 = j \upharpoonright \tau W_{W_0}$ and $h_1 = j \upharpoonright \tau W_{W_1}$. Then $h_0, h_1 \in M$.
Moreover, $h_0 \circ f = j(f) \circ h_1$.

Let $\varphi(f, j(f)) \equiv$ "$\exists h_0, h_1$ as above"

Then $M \models \varphi(f, j(f))$.

$$\begin{tikzcd}
V \arrow[r] \arrow[dr, "\pi_E"'] & M \\
 & \mathrm{Ult}(V,E) \arrow[u, "k"']
\end{tikzcd}$$

factor map

Let $k : \mathrm{Ult}(V,E) \to M$ be the factor map.

Then $k \restriction V_{\delta_1} = id$ and hence $(f, j(\varphi)) \in rng(k)$.

In fact, $k^{-1}(\varphi) = f$ and $k^{-1}(j(\varphi)) = \pi_E(f)$.

Therefore, $\mathrm{Ult}(V,E) \models \varphi(f, \pi_E(\varphi))$.

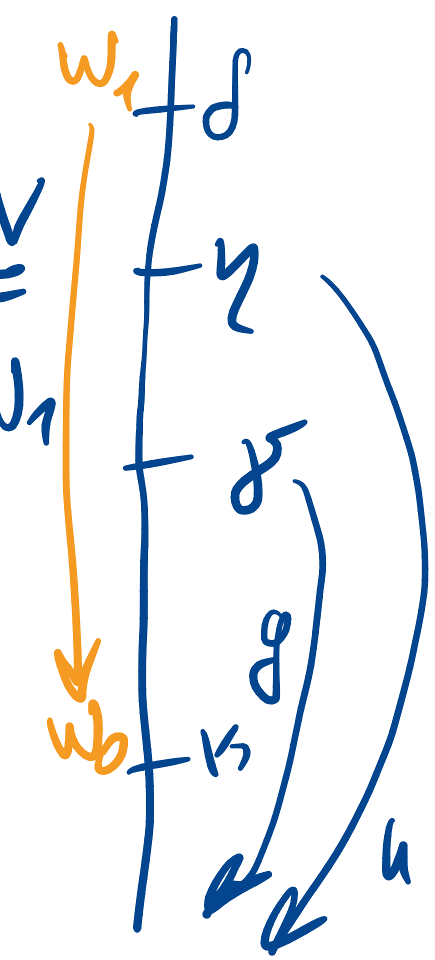
$$\begin{tikzcd}
TW_{w_1} \arrow[r, "u_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & \pi_E(W_{w_1}) \arrow[d, "\pi_E(f)"] \\
TW_{w_0} \arrow[r, "u_0"'] & \pi_E(W_{w_0})
\end{tikzcd}$$

$\boxed{\text{In } \mathrm{Ult}(V,E)}$

$\square$

Aim: Show that uB-preserving extenders in fact preserve the uBness of a given set A to their ultrapower and its generic extensions, even if A is only uB in some $V[g]$.

Sps. there is a proper class of Wdns, let A be uB in $V[g]$, for some $g \subseteq Col(\omega, \gamma)$ for some fixed γ. Moreover, let h be $Col(\omega, \eta)$-generic over $V[g]$ for some fixed $\eta > \gamma$.
Let $\delta > \eta$ be Woodin and let $W_1 \subseteq meas_{\delta^+}$ be measures in V s.t. there is a hom. sys. $\bar{\mu}^{(1)}$ for A in $V[g]$ with meas. from W_1 (... their lift ups to $V[g]$).



By the flipping lemma, there is some $W_0 \subseteq meas_{\kappa^+}$ and a flipping function $f: TW_{W_1} \to TW_{W_0}$.

Let $\boxed{\bar{\mu}^{(0)} = f''\bar{\mu}^{(1)}}$.

Lemma: In the situation above, let E be an $\mathbb{R}$-complete uB-preserving V-extender for (W_0, W_1, f) with cp. κ and length $\geq \delta$. Then A is δ^+-uB in $\mathrm{Ult}(V,E)[g][h]$.

proof: Fix h_0, h_1, witnessing that E is uB-preserving.
Recall that $\bar{\mu}^{(1)}$ witnesses that A is δ^+-hom. Suslin in $V[g]$.
As A is δ^+-uB, we can make sense of $A_h^{V[g][h]}$.

Claim: For any $u \in \mathrm{Ult}(V,E)[g][h] \cap \mathbb{R}$,

$$u \in A_h^{V[g][h]} \iff \left(h_0'' \bar{\mu}^{(0)}_{u \restriction n} \mid n < \omega \right) \text{ is ill-founded}$$

$$\iff \left(h_1'' \bar{\mu}^{(1)}_{u \restriction n} \mid n < \omega \right) \text{ is well-founded}$$

In particular, A is δ^+-hom. Suslin in $\mathrm{Ult}(V,E)[g]$ and

$$A_h^{V[g][h]} \cap \mathrm{Ult}(V,E)[g][h] = A_h^{\mathrm{Ult}(V,E)[g][h]}.$$

Pf of claim: Let M be the collapse of a ctble hull of V_Θ for some large Θ. Take $\bar{g} \in V$ gen over M. Let $\bar{E}$ be the collapse of E.
Let $\bar{h} \in V$ be $Col(\omega, \bar{\eta})$-gen over $M[\bar{g}] \supseteq Ult(M, \bar{E})[\bar{g}]$.

WTS: For any $u \in Ult(M, \bar{E})[\bar{g}][\bar{h}] \cap \mathbb{R}$,

$$u \in A \iff (\bar{h}_0 '' \xi^{(0)}_{u \restriction n} \mid n<\omega) \text{ is ill-fdd}$$
$$\iff (\bar{h}_1 '' \xi^{(1)}_{u \restriction n} \mid n<\omega) \text{ is well-fdd}$$

where $\xi^{(0)}, \xi^{(1)}, \bar{h}_0, \bar{h}_1, \bar{f}$ are the collapses of $\bar{\mu}^{(0)}, \bar{\mu}^{(1)}, h_0, h_1, f$.

Let H be a hull of V_Θ,

(1) $\mathbb{R}^V \subseteq H$

(2) $|H| = |\mathbb{R}^V|$

(3) $M \prec H$

$$\begin{tikzcd}
 & V_\Theta & \\
 & H \arrow[u, "\sigma^H"] \arrow[r] & Ult(H, \sigma^M(\bar{E})) \arrow[ul, dashed, "\text{realization}"'] \\
 & M \arrow[u, "\sigma^M"] \arrow[r] & Ult(M, \bar{E}) \arrow[u, "\sigma^*"']
\end{tikzcd}$$

h_0^*, h_1^*

$\bar{\sigma}^{0}, \bar{\sigma}^{(1)}$

$\xi^{(0)}, \xi^{(1)}$

$\bar{h}_0, \bar{h}_1$

__Subclaim:__ For any $u \in \mathcal{U}^{\mathrm{ext}}(M, \bar{E})[\bar{g}][\bar{h}] \cap \mathbb{R}$,

$$(\bar{h}_0 '' \bar{\xi}^{(0)}_{u\restriction n} \mid n < \omega) \text{ is well-fdd} \iff (\bar{h}_1 '' \bar{\xi}^{(1)}_{u\restriction n} \mid n<\omega) \text{ is ill-fdd}.$$

Pf:

$\pi_E(\bar{f})$ is a flipping function

$$(\bar{h}_1 '' \bar{\xi}^{(1)}_{u\restriction n} \mid n<\omega) \text{ is ill-fdd} \iff (\pi_E(\bar{f}) '' \bar{h}_1 '' \bar{\xi}^{(1)}_{u\restriction n} \mid n<\omega) \text{ is well-fdd}$$

$$\iff (\bar{h}_0 '' \bar{f} '' \bar{\xi}^{(1)}_{u\restriction n} \mid n<\omega) \text{ is well-fdd}$$

$$\iff (\bar{h}_0 '' \bar{\xi}^{(0)}_{u\restriction n} \mid n<\omega) \text{ is well-fdd},$$

$$\begin{tikzcd}
\bar{W}_1 \arrow[r, "\bar{h}_1"] \arrow[d, "\bar{f}"'] \arrow[dr, phantom, "\circlearrowleft"] & \pi_E(\bar{W}_1) \arrow[d, "\pi_E(\bar{f})"] \\
\bar{W}_0 \arrow[r, "\bar{h}_0"'] & \pi_E(\bar{W}_0)
\end{tikzcd}$$

since $\bar{\xi}^{(0)} = \bar{f} '' \bar{\xi}^{(1)}$. $\square$

Note: The same argument yields

$$(\iota '' h_0^* '' \bar{\nu}^{(0)}_{u\restriction n} \mid n<\omega) \text{ is well-fdd} \iff (\iota '' h_1^* '' \bar{\nu}^{(1)}_{u\restriction n} \mid n<\omega) \text{ is ill-fdd}.$$

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$L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h})$ ← Aim: Find a derived model representation of this

uB sets in $V[g][h]$

$\mathbb{R}^{V[g][h]}$

$g \subseteq Col(\omega, 2^{2^\kappa})$ s.c.

→ For notational simplicity, assume that h is trivial.

More precisely: Find some model N s.t. $\omega_1^{V[g]}$ is a limit of Woodin cardinals in N and

$$\boxed{L(\Gamma^\infty_g, \mathbb{R}_g) = L(Hom^*, \mathbb{R}^*)^{N[G]},}$$

where G is $Col(\omega, <\omega_1^{V[g]})$-generic over N.

Let g' be $\mathrm{Col}(\omega, \Gamma_g^\infty)$-gen. over $V[g]$ and let

$(A_i \mid i<\omega)$ be an enumeration of Γ_g^∞ in $V[g*g']$ and

let $(x_i \mid i<\omega)$ be an enumeration of $\mathbb{R}^{V[g]}$.

Let $\kappa < \delta_0 < \delta_1 < \delta_2 < \delta_3 < \delta_4$ be Woodin cards.

As there is a proper class of Woodins, each A_i is hom. Suslin in $V[g]$.

For each $i<\omega$, let $W_2^{(i)} \in V_{\delta_4}$ be a set of measures with $W_2^{(i)} \subseteq \mathrm{meas}_{\delta_3^+}$.

Fix hom. syst. $\bar{\mu}_2^{(i)}$ s.t. $A_i = \{ x \in \omega^\omega \mid ((\bar{\mu}_2^{(i)})_{x \restriction n} \mid n<\omega) \text{ is wfdd} \}$.

By Flipping Lemma, there is $W_1^{(i)} \subseteq \mathrm{meas}_{\delta_2^+}$ and 1-1 Lipschitz function

$f_1^{(i)} : TW_{W_2^{(i)}} \to TW_{W_1^{(i)}}$ s.t. for all $<\delta_2^+$-generics H

and all $\vec{\mu} \in (TW_{W_2^{(i)}})^{V[H]}$, $\vec{\mu}$ is well-fdd iff $f_1^{(i)}(\vec{\mu})$ is ill-fdd.

Again, by the Flipping Lemma, there is some $W_0^{(i)} \leq \mathrm{meas}_{K^+}$ as well as a 1-to-1 Lipschitz function $f_0^{(i)} : TW_{W_1^{(i)}} \to TW_{W_0^{(i)}}$.

Using the flipping functions, let

$$\overline{\mu}_1^{(i)} = f_1^{(i)} \text{"} \, \overline{\mu}_2^{(i)} \quad \text{and}$$

$$\overline{\mu}_0^{(i)} = f_0^{(i)} \text{"} \, \overline{\mu}_1^{(i)}.$$

Pick a suff. large ordinal Θ and a sequence

$(\hat{M}_i, M_i, g_i \mid i<\omega)$ of models $\hat{M}_i$ and M_i as well as M_i-generics g_i

together with elem. emb.

$\hat{\pi}_i : \hat{M}_i \to V_\Theta[g]$ and $\pi_i : M_i \to V_\Theta$ s.t. for all $i<\omega$

(1) $\hat{M}_i = M_i[g_i]$ is ctble in $V[g]$,

(2) $\pi_i \in V[g]$,

(3) $\bar{\mu}_2^{(i)} \in \mathrm{rng}(\pi_i)$, $f_1^{(i)}, f_0^{(i)} \in \mathrm{rng}(\pi_i)$,

(4) $\mathrm{rng}(\pi_i) \subseteq \mathrm{rng}(\pi_{i+1})$, and

(5) $\pi_i \restriction \mathcal{P}(\kappa) = \mathrm{id}$.

Call such a sequence a *block*.

In order to ensure that we can keep iterating these M_i's, we will realize them into a large model W.

Let $j : V \to W$ be a θ-s.c. embedding with c.p. κ. Then $W^\theta \subseteq W$ and $j(\kappa) > \theta$. In part., $V_\theta \subseteq W$ and g is generic over W.

Note that $j \circ \pi_i \in W[g]$ as $W[g]$ is closed under ctble seq. in $V[g]$.

Write $\sigma_i = j \circ \pi_i : M_i \to W_{j(\theta)}$. Then

(1) $\hat{M}_i = M_i[g_i]$ is ctble in $W[g]$

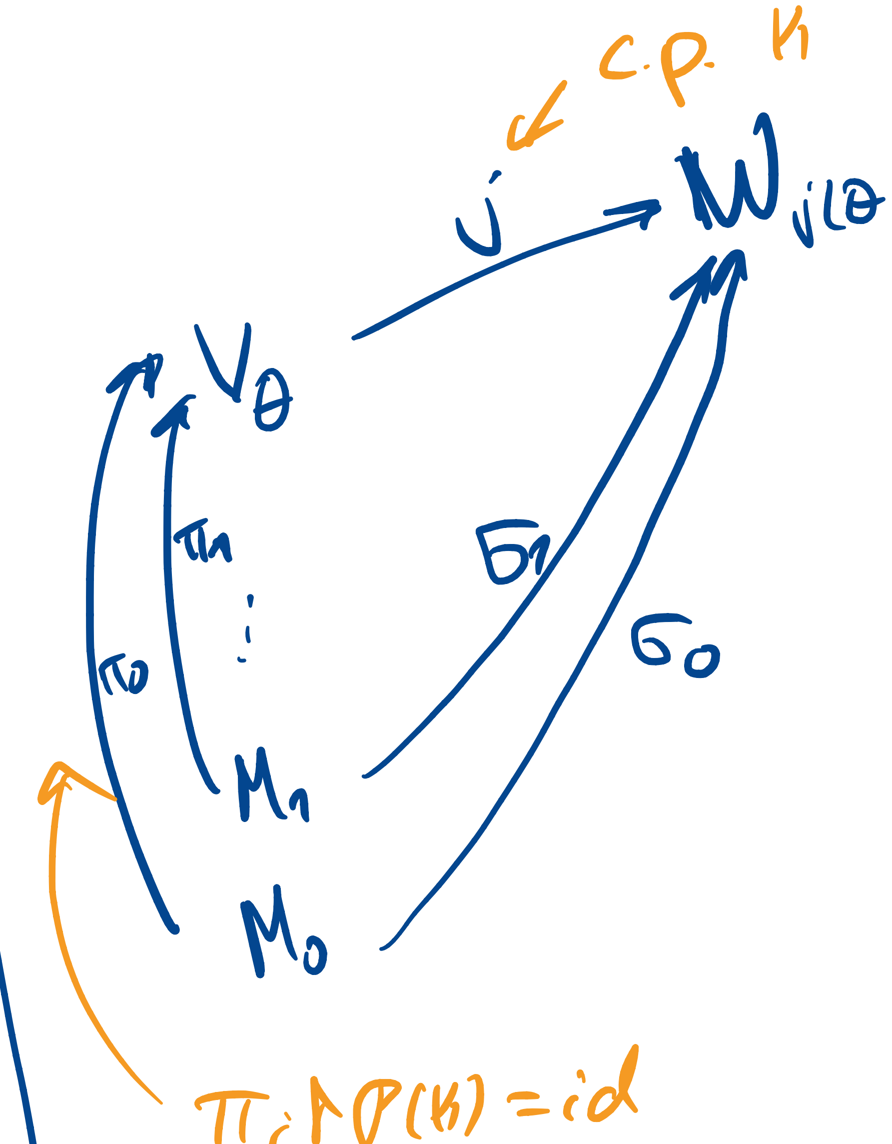
(2) $\mathcal{P}(\kappa)^{M_i} = \mathcal{P}(\kappa)^{M_{i+1}}$,

(3) $\sigma_i \in W[g]$

(4) $j''\bar{\mu}_2^{(i)} \subseteq \mathrm{rng}(\sigma_i)$, $j(f_1^{(i)}), j(f_0^{(i)}) \in \mathrm{rng}(\sigma_i)$, and

(5) $\mathrm{rng}(\sigma_i) \subseteq \mathrm{rng}(\sigma_{i+1})$.

Call this a <u>weak block</u>.



Lemma: For each $i<\omega$, $j''\bar{\mu}_2^{(i)}$ gives rise to a hom. system for A_i in $W[g]$.

proof: Note that $j''\bar{\mu}_2^{(i)} \in W[g]$.

Write $\bar{\mu}_2^{(i)} = (\mu_s \mid s \in \omega^{<\omega})$.

Claim: For each $x \in \mathbb{R}^{V[g]} = \mathbb{R}^{W[g]}$,
$(\mu_{x \restriction n} \mid n<\omega)$ is well-fdd in $V[g]$ iff $(j(\mu_{x\restriction n}) \mid n<\omega)$ is well-fdd in $W[g]$.

proof: "$\Leftarrow$" easy, as $(\mu_{x\restriction n} \mid n<\omega)$ ill-fdd $\Rightarrow (j(\mu_{x\restriction n}) \mid n<\omega)$ ill-fdd.

"$\Rightarrow$" Sps. $(\mu_{x\restriction n} \mid n<\omega)$ is well-fdd in $V[g]$. Recall $f_1^{(i)}$ flipping funct. in V.
Then $f_1^{(i)}(\mu_{x\restriction n} \mid n<\omega)$ is ill-fdd. So $j(f_1^{(i)}(\mu_{x\restriction n} \mid n<\omega))$

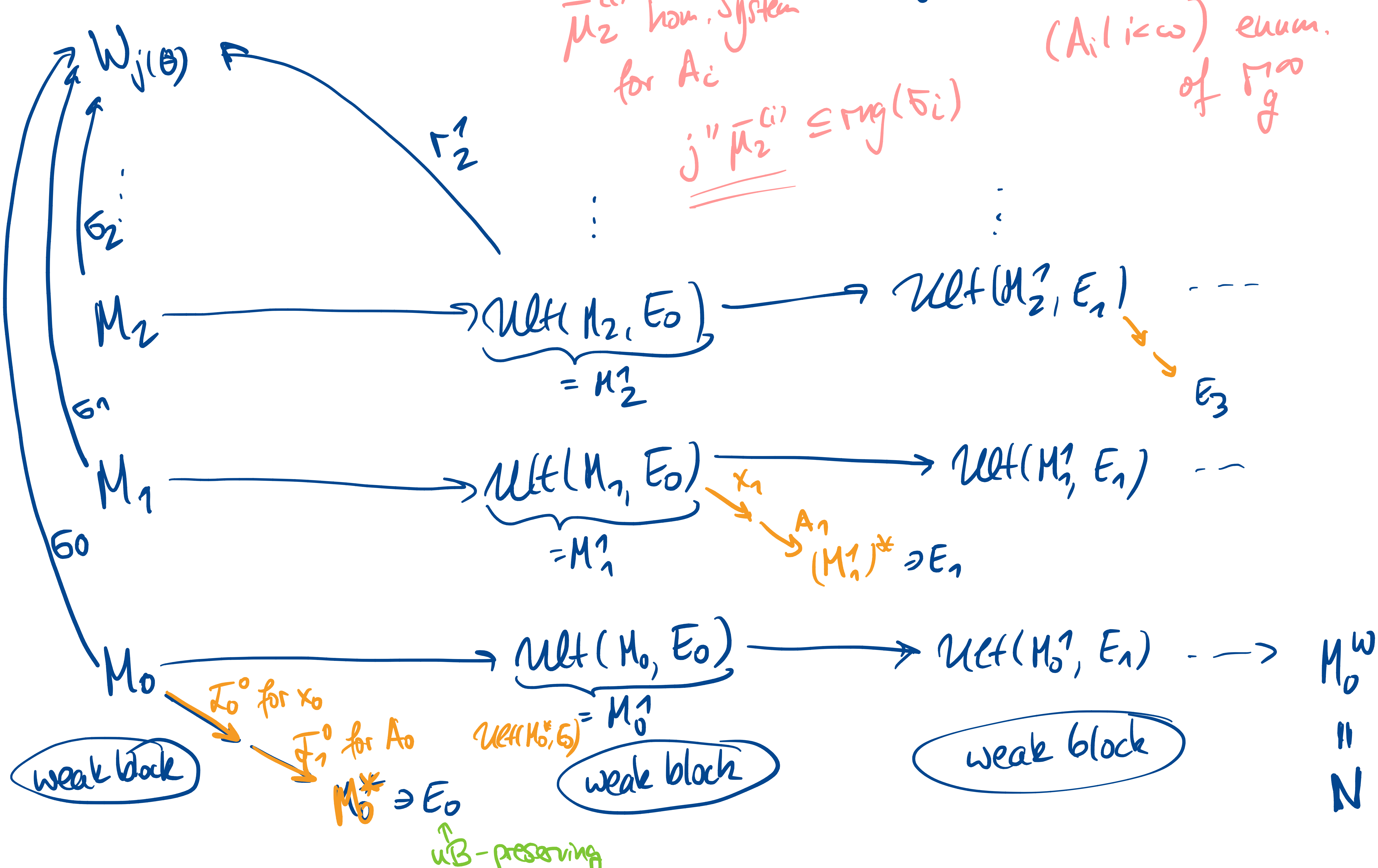
$= \underbrace{j(f_1^{(i)})}_{\uparrow \text{ flipping function in } W}(j(\mu_{x\restriction n}) \mid n<\omega))$ is ill-fdd in $W[g]$.

So $(j(\mu_{x\restriction n}) \mid n<\omega)$ is well-fdd. □

□

Picture of the proof

$g \subseteq Col(\omega, 2^{2^\kappa})$



Fix some $i<\omega$ and let $A = A_i$, $x = x_i$. Write $\bar\mu_2$ for $j''\bar\mu_2^{(i)}$.

Fix a weak block $(\hat{M}_j, M_j, \sigma_j, g_j \mid j<\omega)$. Write f_1, f_0 for the flipping functions.

Let $\{\bar\delta_0, \bar\delta_1, \bar\delta_2, \bar\delta_3, \bar f_1, \bar f_0, \bar W_2, \bar W_1, \bar W_0\} \cup \bar\nu_2 \subseteq M_i$ be the preimages of $\{\ldots\} \cup \bar\mu_2 \subseteq W$ under σ_i.

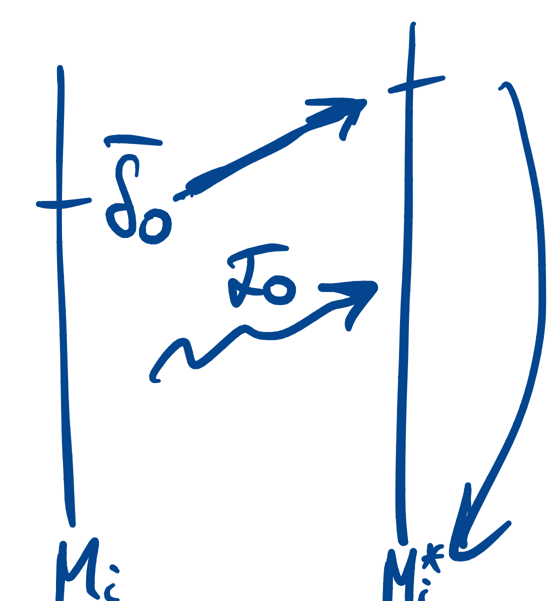
Let $\bar\mu_1 = f_1''\bar\mu_2$,
$\bar\mu_0 = f_0''\bar\mu_1$.

Let $\bar\nu_1 = \bar f_1''\bar\nu_2$ and $\bar\nu_0 = \bar f_0''\bar\nu_1$.

Make a real generic

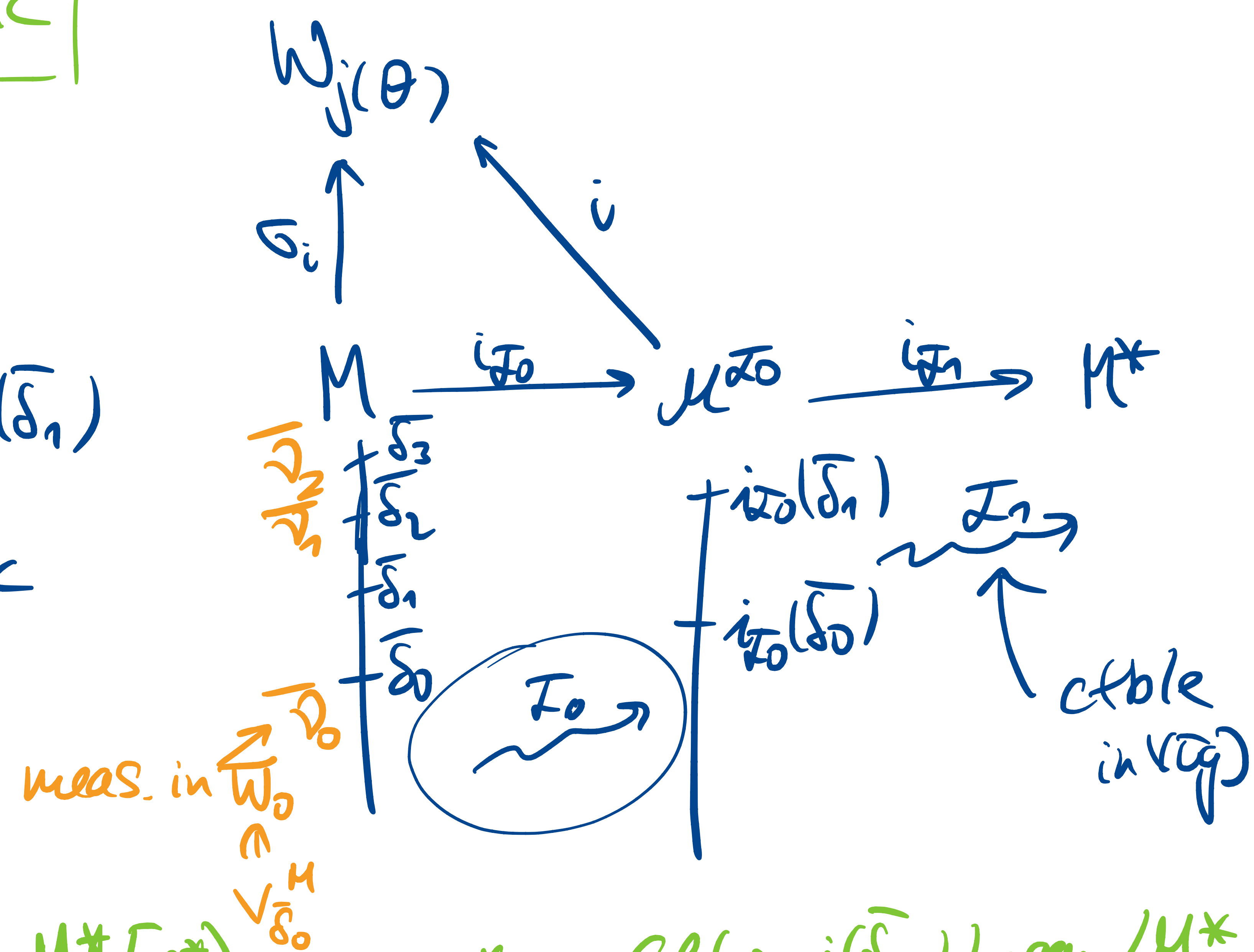
Let $\mathcal{T}_0$ be the ith tree on M_i acc. to the realization strategy (into $W_{j(\theta)}$) resulting from making x generic below $\bar\delta_0$ acc. to Neeman.

Let $\iota : \mathcal{M}^{\mathcal{T}_0} \longrightarrow W_{j(\theta)}$ be the realization.



Make a hom. system generic

Let $\mathcal{I}_1$ be the it. tree on $M^{\mathcal{I}_0}$ acc. to the realization strategy that acts between $i_{\mathcal{I}_0}(\bar\delta_0)$ and $i_{\mathcal{I}_0}(\bar\delta_1)$ and makes $\boxed{i_{\mathcal{I}_0} \upharpoonright V^M_{\bar\delta_0}}$ generic acc. to Neeman.



Let $\underbrace{\boxed{\bar\nu_0^*} = i_{\mathcal{I}_0}\text{"}\bar\nu_0}_{x \in M^*[g^*]}$ $\in M^*[g^*]$ where g^* is $Col(\omega, i(\bar\delta_1))$-gen $/M^*$

- Flip back up: $\boxed{\bar\nu_1^*} = (i(\bar f_0))^{-1}\text{"}\bar\nu_0^*$
  $\boxed{\bar\nu_2^*} = (i(\bar f_1))^{-1}\text{"}\bar\nu_1^*$

"Reversing the flips"

WTS: $\bar\nu_2^*$ witnesses that "A" is hom. Suslin in M^*.

Neeman's genericity iteration

Def: Let $\mathbb{P} \in M$ be a poset. An iteration tree $\mathcal{T}$ on M is said to <u>absorb</u> x to an extension by an image of $\mathbb{P}$ in case for every well-fdd cofinal branch b through $\mathcal{T}$, there is a generic extension $M_b^{\mathcal{T}}[g]$ of $M_b^{\mathcal{T}}$ by the poset $j_{0b}^{\mathcal{T}}(\mathbb{P})$, so that $x \in M_b^{\mathcal{T}}[g]$.

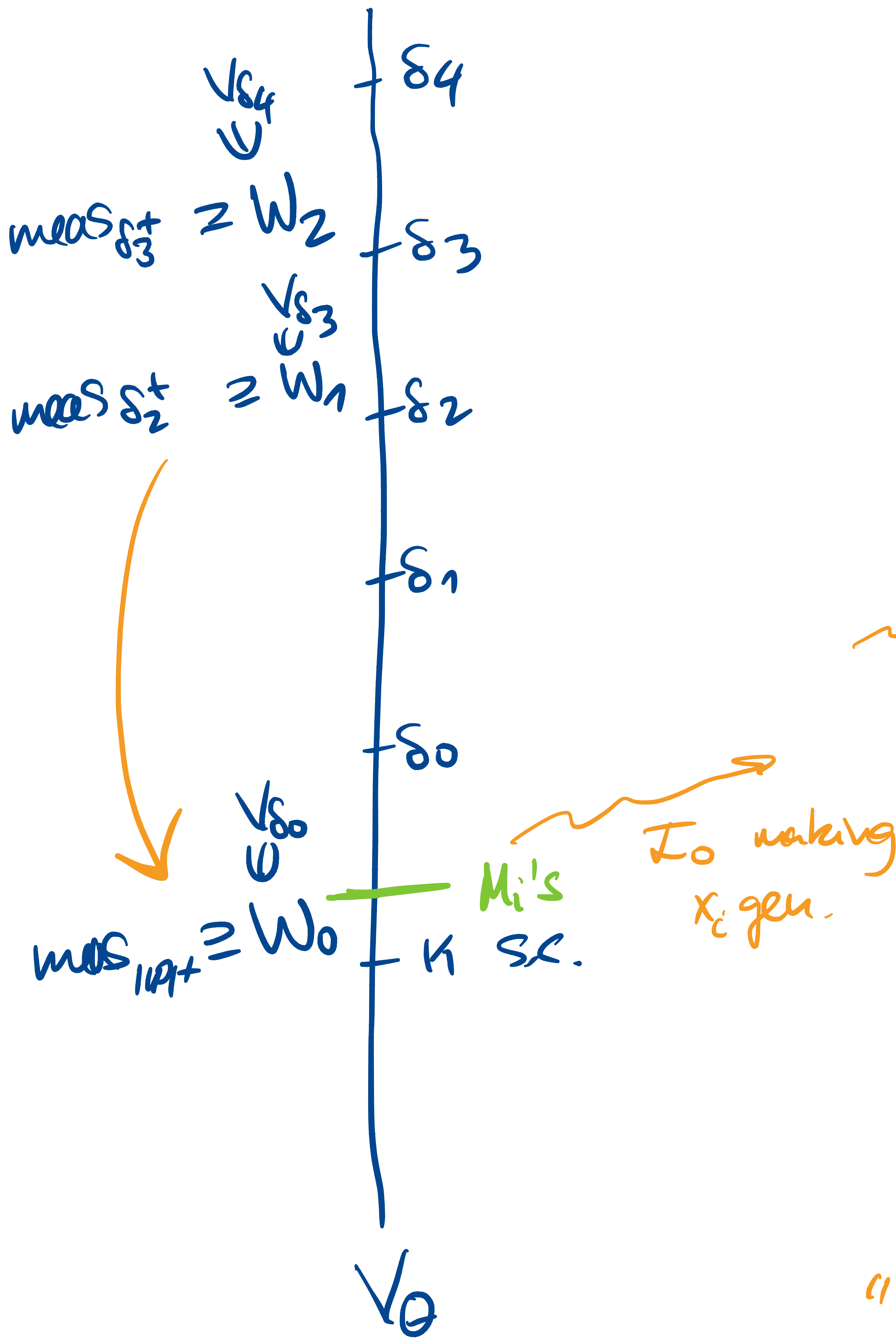
Thm (Neeman): Let M be a model of ZFC, let δ be Woodin in M s.t. $\mathcal{P}^M(\delta)$ is countable in V. Then for every real x there is a length ω iteration tree $\mathcal{T}$ on M which absorbs x into an extension by an image of $Col(\omega, \delta)$.

References: • Neeman, Optimal proofs of determinacy, BSL
↳ Corollary 1.8
• Neeman, Determinacy in $L(\mathbb{R})$, Handbook of Set Theory
↳ Section 7

Woodin's genericity iteration

+ poset δ-cc "extender algebra"

− it requires more iterability, in general, (ω_1+1)-iterability



$\overline{\mu}_2$ with measures from W_2

$\overline{\mu}_1$ with meas. from W_1, a flip of $\overline{\mu}_2$

$\mathcal{I}_1$ "making nom. system for A_i gen."

$\overline{\mu}_0$ with meas. from W_0, a flip of $\overline{\mu}_1$

"collapsed picture" in M_i

We have $\overline{\mathcal{J}}_1^* = (i(\overline{f_0}))^{-1}{}'' \underbrace{\overline{\mathcal{V}}_0^*}_{i_{\overline{\mathcal{J}}_0}{}''\overline{\mathcal{V}}_0} = (i(\overline{f_0}))^{-1}{}''(i''\overline{\mathcal{J}}_0)$ as $\overline{\mathcal{V}}_0$ is below δ_0.

Hence, as $\overline{\mathcal{V}}_0 = \overline{f_0}{}''\,\overline{\mathcal{V}}_1$, $\quad \overline{\mathcal{J}}_1^* = i''\,\overline{\mathcal{V}}_1$.

Similarly, $\overline{\mathcal{J}}_2^* = i''\,\overline{\mathcal{V}}_2$. $\quad$ Let $W_j^* = i(\overline{W}_j)$ for $j \in \{0,1,2\}$.

<u>Lemma</u>: Let $\gamma \geq i(\overline{\delta}_1)$ be a cardinal in M^* ~~with $\gamma \leq i(\overline{\delta}_2)$~~ and let h^* be $\mathrm{Col}(\omega,\gamma)$-generic over $M^*[g^*]$ with $h^* \in V[g]$. Then for any $u \in M^*[g^*][h^*] \cap \mathbb{R}$,

$$u \in A \iff ((\overline{\mathcal{J}}_2^*)_{u \restriction n} \mid n<\omega) \text{ is wfd} \iff ((\overline{\mathcal{J}}_1^*)_{u\restriction n} \mid n<\omega) \text{ is ill-fdd.}$$

Pf: The second equivalence follows from the facts that $\overline{\mathcal{J}}_1^* = i(\overline{f}_1)''(\overline{\mathcal{J}}_2^*)$ and $i(\overline{f}_1)$ is a flipping function.

For the first equivalence, recall that $\bar{\mu}_2 \cup \{f_1, f_0\} \subseteq \mathrm{rng}(\sigma_i)$ and hence $\bar{\mu}_1 \subseteq \mathrm{rng}(\sigma_i)$. We have

$$
\begin{aligned}
u \in A &\iff ((\bar{\mu}_2)_{u\restriction n} \mid n<\omega) \text{ is well-fdd.}\\
&\iff ((\bar{\mu}_1)_{u\restriction n} \mid n<\omega) \text{ is ill-fdd.}\\
&\implies (\sigma_i^{-1}((\bar{\mu}_1)_{u\restriction n}) \mid n<\omega) \text{ is ill-fdd.}\\
&\iff ((\bar{\nu}_1)_{u\restriction n} \mid n<\omega) \text{ is ill-fdd.}\\
&\implies (i((\bar{\nu}_1)_{u\restriction n}) \mid n<\omega) \text{ is ill-fdd.}\\
&\iff ((\bar{\nu}_1^*)_{u\restriction n} \mid n<\omega) \text{ is ill-fdd.}
\end{aligned}
$$

$$
\begin{tikzcd}
\bar{\mu}_1, \bar{\mu}_2 & W_{j(\theta)} & \\
\bar{\nu}_1, \bar{\nu}_2 & M_i \arrow[u, "\sigma_i"] \arrow[r, "i"'] & M^* \quad \bar{\nu}_1^*, \bar{\nu}_2^*
\end{tikzcd}
$$

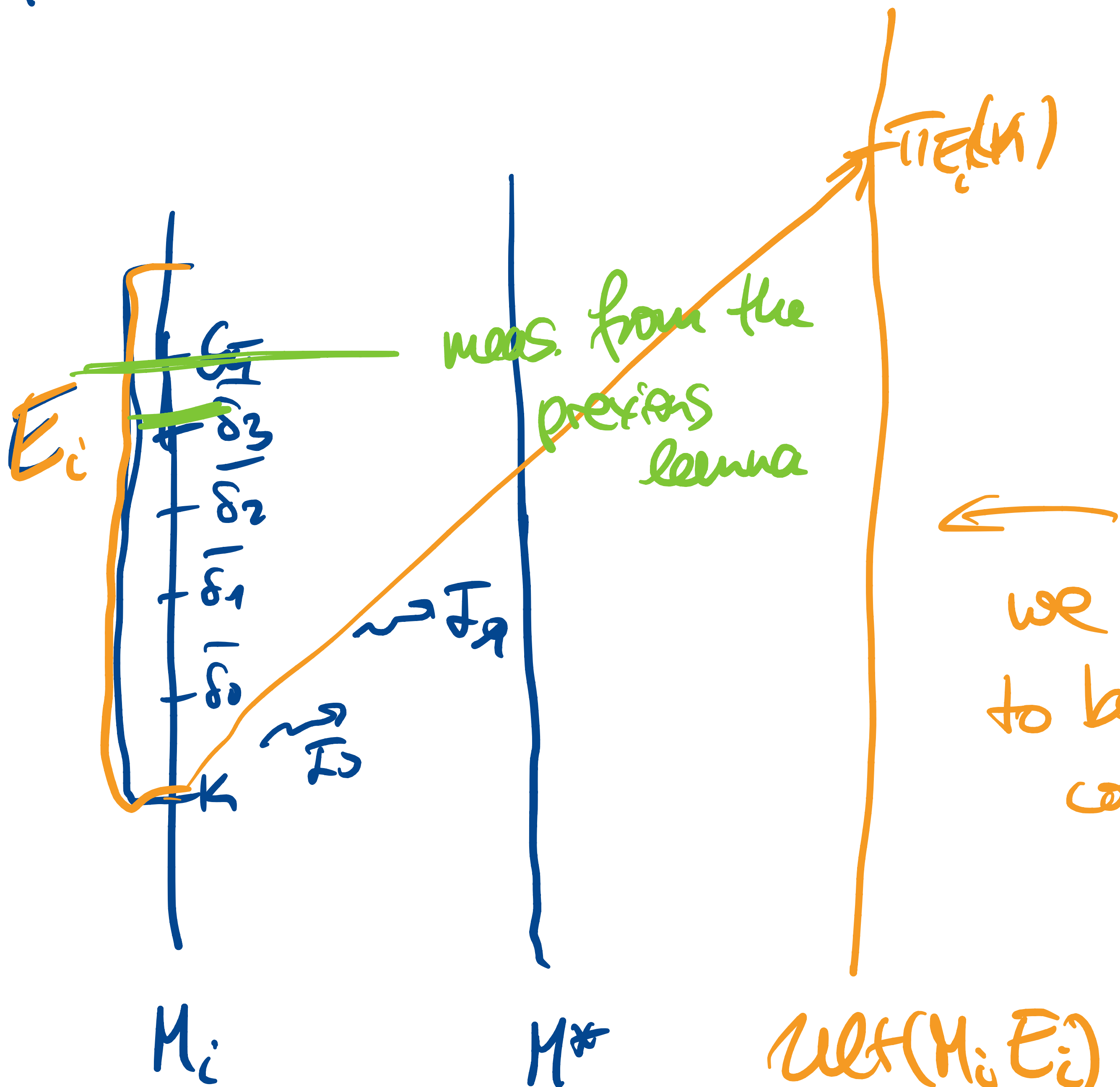
Moreover, $u \notin A \implies ((\bar{\nu}_2^*)_{u\restriction n} \mid n<\omega)$ is ill-fdd, similarly.

$\square$

Next goal: Extend the characterization in the previous lemma from M^* to $Ult(M^*, E)$ by some short extender $E \in M^*$.

Key point: We want such an extension with systems of measures that are $<\pi_E(\kappa)$-complete, even if $\pi_E(\kappa) > i(\delta_4)$.

Why?



← So here we need them to be $<\pi_{E_i}(\kappa)$-complete

To show that A is in $((\mathrm{Hom}^*)^{M^0_\infty[G]}$ for $G \subseteq Col(\omega, <\kappa_\infty)$

we need hom. systems that are $<\kappa_\infty$-complete!

eventual image of κ

M^0_ω = dir lim

→ To find these new suff. complete hom. systems we will use uB-preservation!