MINIMAL ACTIONS OF HOMEOMORPHISM GROUPS

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Abstract. Let X be a closed manifold of dimension 2 or higher or the Hilbert cube. Following Uspenskij one can consider the action of $Homeo(X)$ equipped with the compact-open topology on $\Phi \subset 2^{2^X}$, the space of maximal chains in 2^X, equipped with the Vietoris topology. We show that if one restricts the action to $M \subset \Phi$, the space of maximal chains of continua then the action is minimal but not transitive. Thus one shows that the action of $Homeo(X)$ on $U_{Homeo(X)}$, the universal minimal space of $Homeo(X)$ is not transitive (improving a result of Uspenskij in [Usp00]). Additionally for X as above with $dim(X) \geq 3$ we characterize all the minimal subspaces of $V(M)$, the space of closed subsets of M, and show that M is the only minimal subspace of Φ. For the case $dim(X) \geq 3$, we also show that $(M, Homeo(X))$ is strongly proximal.
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1. Introduction

We consider here compact G–spaces with G a Polish group and the action assumed to be continuous as a function of both variables. Such a G–space X is said to be **minimal** if X and $\emptyset$ are the only G-invariant closed subsets of X. By Zorn's lemma each G–space contains a minimal G-subspace. These minimal objects are in some sense the most basic ones in the category of G–spaces. For various topological groups G they have been the object of vast study. Given a topological group G one is naturally interested in trying to describe all of them up to isomorphism. Such a description is given by the following construction: one can show there exist a minimal G–space U_G unique up to isomorphism such that if X is a minimal G–space then X is a factor of U_G, i.e, there is a continuous G-equivariant mapping from U_G onto X. U_G is called the **universal minimal G–space** (for existence and uniqueness see [Usp00]). The task of calculating explicitly this minimal universal space is very hard. For some groups the space itself is complicated, e.g. by a known theorem the universal minimal flow of a non-compact locally compact group is non-metrizable (see [KPT05], Theorem A2.2.).

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For non locally compact groups the universal minimal space may reduce to one point. Such groups are called **extremely amenable**. Since every G–space X contains a minimal G–space, these groups may be characterized by a fixed point property, i.e. any G–space X has a G–fixed point. Using novel and original techniques Pestov in [Pes98] showed that the following groups had this property: the group of order preserving automorphisms of the rational numbers, equipped with the topology in which a subbasis of the identity consists of (all) stabilizers of a finite number of points, and the groups $Homeo_+(I)$ and $Homeo_+(\mathbb{R})$ of orientation preserving homeomorphisms of the unit interval and of the real line, respectively, equipped with the compact-open topology. The universal minimal space may be metrizable without being a single point, for example: the circle S^1 is the universal minimal space of the group $Homeo_+(S^1)$ of its orientation preserving homeomorphisms, equipped with the compact-open topology.

Motivated by the last result Pestov asked in the last section of [Pes98] if the Hilbert cube $Q = [-1, 1]^{\mathbb{N}}$ with the natural action of $Homeo(Q)$, equipped with the compact-open topology, is the universal minimal space for $Homeo(Q)$. In [Usp00] Uspenskij answered Pestov's question in the negative by showing that for every topological group G, the action of G on the universal minimal space U_G is not 3-transitive, i.e., there exist triples (a_1, a_2, a_3) and (b_1, b_2, b_3) of distinct points of U_G such that no $g \in G$ satisfies $g(a_i) = b_i$ for $i = 1, 2, 3$. In order to do so Uspenskij introduced the space of maximal chains of a given topological space. We now review this notion. Given a compact space K, let $V(K)$ be the space of all non-empty closed subsets of K, equipped with the Vietoris topology (see definition 1.1 in [IN99]). A subset $C \subset V(K)$ is a **chain** in $V(K)$ if for any $E, F \in C$ either $E \subset F$ or $F \subset E$. A chain is **maximal** if it is maximal with respect to the inclusion relation. One verifies easily that a maximal chain in $V(K)$ is a closed subset of $V(K)$, and that Φ the space of all maximal chains in $V(K)$ is a closed subset of $V(V(K))$, i.e. $\Phi \subset V(V(K))$ is a compact space. Note that a G-action on K naturally induces a G-action on $V(K)$ and $\Phi(K)$. This is true in particular for $K = U_G$. Therefore there is a continuous G-equivariant mapping $f : U_G \to \Phi(U_G)$. By cleverly investigating this mapping Uspenskij achieved the aforementioned result.

Motivated by Uspenskij's idea of looking on the maximal chains space of the universal minimal space, Glasner and Weiss studied in [GW03] the maximal chains space of the Cantor set K, and showed that it is the universal minimal space for $Homeo(K)$, equipped with the compact-open topology. It is important to point out that whereas Uspenskij used the (abstract) existence of the space of maximal chains in $V(U_G)$, Glasner and Weiss' method is constructive. The first steps consist of constructing the maximal chains space of the Cantor set and analyzing its properties.

In a recent article Pestov asked (albeit while attributing the questions to Uspenskij) for an explicit description of the universal minimal space of the group of homeomorphisms $Homeo(X)$ (equipped with the compact-open topology), X being a closed manifold of dimension 2 or higher or the Hilbert cube (see [Pes05] section 5.2, Open Questions 28 & 29). Here and elsewhere, the term **closed manifold** refers to a compact manifold without boundary. Motivated by these and similar questions (where X is allowed to be an even more general topological space) we apply the constructive method to a large class of groups of homeomorphisms of topological spaces (equipped with the compact-open topology). This class includes in particular the group of homeomorphisms of X, where X is any closed manifold of dimension 2 or higher or the Hilbert cube.

It is important to note, that the specific ideas Glasner and Weiss used in [GW03], heavily depend on the fact that K is zero-dimensional. For higher dimensions new ideas are needed. The scheme we would ideally like to use is to start with the given space X, then characterize all minimal subspaces of $V(X)$, next continue with characterizing the minimal subspaces of the iteration $V^2(X) = V(V(X))$ and so on, characterizing the minimal subspaces of $V^n(X)$ for each $n \in \mathbb{N}$. This scheme would include the analysis of the space of maximal chains in $V(X)$ and much more, but unfortunately it turns out to be very difficult to carry out.

We managed to obtain results for the "first three levels". It is easy to see that the only minimal subspaces of $V(X)$ are $\{X\}$ and $\big\{\{x\} \,|\, x \in X\big\}$. Characterizing all minimal subspaces of $V^2(X)$ already turns out to be rather hard. However one encounters a new and interesting phenomenon involving **continua**, i.e. non-empty compact, metric and connected spaces. Indeed Φ, the space of maximal chains in $V(X)$, is not minimal, but rather $M \subset \Phi$, the space of maximal chains (consisting only) of continua of X. This space can also be shown to coincide with the space of connected (w.r.t $V(X)$) maximal chains (see Lemma 2.3). Put formally:

Theorem 1.1. *If X is a closed manifold of dimension 2 or higher, or the Hilbert cube, then M, the space of maximal chains of continua is minimal under the action of $Homeo(X)$ on Φ.*

This theorem enables us to improve on Uspenskij's result and we prove:

Theorem 1.2. *If X is a closed manifold of dimension 2 or higher, or the Hilbert cube, and $G = Homeo(X)$, then the action of G on the universal minimal G–space U_G, is not transitive.*

Interestingly for a large class of spaces X one has that M, the space of maximal chains of continua, is the only minimal $Homeo(X)$-subspace of Φ. In particular, we prove:

Theorem 1.3. *If X is a closed manifold of dimension 2 or higher, or the Hilbert cube, then M, the space of maximal chains of continua, is the only minimal subspace of the $Homeo(X)$-space Φ.*

Analyzing all minimal subspaces of $V^3(X)$ turned out to be rather difficult. However we managed to classify all minimal subspaces of $V(M) \subset V^3(X)$:

Theorem 1.4. *If X is a closed manifold of dimension 3 or higher, or the Hilbert cube, then the action of $Homeo(X)$ on $V(M)$, the space of non-empty closed subsets of the space of maximal chains of continua has exactly the following minimal subspaces:*

(1) $\{M\}$
(2) $\{M_x\}_{x\in X}$, *where* $M_x = \{c \in M(X) \mid \bigcap\{c_\alpha \mid c_\alpha \in c\} = \{x\}\}$,
(3) $\big\{\{c\} \mid c \in M\big\}$

M is said to be **strongly proximal** under $G = Homeo(X)$ if for any Borel probability measure μ on M, there exists a sequence (g_n) of elements of G such that $[g_n]_*(\mu)$ converges to the measure concentrated at a singleton. We prove:

Theorem 1.5. *If X is a closed manifold of dimension 3 or higher, or the Hilbert cube, then $(M, Homeo(X))$ is strongly proximal.*

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Contents

2. Preliminaries

In this paper an effort is made to state theorems and lemmas in their broadest generality. We use the symbol X to denote the space we are working with. (X,d) is always assumed to be compact, metric, connected and non-trivial (by which we mean it contains more than one point, hence it contains infinitely many points). By an **ϵ-net** of a set $D \subset X$ we mean a finite collection $A = \{a_i\}_{i=1}^{K} \subset D$ such that for all $p \in D$, $dist(p,A) = min_{i=1,\ldots,K} d(p,a_i) < \epsilon$. Let $V(X)$ denote the space of non-empty closed subsets of X. We equip $V(X)$ with the Hausdorff metric:

$$d_{V(X)}(A_1,A_2) = d(A_1,A_2) = \inf\{\epsilon > 0 \mid A_2 \subset B(A_1,\epsilon) \text{ and } A_1 \subset B(A_2,\epsilon)\} \tag{2.1}$$

where A_1, A_2 are two non-empty closed sets and $B(A,\epsilon) = \{x \in X \mid \exists a \in A \ni d(x,a) < \epsilon\}$. The Hausdorff metric induces the Vietoris topology on $V(X)$ (see Theorem 3.1 of [IN99]). We define $V^n(X)$ for all $n \in \mathbb{N}$ using the natural definition $V^n(X) = V(V^{n-1}(X))$. A warning is due here: along the paper the notation $d(\cdot,\cdot)$ will be used to denote distance in various metric spaces. The reader should keep in mind that $d(\cdot,\cdot)$ denotes mostly the distance of the underlying space X or the Hausdorff distance in $V(X)$, $V^2(X)$ or $V^3(X)$, where the choice should be clear from the context.

Let $C(X)$ be the subspace of $V(X)$ consisting of all subcontinua of X. Here are two definitions that will play an important role in the paper:

$$\Phi(X) = \text{ the collection of maximal chains in } V(X)$$
$$M(X) = \text{ the collection of maximal chains in } C(X)$$

If the underlying space X is clear from the context we write Φ, M respectively. In Lemma 2.3 we show that M can be characterized as the space of connected (w.r.t $V(X)$) maximal chains. If $c \in M(X)$ and $D \in c$ then we define **the initial segment of c ending at D** to be $c' = \{R \in c \mid R \subset D\}$. Notice $c' \in M(D)$. Let

$$r(c) = \bigcap\{c_\alpha \mid c_\alpha \in c\}$$

while the set of all chains rooted at $\{x\}$ is denoted by M_x and Φ_x, respectively:

$$M_x = \{c \in M \mid r(c) = \{x\}\}, \quad \Phi_x = \{c \in \Phi \mid r(c) = \{x\}\}.$$

Lemma 2.1. *For $c = \{c_\alpha\}_{\alpha \in A} \in M$ and $F \subset c$ one has $\mathrm{Cl}_X\{\bigcup_{c_\alpha \in F} c_\alpha\} \in c$ and $\bigcap_{c_\alpha \in F} c_\alpha \in c$.*

Proof. This follows from the maximality of c as a subset of $C(X)$. $\square$

Given $c = \{c_\alpha\}_{\alpha\in A} \in M$ and $D \in V(X)$ with $r(c) \subset D$ we call the set $c_D = \bigcup\{c_\alpha | \, c_\alpha \subseteq D\}$ **the maximal element of c inside D.**

Lemma 2.2. *We have the following:*

(1) $c_D \subseteq D$.
(2) *If N is an open set so that $r(c) \subset N$ and $\partial N \neq \emptyset$ then $c_{\overline{N}} \cap \partial N \neq \emptyset$*

Proof. (1) is a direct consequence of lemma 2.1 and the fact that D is closed. (2) Let $I = \bigcap\{c_\alpha \,|\, c_\alpha \cap \partial N \neq \emptyset\}$. By standard compactness arguments $I \cap \partial N \neq \emptyset$ and thus it is enough to show $I \subseteq c_{\overline{N}}$. Assume not. Then $c_{\overline{N}} \subsetneq I$. According to Theorem 15.2 of [IN99] there exist $F \in c$ so that $c_{\overline{N}} \subsetneq F \subsetneq I$. Since $F \cap N \supset r(c) \neq \emptyset$ and F is connected, it follows that $F \subset N$ and thus $F \subseteq c_{\overline{N}}$ – a contradiction. $\square$

Lemma 2.3. $M = \Phi \cap C(V(X))$.

Proof. Let $c \in M$. According to Lemma 14.4 of [IN99] c is an **order arc in** $C(X)$, i.e there exists a homeomorphism $i : [0,1] \to C(X)$ so that $i([0,1]) = c$ and $0 \leq t_1 < t_2 \leq 1$ implies $i(t_1) \subsetneq i(t_2)$. In particular one concludes c is connected. Conclude $M \subset \Phi \cap C(V(X))$. In order to prove the opposite inclusion assume $c \in \Phi \cap C(V(X))$ and some $D \in c$ is not connected, i.e. $D = D_1 \cup D_2$, D_1, D_2 disjoint closed sets. Every member of c is either contained in D_1 or meets D_2. This implies c is not connected, contradicting the initial assumption. $\square$

Lemma 2.1. $\{M_x\}_{x\in X} \in C(V(M))$ *and the function* $m : X \to \{M_x\}_{x\in X}$ *given by* $m(x) = M_x$ *is a homeomorphism.*

Proof. Recall that r is the continuous function $r : M \to X$ given by $r(c) = \bigcap_{c_\alpha \in c} c_\alpha$. Notice that for $x \in X$, $M_x = r^{-1}(\{x\})$ which implies $M_x \in V(M)$ and the function $m^{-1} : Z \to X$, where $Z = \{M_x\}_{x\in X} \subset V(M)$, is continuous and 1-to-1. Now, the set Z is closed in $V(M)$ and therefore is compact, and thus this function is a homeomorphism. Hence $m = r^{-1}$ is a homeomorphism and X being connected, so is Z. $\square$

3. Local Transitivity, Strong Arcwise-Inseparability & Strong $\mathbb{R}$-Inseparability

In this section we introduce important topological assumptions used throughout the article and discuss some examples. Our actions of a group G on X will always be induced by G being a subgroup of the group $Homeo(X)$, equipped with the compact–open topology (which is in this setting the same as the uniform convergence topology, see [Mun75] p. 286). The action of G on X induces a natural action on $V^n(X)$ for all $n \in \mathbb{N}$. Given $g \in G$

and $A \in V(X)$ (i.e. $A \subset X$ is a closed set) one defines $gA = \{ga|\, a \in A\}$. The action of G on V^n for general n is defined inductively, based on the equality $V^n(X) = V(V^{n-1}(X))$. We assume that the group (or the action) is **locally transitive** in the sense that for any open set $U \subset X$ and $x \in U$ the set $\{gx \,|\, g \in G_U\}$ is a neighborhood of x, where

$$G_U = \{g \in G \,|\, gx = x \,\text{for}\, x \notin U\}.$$

For a compact interval $I \subset \mathbb{R}$ we denote by $C_s(I, X)$ the collection of continuous simple (injective) paths $p : I \to X$. We call such paths **arcs**. As it is usually done in the literature, the images of arcs are called arcs as well. A space X is called **strongly arcwise-inseparable** (SAI) iff any non-empty open and connected set $U \subset X$ and for any arc $p \in C_s([a, b], U)$ the set $U \setminus p([a, b])$ is connected and nonempty. A space X is called **strongly $\mathbb{R}$-inseparable** (SRI) iff for any non-empty open and connected set $U \subset X$ and any arc $p \in C_s([a, b], X)$, the set $U \setminus p([a, b])$ is connected and non-empty. Notice property (SRI) implies property (SAI). Throughout the article X is assumed to be either strongly arcwise-inseparable or strongly $\mathbb{R}$-inseparable. Here are the basic facts the reader should keep in mind. Closed manifolds of dimension 2 are strongly arcwise-inseparable (see Theorem A.1). Closed manifolds of dimension 3 or higher and the Hilbert Cube are $\mathbb{R}$ -inseparable (see Theorem A.3). We present a list of locally transitive groups (except the case when X is the Hilbert cube, which is proven in Lemma A.2, the other examples are simple and therefore the proofs are omitted):

Examples of Locally transitive Groups 3.1. *Let X be the Hilbert cube or a closed manifold of dimension 2 or higher. Then, any group containing one of the following groups is locally transitive:*

(1) $G = Homeo_0(X)$*, the arcwise connected component of the identity in* $Homeo(X)$*.*

For X an orientable manifold:

(2) $G = Homeo_+(X)$*, the group of orientation preserving homeomorphisms.*

For X a smooth manifold:

(3) $G = Diffeo_0(X)$*, the arcwise connected component of the identity in the group of diffeomorphisms of* X*.*

4. The Minimal Subspaces of $V(X)$

Let X be a Peano continuum, (i.e. X is compact, metric, connected and locally connected) with the property that the removal of a point from an open and connected set does not affect its connectivity. In other words assume that if $U \subset X$ is open and connected and $p \in U$ then $U \setminus \{p\}$ is connected. We now characterize the minimal subspaces of $V(X)$. Let us recall that the action of G on X is called **n-transitive** if $|X| \geq n$ and for any two n-tuples of distinct points $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in X^n$, there exist $g \in G$ so that $g(a_i) = b_i$ for $i = 1, 2, \dots, n$.

Lemma 4.1 (Global Transitivity)**.** *Suppose X is a continuum such that for each connected open set $U \subset X$ and each $p \in X$ the set $U \setminus \{p\}$ is connected. If the action of G on X is locally transitive then for any open and connected set $U \subset X$, the action of G_U on U is n-transitive, for all $n \in \mathbb{N}$.*

Proof. By induction. The case $n = 1$ follows from the assumption, as given $x \in U$ the set $\{gx : g \in G_U\}$ is open and closed in U, and hence is equal to U. Let now $n > 1$. Let $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ be two n-tuples of distinct points of U. By assumption there exists an $f \in G_U$ so that $f(a_i) = b_i$ for $i = 1, \dots, n-1$. Let $V = U \setminus f(\{a_1, \dots, a_{n-1}\})$. Notice V is open and connected. Apply the case $n = 1$ of the induction to find a $g \in G_V$ so that $g(f(a_n)) = b_n$. Define $h = g \circ f$. Notice $h \in G_U$ and $h(a_i) = b_i$ for $i = 1, \dots, n$. □

Theorem 4.2. *Under assumptions of the preceding lemma the only minimal subspaces of $V(X)$ are:*

(1) $\{X\}$
(2) $\big\{\{x\} \,|\, x \in X\big\}$

Proof. It is clear that the two presented subspaces are minimal. To show they are the only minimal subspaces it is enough to show that any element of $V(X)$ has the property that the closure of its orbit intersects one of these subspaces. Let then $A \in V(X)$. If $|A| < \infty$ then by Lemma 4.1 one can find $g_n \in G, n \in \mathbb{N}$ and $z \in X$ so that $g_n(A) \to_{n\to\infty} \{z\}$. i.e. the closure of the orbit of A intersects the second subspace. If $|A| = \infty$ we will show that the closure of the orbit of A intersects the first subspace. Let $\epsilon > 0$ be given. Let $\{x_i\}_{i=1}^{l} \subset X$ be an ϵ-net of X. Find $\{y_i\}_{i=1}^{l} \subset A$ and $g \in G$ so that $g(y_i) = x_i$. Conclude that $d(g(A), X) < \epsilon$. □

5. Approximation of M by Ray-induced Chains

From now onward we assume that X is a Peano continuum, (i.e. X is compact, metric, connected and locally connected) which is strongly arcwise-inseparable. In this section we will show that the chains in M can be approximated by the so-called ray-induced chains.

Definition 5.1. We call the members of $C_s([0,\infty), X)$ **rays**. By an **$\overline{\mathbb{R}}_+$-chain** we mean any element c of M such that $c = (c_t)_{t\in[0,\infty]}$ and there exists a ray γ with $c_t = \gamma([0,t])$ for all $t < \infty$. When the last condition is satisfied, we say that c is **induced** by the ray γ. Let us observe that then one has $c_\infty = X$ (this follows from maximality of c) and thus the ray γ is dense in X (in the sense that $\gamma([0,\infty))$ is dense in X), again by the maximality of c. We denote

$$\mathfrak{R} = \{c \in M \mid c \text{ is a } \overline{\mathbb{R}}_+\text{−chain}\} \quad \text{and} \quad \mathfrak{R}_x = \mathfrak{R} \cap M_x \quad (x \in X)$$

Lemma 5.2. *Let $\gamma \in C_s([0,k],U)$, where U is an open connected set in X. Then given $\epsilon > 0$ and $x_1,\ldots,x_l \in U$ there exists an arc $\gamma' \in C_s([0,k+l],U)$ such that $\gamma'(t) = \gamma(t)$ for $t \leq k-1$, $d(\gamma'(t),\gamma(t)) < \epsilon$ for $t \in [k-1,k]$ and $x_1,\ldots,x_l \in B_\epsilon(\gamma'([k,k+l])$.*

Proof. An easy induction shows that it suffices to consider the case when $l = 1$. If $x_1 \in \gamma([0,k])$ we are done, so assume not. Pick $s \in (k-1,k)$ with $\operatorname{diam}\{\gamma([s,k])\} < \frac{\epsilon}{2}$, and using connectedness of the set $U \setminus \gamma([0,s])$ pick an arc p in it from $\gamma(k)$ to x_1. The desired arc γ' is obtained by first traveling along the arc γ until we hit p and from then on traveling along p. (Parametrization needs to be adjusted so that $\gamma'(t) = \gamma(t)$ for $t \leq s$ and $\gamma(t)$ stays close to the point $\gamma(s) = \gamma'(s)$ for $t \in [s,k]$). □

Theorem 5.3. *$\overline{\mathfrak{R}} = M$ (for X a Peano continuum which is SAI).*

Proof. Let $c \in M$ and $\epsilon > 0$ be given. By Lemma 14.4 of [IN99] there is an embedding j of $[0,\infty]$ into $C(X)$ such that $c = j([0,\infty])$ and $j(t_1) \subset j(t_2)$ for $t_1 < t_2$. Subdivide $[0,\infty]$ into infinitely many intervals, each mapped under j to a set of diameter $< \epsilon$. By changing the units in the domain one can assume for simplicity that the intervals have diameter less than 1 and thus that $d(j(t),j(k)) < \epsilon$ for all reals $t \geq 0$ and integers k such that $k-1 \leq t \leq k$. Denote by $Con_\epsilon(j(k)) \subset B_\epsilon(j(k))$ the connectivity component of $B_\epsilon(j(k))$ which contains $j(k)$, it is open in X by the local connectedness of X. Inductively we construct arcs $\gamma_k : [0,k] \to X$ such that:

(1) $\{\gamma_k(0)\} = r(c)$, $\gamma_k([k-1,k]) \subset Con_\epsilon(j(k))$ and $j(k) \subset B_{\frac{\epsilon}{k}}(\gamma_k([k-1,k]))$,
(2) $\gamma_k(t) = \gamma_{k-1}(t)$ for $t \in [0,k-2]$,
(3) $d(\gamma_k(t),\gamma_{k-1}(t)) < \epsilon$ for $t \in [k-2,k-1]$

To this end suppose first that $k \geq 2$ and γ_k has already been constructed. Let $U = Con_\epsilon(j(k))$. Let $x_1,\ldots,x_l$ be an $\frac{\epsilon}{k}$-net of $j(k)$. Applying Lemma 5.2 with $\gamma = \gamma_k$ and then changing the parameter set of γ' from $[0,k+l]$ to $[0,k+1]$ we get the desired arc γ_{k+1}. This takes care of the inductive step, the case $k = 1$ is handled similarly. (Conditions (2) and (3) are then void.) Put $\gamma(t) = \lim_{k\to\infty} \gamma_k(t)$ and $a_t = \gamma([0,t])$ for $t \geq 0$. Since $\bigcup_k j(k) = X$ we infer from (1) that the image of γ is dense in X. Moreover $a_k = \gamma_{k+1}([0,k])$ by (2), and thus from (1) , (3) and the monotonicity of the sequence $(j(n))$ it follows that a_k is contained in the 2ϵ-neighborhood of $j(k)$ and contains the set $S = \gamma_{k+1}([k-1,k])$ such that $j(k) \subset B_{2\epsilon}(S)$. Hence $d(a_k, j(k)) < 2\epsilon$ for all k. Now if $t \geq 0$, say $t \in [k, k+1]$ for some integer k, then

$$a_t \subset a_{k+1} \subset B_{2\epsilon}(j(k+1)) \subset B_{3\epsilon}(j(t)) \text{ and } B_{3\epsilon}(a_t)) \supset B_{3\epsilon}(a_k) \supset B_\epsilon(j(k)) \supset j(t)$$

Thus the ray-induced chain $\{a_t\}_{t\geq 0} \cup \{X\}$ is 3ϵ-close to c, completing the proof. □

Until now we assumed X was strongly arcwise-inseparable. Suppose X is strongly $\mathbb{R}$-inseparable. Under this assumption we first present a lemma which is a generalization of Lemma 5.2, then a definition which is a generalization of Definition 5.1. Finally we state Theorem 5.6 which generalizes Theorem 5.3. The proof of Theorem 5.6 is omitted as it can be filled in by a reader who understood the proof of Theorem 5.3.

Lemma 5.4. *Let $\gamma_1, \ldots, \gamma_N \in C_s([0,k], U_i)$ be **disjoint** arcs, where $U_1, \ldots, U_N$ are open connected sets in X. Then, given numbers $\epsilon, \delta > 0$, there exist $a > k$ and **disjoint** arcs $\gamma_i' \in C_s([0,a], U_i)$ such that $\gamma_i'(t) = \gamma_i(t)$ for $t \leq k-\delta$, $d(\gamma_i'(t), \gamma_i(t)) < 2\epsilon$ for $t \in [k-\delta, k]$ and each arc $\gamma_i'([0,a])$ is $\epsilon/2$–dense in X.*

Moreover, if for all i one has $\operatorname{diam} U_i < \epsilon$ and $\gamma_i(k) \in U_i \setminus F_i$ for some closed set F_i with $U_i \setminus F_i$ connected, then the arcs γ_i' can be constructed so that $\gamma_i'(t) \notin F_i$ for $t \geq k-\delta$.

Proof. (sketch) By considering the arcs $t \mapsto \gamma_i(Ct)$ for C large enough, and eventually switching back to the original parametrization, one can assume without loss of generality that $\delta = 1$. The first part is then proved as in Lemma 5.2, using the (SRI) property of X to make the approximating arcs disjoint and taking for $x_1, \ldots, x_l$ an $\epsilon/2$–net in X.

The idea of the proof of the "moreover" part is first to use the above one with ϵ replaced by $\epsilon/2$ and then, for each i, to subdivide the segment $[k-\delta, a]$ into finitely many segments, so small that they are mapped by γ_i' into $X \setminus F_i$ or into U_i. Then, possibly combining adjacent segments which are mapped to U_i, one can assume that their endpoints are mapped to $U_i \setminus F_i$. The connectivity of $U_i \setminus F_i$ now allows to modify γ_i' on such a segment so that the altered γ_i' takes values in $U_i \setminus F_i$. Since $\operatorname{diam} U_i < \epsilon$, the modifications will stay ϵ-close to the unaltered γ_i''s, and hence 2ϵ–close to γ_i on $[k-\delta, k]$. □

Definition 5.5 (The set $\mathfrak{R}_*^N$)**.** Let $N \in \mathbb{N}$. Equip M^N with the product topology. We define the subspace $\mathfrak{R}_*^N \subset M^N$ as follows: $(c_1, \ldots, c_N) \in \mathfrak{R}_*^N$ iff:

(1) $c_i = \{\gamma_i([0,t])\}_{t\in\mathbb{R}_+} \cup \{X\} \in \mathfrak{R},\ i = 1, \ldots, n$
(2) $\gamma_i(\mathbb{R}_+) \cap \gamma_j(\mathbb{R}_+) = \emptyset,\ 1 \leq i < j \leq n$

Theorem 5.6. $\overline{\mathfrak{R}_*^N} = M^N$ *(for X a Peano continuum which is SRI).*

6. The Minimality of M

Let X be a Peano continuum which is strongly arcwise-inseparable. Our goal in this section is to show M is minimal under the action of G. We start with a definition:

Definition 6.1. T is a **δ-tube for** $p \in C_s([0,1], X)$ iff for some $0 = t_0 < t_1 < \ldots < t_l = 1$ there exist open connected subsets $U_1, U_2, \ldots, U_l$ (the "links" of the tube) such that $T = \bigcup_{i=1}^{l} U_i$ and

(1) $\operatorname{diam}\{U_i\} < \delta$ and $p([t_{i-1}, t_i]) \subset U_i$ for $i = 1, \ldots, l$.
(2) $\mathrm{Cl}_X\{U_i\} \cap \mathrm{Cl}_X\{U_j\} \neq \emptyset$ iff $|i - j| \leq 1$.

Lemma 6.2. *Let $p \in C_s([0,1], X)$, let $\delta > 0$. Then p has a δ-tube contained in a given neighbourhood U of $p([0,1])$.*

Proof. Using simplicity of p we may choose points $0 = t_0 < \cdots < t_l = 1$ such that $\operatorname{diam} p([t_{i-1}, t_i]) < \delta/2$ for each $i = 1, \ldots, k$. Then, there is a $\rho < \delta/2$ such that for $B_i = B(p([t_{i-1}, t_i]), \rho)$ we have $B_i \cap B_j = \emptyset$ for $|i - j| > 1$. We define U_i to be the connected component of $p([t_{i-1}, t_i])$ in B_i. By local connectivity, each U_i is open and so $T = \bigcup_i U_i$ is a δ–tube for p. Also, if δ is small enough then $T \subset U$. $\square$

Introduce the notation:

$$dist(A, B) = \inf_{x \in A, y \in B} d(x, y) \quad \text{and} \quad dist(x, A) = dist(\{x\}, A)$$

for $A, B \subset X$. Notice that in general $dist(x, A) < d(\{x\}, A)$ and $dist(A, B) < d(A, B)$.

Lemma 6.3. *Let $T = \bigcup_{i=1}^{n} U_i$ be an ϵ-tube around an arc $p : [0, a] \to X$, then:*

(1) *For every continuum $K \subset T$ such that $p(0) \in K$ there exists a $t \in [0, a]$ with $d(K, p([0, t])) < \epsilon$.*
(2) *If $q : [0, b] \to T$ is an arc satisfying $q(0) = p(0)$ and $q(b)$ belongs to a sufficiently small neighborhood of $p(a)$, then for each $s \in [0, a]$ there exists a $t \in [0, b]$ with $d(p([0, s]), q([0, t])) < \epsilon$.*
(3) *If a chain $c \in M_{p(0)}$ satisfies that $p(a)$ belongs to a sufficiently small neighborhood of $C \subset T$ for some $C \in c$, then given $s \in [0, a]$ there exists a $C' \in c$ such that $d(p([0, s]), C') < 2\epsilon$.*

Proof. (1) Let $j = n$ if $K \cap U_n \neq \emptyset$ and $j = \min\{i |\, K \cap U_i = \emptyset\}$ otherwise. Then $t = t_j$ does the job.
(2) For the above mentioned sufficiently small neighborhood of $p(a)$ we take U_n from the definition of a tube. Let first $s = t_j$ for some $j \geq 1$, then, with $t = \inf\{t' \in [0, b] |\, q(t') \notin U_{j-1}\}$ it is easy to see that $d(q([0, t]), p([0, t_j]) < \epsilon$. The general case follows similarly.

(3) c) Let $B_{2\delta}(p(a))$ be sufficiently small neighborhood of $p(a)$ as desired in (2) for some $0 < \delta < \epsilon$. By Theorem 5.3 there exists a chain $c' \in M_{p(0)}$ induced by a ray $\gamma : [0,1] \to X$ with dense image and $\gamma(0) = p(0)$, such that $d(c, c') < \delta$; here we request that $\delta < dist(p([0,a]), X \setminus T)$. Then for some $q > 0$ one has $\gamma([0,q]) \subset T$ and $d(p(a), \gamma(q)) < 2\delta$. By (2), there exists a $t \in [0,q]$ such that $d(p([0,s]), \gamma([0,t])) < \epsilon$. Thus it remains to take C' so that $d(\gamma([0,q]), C') < \epsilon$.

□

Recall $M_x = \{c \in M | \, r(c) = \{x\}\}$.

Lemma 6.4. *Let $x \in X$. Let $\epsilon > 0$. Let $c \in M_x$. Let $f = \{\gamma([0,t])\}_{t \in \mathbb{R}_+} \cup \{X\} \in \mathfrak{R}_x$. Let $a \in \mathbb{R}$ be such that $d(X, \gamma([0,a]) < \epsilon$. Let $U \subset X$ be an open subset such that $\gamma([0,a]) \subset U$. Then there is $g \in G_U$ so that $d(g(c), f) < 3\epsilon$.*

Proof. As $\gamma_{|[0,a]} \subset U$ is a simple curve, Lemma 6.2 tells us $\gamma_{|[0,a]}$ has T, an ϵ-tube in U. Using Lemma 2.2 one can choose $C \in c$ so that $C \neq \{x\}$ and $C \subset T$. Choose $q \in C \setminus r(c)$. Using Lemma 4.1 choose $g \in G$ so that $g(q) = \gamma(a)$, $g(\gamma(0)) = \gamma(0)$ and $g_{|T^c} = Id$, which implies $g_{|U^c} = Id$. In an unprecise manner one can say that g "stretches" C along $\gamma([0,a])$. Therefore it should not come as a surprise that we will now be able to show that $d(g(c), f) < 3\epsilon$. In fact, this inequality follows directly from parts (1) and (3) of Lemma 6.3 if one takes into account that for $s > a$ and $D \in c$ with $C \subset D$ the sets $\gamma([0,s])$ and $g(D)$ are 2ϵ-dense in X (The latter contains $g(C)$ which contains a set 3ϵ-close to $\gamma([0,a])$. □

Theorem 6.5. *Let G act locally transitively on a Peano continuum X which is strongly arcwise-inseparable. Then the action of G on M(X) is minimal.*

Proof. Let $c, f \in M$ and $\epsilon > 0$. Using transitivity of G (Lemma 4.1) one can assume without loss of generality $\{r\} = r(c) = r(f)$. Using Theorem 5.3 one can assume $f = \{\gamma([0,t])\}_{t \in \mathbb{R}_+} \cup \{X\} \in \mathfrak{R}$. Choose $a > 0$ so that $d(\gamma([0,a]), X) < \epsilon$. Now invoke Lemma 6.4 with $U = X$ to conclude there is a $g \in G$ so that $d(g(c), f) < 3\epsilon$. □

Corollary 6.6. *Under assumptions of Theorem 6.5, the action of G on the universal minimal G–space U_G is not 1–transitive.*

Proof. It is enough to show that the minimal G–space M is not 1-transitive. Let $c \in \mathfrak{R}$. c is induced by some ray γ. Let $r \in X$. Define $v = \{\overline{B}(r,t)\}_{t \in \mathbb{R}}$. Since no arc is SAI it is easy to show one cannot map balls $\overline{B}(r,t)$ (homeomorphically) onto arcs of the form $\gamma([0,a])$. This implies there does not exits $g \in G$ so that $g(v) = c$, from which we conclude the action of G on M is not 1-transitive. Q.E.D. □

7. The uniqueness of M as a minimal subspace of Φ.

Definition 7.1. Let $x \in X$. Let $\epsilon > 0$. A sequence $\mathfrak{B}$ of open sets $x \in B_1 \subset \overline{B_1} \subset B_2 \subset \ldots \subset \overline{B}_{N-1} \subset B_N$ is an (N, ϵ) **annuli telescope around** x if there exist an open set U with $\overline{B}_N \subset U \subset B(x, \epsilon)$ so that $U \setminus \overline{B}_i$ is connected for $i = 1, \ldots, N$. Notice that if V is open and connected with $U \subset V$ then $V \setminus \overline{B}_i$ is connected for $i = 1, \ldots, N$. For convenience one defines $A_1 = \overline{B}_1$ and $A_i = \overline{B}_i \setminus \overline{B}_{i-1}$, $i = 2, \ldots, N$. $\{A_i\}_{i=1}^N$ is called the **accompanying telescope decomposition**.

Moreover, we say that $c \in \Phi_x$ is **$\mathfrak{B}$-compatible** iff there exist $\{C_i\}_{i=1}^N \subset c$ so that $\{x\} = C_1 \subset C_2 \subset \ldots \subset C_N$, $C_i \subset \overline{B_i}$ and $C_i \cap A_i \neq \emptyset$ for $i = 1, \ldots, N$.

We say that X has the **Telescoping Annuli Property** if for any $x \in X, \epsilon > 0, N \in \mathbb{N}$, there is a $(1, \epsilon)$ annuli telescope around x, which implies by a simple argument that for any $\epsilon > 0, N \in \mathbb{N}$ there is an (N, ϵ) annuli telescope around x.

Theorem 7.2. *If X is a Peano continuum which is SAI and has the telescoping annuli property, then the only minimal subspace of $\Phi(X)$ is $M(X)$.*

Proof. Let $c \in \Phi$. Our goal will be to show that the closure of the orbit of c intersects M. Let $\epsilon > 0$. Let $f = \{\gamma([0, t])\}_{t \in \mathbb{R}_+} \cup \{X\} \in \mathfrak{R}_{r(c)}$. Let $a \in \mathbb{R}$ so that $d(X, \gamma([0, a]) < \epsilon$. Let $T = \bigcup_{i=1}^N U_i$ be an ϵ-tube of $\gamma([0, a])$ with $\operatorname{diam}\{U_i\} < \epsilon$ and $0 = t_0 < t_1 < \ldots < t_l = a$ so that $p([t_{i-1}, t_i]) \subset U_i$. Denote $T_k = \bigcup_{i=1}^k U_i$, $k = 1, \ldots, N$. As X has the telescoping annuli property one can choose $\mathfrak{B}_N = \{B_i\}_{i=1}^N$ to be an annuli telescope around $r(c)$ such that $r(c) \subset B_1 \subset B_2 \subset \ldots \subset B_N \subset V \subset U_1$, with V an open set so that $V \setminus \overline{B}_i$ is connected for $i = 1, 2, \ldots, N$. Let $\{A_i\}_{i=1}^N$ be its accompanying telescope decomposition. Using induction we will find $g_N \in G$ such that $g_N(c)$ is $\mathfrak{B}_N$-compatible. Define $\mathfrak{B}_k = \{B_i\}_{i=1}^k$ for $k = 1, \ldots, N$. Notice that for $g_1 = Id$, $g_1(c)$ is $\mathfrak{B}_1$-compatible. This is the base step of the induction. Assume we have found $g_k \in G, k < N$ so that $g_k(c)$ is $\mathfrak{B}_k$-compatible. We will now construct $g_{k+1} \in G$ so that $g_{k+1}(c)$ is $\mathfrak{B}_{k+1}$-compatible. Let $g_k(C_1) \subset g_k(C_2) \subset \ldots \subset g_k(C_k) \in g_k(c)$ so that $g_k(C_i) \subset \overline{B_i}$ and $g_k(C_i) \cap A_i \neq \emptyset$ for $i = 1, \ldots, k$. Let $R = [g_k(c)]_{\overline{B}_{k+1}}$. If $R \cap A_{k+1} \neq \emptyset$, let $g_{k+1} = g_k$ and $C_{k+1} = g_{k+1}^{-1}(R)$. If $R \cap A_{k+1} = \emptyset$, define $R_+ = \bigcap \{g_k(c_\alpha) |\, g_k(c_\alpha) \cap B_{k+1}^c \neq \emptyset\}$. As $dist(B_{k+1}^c, \overline{B}_k) > 0$, the maximality of c implies that $R_+ = R \cup \{p\}$ for some $p \in B_{k+1}^c$. By Lemma 4.1 as $X \setminus \overline{B}_k$ is connected one can find $h \in G, y \in A_{k+1}$ so that $h(p) = y$ and $h_{|\overline{B}_k} = Id$. Define $g_{k+1} = h \circ g_k$. Notice that $g_{k+1}(C_i) = g_k(C_i)$ for $i = 1, \ldots, k$. Moreover $g_{k+1} \circ g_k^{-1}(R_+) \subset \overline{B}_{k+1}$ and $g_{k+1}(R_+) \cap A_{k+1} \neq \emptyset$. This finishes the induction. We now choose distinct $y_i \in g_N(C_i) \cap A_i, z_i \in U_i$, $i = 1, \ldots N$.

Denote $B_0 = \emptyset$. As $T_k \setminus \overline{B}_{k-1}$ is open and connected, using property (SAI), one can choose disjoint arcs $p_k \in C_s([0,1], T_k \setminus \overline{B}_{k-1})$ with $p_i(0) = y_i$ and $p_i(1) = z_i$. Conclude one can find disjoint open connected subsets $p_i([0,1]) \subset W_i \subset T_k \setminus \overline{B}_{k-1}$, $i = 1, \ldots, N$ and therefore by Lemma 4.1 we can find $q_i \in G$ so that $q_i(y_i) = z_i$ and $[q_i]_{|W_i^c} = Id$. Let $q = q_1 \circ \cdots \circ q_N \circ g_N$. Notice $q(C_i) \subset T_i \subset B(\gamma([0,t_i]), \epsilon) \subset B(q(C_i), 2\epsilon)$ for $i = 1, 2, \ldots, N$. We claim this implies $d(q(c), f) < 2\epsilon$. Indeed for $D \in c$ with $C_i \subset D \subset C_{i+1}$ $d(q(D), \gamma([0,t_i]) < 2\epsilon$. For $t_i \leq s \leq t_{i+1}$ one has $d(\gamma[0,s], q(C_i)) < 2\epsilon$. For $s \geq a$ one has $d(\gamma[0,s], X) < \epsilon$. Finally for $C_N \subset D$ for $D \in c$ one has $d(\gamma([0,a], D) < 2\epsilon$. □

8. The Minimal Subspaces of V(M)

In this section we assume X is a Peano continuum which is strongly $\mathbb{R}$-inseparable. For Lemma 8.4, Corollary 8.5 and Theorem 8.6 we assume (X, G) has the boundary shrinking property (to be defined in this section). Our goal is to find all minimal subspaces of $V(M)$. Three minimal subspaces are evident. These are $S_s = \{M\}$, $S_f = \{M_x\}_{x \in X}$ and $S_p = \big\{\{c\} \,|\, c \in M\big\}$. The surprising conclusion of this section is that these are the only minimal subspaces of $V(M)$. For $F \in V(M)$ let

$$GF = \{g(F) \mid g \in G\}$$

$$r(F) = \{r(c) \mid c \in F\}$$

In order to facilitate the statement of various theorems we call $F \in V(M)$ **space-like**, **fiber-like** or **point-like**, iff respectively $\mathrm{Cl}_{V(M)}(GF) \cap S_s \neq \emptyset$, $\mathrm{Cl}_{V(M)}(GF) \cap S_f \neq \emptyset$ or $\mathrm{Cl}_{V(M)}(GF) \cap S_f \neq \emptyset$. We start with an easy lemma:

Lemma 8.1. *If $|r(F)| = \infty$ then F is space-like.*

Proof. Let $F \subset V(M)$ be such that $r(F) = \infty$. Let $\epsilon > 0$. Let $\{f^i\}_{i=1}^N \subset M$ be an ϵ-net of M. Using Lemma 2.1 and as X is non-trivial and connected, one can assume without loss of generality that $r(f^1), \ldots, r(f^N)$ are distinct. Choose $c^i \in F$, $i = 1, \ldots, N$ so that $r(c^i) = \{r^i\}$, $i = 1, \ldots, N$ are distinct. Using the N-transitivity of G (Lemma 4.1) one can assume without loss of generality $r(f^i) = \{r^i\}$. We will now find $g \in G$ so that $d(g(c^i), f^i) < 3\epsilon$ for $i = 1, \ldots, N$. By Lemma 5.6 one can assume $(f^1, \ldots, f^N) \in \mathfrak{R}_*^N$, in particular: $f^i = \{\gamma^i([0,t])\}_{t \in \mathbb{R}_+} \cup \{X\} \in \mathfrak{R}$, $i = 1, \ldots, N$, $\gamma^i \in C_s(\mathbb{R}_+, X)$. Let $a \in \mathbb{R}_+$ so that $d(\gamma^i([0,a]), X) < \epsilon$, $i = 1, \ldots, N$. Find disjoint open sets $U_1, \ldots, U_N$ so that $\gamma^i([0,a]) \subset U^i$. Using Lemma 6.4 conclude there are $g^j \in G_{U_j}$, $j = 1, \ldots, N$ so that $d(g^j(c^j), f^j) < 3\epsilon$. Define $g = g^1 \circ g^2 \circ \cdots g^N$. Notice $d(g(c^j), f^j) < 3\epsilon$, $j = 1, \ldots, N$. We conclude $d_{V(M)}(G(F), M) < 4\epsilon$. □

Definition 8.2. Let $\epsilon > 0$. A non-empty connected open set A is **ϵ-encircling** a connected closed subset $B \subset X$ if $A \cap B = \emptyset$ and $A \cup B$ is open and connected with $\operatorname{diam}\{B \cup A\} < \epsilon$. Notice that the fact that $A \cup B$ is open implies $\partial B \subset \partial A$ thus "A is encircling B".

Definition 8.3. Let $\epsilon > 0$ and $x \in X$. An open connected subset $A \subset X$ has the **(G, ϵ, x)-Boundary Shrinking Property** if:

- The boundary ∂A is connected and has at least two points.
- For any closed $W \subsetneq \partial A$, $\delta > 0$ and $y \in A$ with $y \neq x$, there exists $h \in G_{B(A,\delta)}$ so that $h(x) = x$, $h(W) \subset B(y, \delta)$.
- There exists an open connected set E which is ϵ-encircling $\overline{A}$.

We say that the G–space X has the **Boundary Shrinking Property** (BSP) if for any $x \in X$ and $\epsilon > 0$ there exists an open connected set A with $x \in A$ which has the (G,ϵ,x)-boundary shrinking property.

Lemma 8.4. *Let the Peano continuum X be SRI and let (X, G) have the boundary shrinking property. Suppose $x \in X$ and $F \in V(M_x)$. Then F is either point-like or fiber-like.*

Proof. Let $\epsilon > 0, e \in M$. We say F is **(ϵ, e)-point-like** if there exists $g \in G$ so that $d(g(F), \{e\}) < \epsilon$. We say F is **ϵ-fiber-like** if there exists $g \in G$ so that $d(g(F), M_x) < \epsilon$ for some $x \in X$. We will prove that for a given $\epsilon > 0$, F is either 3ϵ-fiber-like or $(2\epsilon, e(\epsilon))$-point-like for some $e(\epsilon) \in M$. This will of course imply the statement of the lemma. Let $\epsilon > 0$. Using property (BSP) of (X, G) choose B with the (G,ϵ,x)-boundary shrinking property. Let V be an open connected set ϵ-encircling $\overline{B}$. Choose Z with the $(G, \frac{\epsilon}{2}, x)$-boundary shrinking property. Let A be an open connected set $\frac{\epsilon}{2}$-encircling $\overline{Z}$. We arrange so that $\operatorname{Cl}_X\{Z \cup A\} \subset B$, which implies there is $\delta_1 > 0$ so that $B(Z, \delta_1) \subset B$. Let $f = \{f_\kappa\}_{\kappa \in K} \in F$. Recall $f_{\overline{Z}} = \bigcup\{f_\alpha |\, f_\alpha \subset \overline{Z}\}$. Define $S(f) = f_{\overline{Z}} \cap \partial Z$. By Lemma 2.2 $S(f) \neq \emptyset$. Define $H = \{f \in F|\, S(f) \neq \partial Z\}$. We first assume $H \neq \emptyset$. Let $f_1, \ldots, f_N \in F$. Define:

$$I(f_1, \ldots, f_N) = \{f \in H|\, \exists i \in \{1, \ldots, N\}\ S(f_i) \cap S(f) \neq \emptyset\} \tag{8.1}$$

$$I^c(f_1, \ldots, f_N) = H \setminus I(f_1, \ldots, f_N) \tag{8.2}$$

We are now going to choose a certain sequence of distinct elements $\{f_i\}_{i=1}^{i=N} \subset F$ where $N \in \mathbb{N} \cup \{\infty\}$. Start by choosing an arbitrary $f_1 \in H$. If $I^c(f_1) = \emptyset$ stop. If not choose $f_2 \in I^c(f_1)$. Clearly $f_2 \neq f_1$. If $I^c(f_1, f_2) = \emptyset$ stop. If not choose $f_3 \in I^c(f_1, f_2)$. Continue in this manner. The inductive process results in one of the following two possibilities: (1) $N \in \mathbb{N}$ (2) $N = \infty$. In case (1) we claim that F is $(2\epsilon, e(\epsilon))$-point-like for some $e(\epsilon) \in M$. Let $W = \bigcup_{i=1}^N S(f_i)$ be a closed set. Notice that as ∂Z is connected and $S(f_i) \cap S(f_j) = \emptyset$ for $1 \leq i < j \leq N$ we have $W \subsetneq \partial Z$. As $\overline{B} \cup V$ is open and connected, one can find $\zeta \in C_s([0,1], \overline{B} \cup V)$ with $\zeta(0) = x$ and $\zeta(1) \in V$ (in particular $\zeta(1) \notin \overline{B}$). Invoke Lemma 5.4 (one takes in the lemma $k = 1,\ F_1 = \overline{B}$ and $U_1 = \overline{B} \cup V$) to find $\overline{\zeta} \in C_s(\mathbb{R}_+, X)$, $a, \delta > 0$ so that $e(\epsilon) = c = \{\overline{\zeta}([0,t])\}_{t\in\mathbb{R}} \cup \{X\} \in \mathfrak{R}_x$, $\overline{\zeta}([0, 1-\delta]) \subset \overline{B} \cup V$, $\overline{\zeta}(t) \notin \overline{B}$ for $1 - \delta \leq t \leq a$ and $d(\overline{\zeta}([0,a]), X) < \epsilon$. Let $T = \bigcup_{k=1}^{R} U_k$ be a ϵ-tube of $\overline{\zeta}_{|[0,a]}$ so that $U_1 = B$ (this can be easily be arranged by redefining ζ inside $\overline{B} \cup V$). Using the boundary shrinking property one can find $h \in G_B$, $y \in Z$, $\delta_0 > 0$ so that $h(W) \subset B(y, \delta_0)$, $h(x) = x$. $\delta_0 > 0$ can be chosen small enough so that Lemma 4.1 implies there is $g_1 \in G_T$ with $g_1(x) = x$ and $g_1(B(y, \delta_0))$ is inside a sufficiently small neighborhood of $\zeta(a)$ (in the sense of Lemma 6.3). Let $g = g_1 \circ h$. We will now show that $d(g(F), c) < 2\epsilon$. It is enough to show for all $f \in F$ one has $d(g(f), c) < 2\epsilon$. Let $f \in F$. If $f \in H$ then $S(f) \cap W \neq \emptyset$. If $f \notin H$, then clearly the same conclusion holds. Notice $h(f) \in M_x$, $dist(h(f_{\overline{Z}}), y) < \delta_0$ and $h(f_{\overline{Z}}) \subset B$. Finally notice $\overline{\zeta}(0) \in g(f_{\overline{Z}}) \subset T$ is sufficiently close to $\overline{\zeta}(a)$ and therefore parts (1) & (3) of Lemma 6.3 imply the desired conclusion. We now turn our attention to the case $N = \infty$. We claim that in this case F is 3ϵ-fiber-like. Let $\{c_i\}_{i=1}^{L} \subset M_x$ be an ϵ-net of M_x. Choose $f_i \in F$, $i = 1, \ldots, L$ so that $S(f_i) \cap S(f_j) = \emptyset$ for $i \neq j$. The idea now will be to approximate the c_is by $\mathbb{R}_+$-chains s_is and then act on the f_is with an element $g \in G$ so that $g(f_i)$ will approximate s_i. We will make an essential use of the fact that the f_is intersect ∂Z in disjoint locations in order to construct the above-mentioned $g \in G$. Choose $y_i \in S(f_i)$ and $\mu > 0$ so that $\mu < \min\{d(S(f_i), [f_j]_{\overline{Z}}) \mid 1 \leq i < j \leq L\}$. As X is locally connected one can choose open connected subsets C_i with $y_i \in C_i \subset B(y_i, \mu) \cap (A \cup \overline{Z})$, $i = 1, \ldots, L$, where we use the fact that $A \cup \overline{Z}$ is open. As C_i are open and connected one can find simple paths $\gamma_i \in C_s([0,1], C_i)$, $i = 1, \ldots, L$, so that $\gamma_i(0) = y_i$ and $\gamma_i(1) \in A$. Now invoke Lemma 5.4 (one takes in the lemma $k = 1,\ F_1 = \cdots = F_L = \overline{Z}$ and $U_1 = \cdots = U_L = \overline{Z} \cup A$) to find $a > 0$ and $(s_1, \ldots, s_L) \in \mathfrak{R}_*^L$ with $r(s_i) = \{y_i\}$, $i = 1, \ldots, L$ represented as $s_i = \{\xi_i([0,t])\}_{t\in\mathbb{R}_+} \cup \{X\}$, $i = 1, \ldots, L$ so that there exist $\delta > 0$ so that $\xi_i([1-\delta, a] \cap \overline{Z} = \emptyset$, $[\xi_i]_{|[0,1-\delta]} = [\gamma_i]_{|[0,1-\delta]}$, $d(X, \xi_i([0,a]) < \frac{\epsilon}{2}$ for $i = 1, \ldots, L$ and

$$d(c_i, s_i) < \epsilon \ \ i = 1, \ldots, L \tag{8.3}$$

While defining the ξ_i one can construct disjoint ϵ-tubes $T_i = \bigcup_{k=1}^{q_i} U_i^k$, $i = 1, \ldots, L$ with $U_i^1 = Z$, $U_i^2 = C_i$ and $\zeta_i([0,a]) \subset T_i$. Notice that $Q_i = \bigcup_{k=2}^{q_i} U_i^k$ is a ϵ-tube (for $\zeta_{|[1-\delta,a]}$) so one can choose $g_i \in G_{Q_i}$ so that $g(y_i) = \xi_i(a)$. Let f_i' be the initial segment of f_i ending at $[f_i]_{\overline{Z}}$. By Lemma 6.3:

$$d(g_i(f_i'), s_i) < 2\epsilon, \ i = 1, \ldots, L \tag{8.4}$$

Define $g = g_1 \circ \ldots \circ g_L$. As $g_i \in G_{Q_i}$ we have $[g_i]_{|[f_j]_{\overline{Z}}} = Id$ for $1 \leq i < j \leq L$. From this and conditions (8.3) and(8.4) conclude that $d(c_i, g(f_i)) < 3\epsilon$ for $i = 1, \ldots, L$. In particular $d(g(F), M_x) < 3\epsilon$. Finally if $H = \emptyset$ we choose $\{*\} = W \subsetneq \partial Z$ and repeat the same construction used in the case $H \neq \emptyset$ and $N = 1$. □

Corollary 8.5. *Under the assumptions of Lemma 8.4, if* $F \in V(M_x)$ *and* $|F| < \infty$, *then* F *is point-like.*

Theorem 8.6. *Let the Peano continuum* X *be SRI and let* (X, G) *have the boundary shrinking property. Then, the only minimal subspaces of* $V(M)$ *are:*

(1) $\{M\}$
(2) $\{M_x\}_{x \in X}$
(3) $\big\{\{c\} \,|\, c \in M\big\}$

Proof. The G invariance of all three presented subspaces is clear. The fact that $\{M\}$ and $\big\{\{c\} \,|\, c \in M\big\}$ are closed, is trivial. The fact that $\{M_x\}_{x \in X}$ is closed is proven in Lemma 2.1. The minimality of $\{M\}$ is trivial. The minimality of $\{M_x\}_{x \in X}$ is a consequence of Lemma 2.1 and the transitivity of the action of G on X (Lemma 4.1). The minimality of $\big\{\{c\} \,|\, c \in M\big\}$ is a consequence of Theorem 6.5. To show that the presented subspaces are the only minimal subspaces it is enough to show that any $F \in V(M)$ is either space-like, fiber-like or point-like. Let $F \in V(M)$. If $|r(F)| = \infty$, then by Lemma 8.1 F is space-like. If $|r(F)| \in \mathbb{N}$ one can assume without loss of generality $|r(F)| = 1$. By Lemma 8.4 F is either point-like or fiber-like. □

9. The Strong Proximality of M

The goal of this section is to prove that M is proximal under the assumption of the previous section and strongly proximal under additional assumptions. Let us start with the definition of these two terms. M is said to be **proximal** under G if for any $c, f \in M$ one can find $g_n \in G$ so that $\lim_{n \to \infty} d(g_n(c), d(g_n(f)) = 0$. M is said to be **strongly proximal** under G if for any Borel probability measure μ on M, there exists a sequence (g_n) of elements of G such that $[g_n]_*(\mu)$ converges to the measure concentrated at a singleton.

Theorem 9.1. *Let the Peano continuum* X *be SRI and let* (X, G) *have the boundary shrinking property. Under these conditions* (M, G) *is proximal.*

Proof. This theorem can be proven using only the assumptions of Section 6, but here we will use instead the method of Lemma 8.4. Let $c, f \in M$. One can assume without loss of generality that $c, f \in M_x$ for some $x \in X$. Define $F = \{c, f\} \in V(M)$. As $|F| = 2$, by Corollary 8.5 F is point-like, i.e., there exist $g_n \in G$ so that $\operatorname{diam}\{g_n(F)\} \to_{n\to\infty} 0$, which is equivalent to the proximality of the pair (c, f). $\square$

Theorem 9.2. *Let the Peano continuum X be SRI and let (X,G) have the boundary shrinking property. If (X,G) is strongly proximal then (M,G) is strongly proximal.*

Proof. Let μ be a Borel probability measure on M. Let $r_*(\mu)$ be the projection of μ under the map $r : M \to X$. Using the strong proximality of (X,G) one can assume without loss of generality that $r_*(\mu)(\{x\}) = 1$ for some $x \in X$. Let $\epsilon > 0$. We will prove that one can find $g \in G$ and $c \in M$ so that $g_*(\mu)(B(c, 2\epsilon)) > 1 - \epsilon$. By standard compactness arguments this will show (M,G) is strongly proximal. Using property (BSP) of (X,G) choose Z with the $(G, \frac{\epsilon}{2}, x)$-boundary shrinking property such that $x \in Z \subset B(x, \frac{\epsilon}{2})$. Let $f \in M_x$. Define $S(f) = f_{\overline{Z}} \cap \partial Z$. Let $W \subsetneq \partial Z$ be a closed subset. Repeating an argument appearing in Lemma 8.4, we can find $c \in \mathfrak{R}_x$ and $h_W \in G$ so that $f \in M_x$ with $S(f) \cap W \neq \emptyset$ implies $d(h_W(f), c) < 2\epsilon$. Define $E_W = \{f \in M_x \,|\, S(f) \cap W \neq \emptyset\}$ and $F_W = \{f \in M_x \,|\, d(h_W(f), c) < 2\epsilon\}$. Notice F_W is open in M_x and that $E_W \subset F_W$. Another useful property is that if $W_0, W_1 \subsetneq \partial Z$ are closed subsets so that $\partial Z \setminus W_0$ and $\partial Z \setminus W_1$ are disjoint then $M_x \setminus F_{W_0}$ and $M_x \setminus F_{W_1}$ are also disjoint. Indeed if $f \in M_x \setminus F_{W_i}$ then $S(f) \subset \partial Z \setminus W_i$. This implies $S(f) \cap W_{\bar{i}} \neq \emptyset$, i.e. $f \in E_{W_{\bar{i}}} \subset F_{W_{\bar{i}}}$, which implies $f \notin M_x \setminus F_{W_{\bar{i}}}$ (here we use teh convention $\overline{0} = 1, \overline{1} = 0$). Let $n \in \mathbb{N}$ so that $\frac{1}{n} \leq \epsilon$. As ∂Z is connected and has at least two points one can choose n non-empty pairwise disjoint open subsets $O_1, \ldots, O_n \subset \partial Z$. Define $W_i = \partial Z \setminus O_i, i = 1, \ldots, n$. Conclude that the closed sets $M_x \setminus F_{W_i}, i = 1, \ldots, n$ are pairwise disjoint. Conclude that there exist $1 \leq j \leq n$ so that $\mu(M_x \setminus F_{W_j}) \leq \frac{1}{n} < \epsilon$. Conclude that $\mu(F_{W_j}) > 1 - \epsilon$, i.e. $[h_{W_j}]_*(\mu)(B(c, 2\epsilon)) > 1 - \epsilon$. $\square$

We call (X,G) **base-wise shrinkable** iff X has a basis $\{U_\alpha\}_{\alpha \in A}$ (called a **shrinkable basis**) so that for any pair of open subsets $V \subset \overline{V} \subset U_\alpha, W \subset U_\alpha$ there is $g \in G_{U_\alpha}$ so that $g(V) \subset W$. It turns out that for such spaces one can prove strong proximality.

Lemma 9.3. *Suppose X is a Peano continuum such that for each connected open set $U \subset X$ and each $p \in X$ the set $U \setminus \{p\}$ is connected. If (X,G) is base-wise shrinkable then (X,G) is strongly proximal.*

Proof. Let $\mathfrak{M}(X)$ be the space of Borel probability measures of X. Let $\epsilon > 0$. We will show there exist an open set U_ϵ with $\operatorname{diam}(U_\epsilon) < \epsilon$ and $g_\epsilon \in G$ so that $\mu(g_\epsilon(U_\epsilon)) > 1 - \epsilon$. Cover X by elements from a shrinkable basis $\{U_k\}_{k=1}^{N-1}$ so that $\operatorname{diam}\{U_k\} < \epsilon$ for $k = 1, \ldots, N-1$. Assume without loss of generality that there exist a non-empty open subset $U_N \subset U_1 \setminus \bigcup_{k=2}^{N-1} U_k$. Define

$U_k^r = \{y \in U_k \,|\, dist(y, U_k^c) > \frac{1}{r}\}, r \in \mathbb{N}$, $1 \leq k \leq N$. Notice $U_k = \bigcup_{r=1}^{\infty} U_k^r$. If there is $g \in G$ so that $\mu(g(U_1)) > 1 - \epsilon$, we are done. Assume not. Let $s = \sup_{g \in G} \mu(g(U_N))$. As $U_N \subset U_1$, $s \leq 1 - \epsilon$. Using the fact that U_1 is part of a shrinkable basis one can assume without loss of generality $s - \frac{(1-s)}{2N} < \mu(U_N) \leq s$. As $\mu(U_1) \leq 1 - \epsilon$, there is $2 \leq k \leq N - 1$ so that $\mu(U_k) > \frac{(1-s)}{2N}$, in particular there is $l \in \mathbb{N}$ so that $\mu(U_k^l) > \frac{(1-s)}{2N}$. Choose $q \in \mathbb{N}$ so that $s - \frac{(1-s)}{2N} < \mu(U_N^q) \leq s$. As X is arcwise connected there is $p \in C_s([0,1], X \setminus U_N^q)$ so that $p(0) \in U_k$ and $p(1) \in U_N$. Let T be a δ-tube for p for some $\delta > 0$ so that $T \subset X \setminus U_N^q$. Using Lemma 4.1 find $h \in G_T$ so that $h(p(0)) = p(1)$. Find an open subset $p(0) \in Z \subset U_k$ so that $h(Z) \subset U_N$. As (X, G) is base-wise shrinkable one can find $e \in G_{U_k}$ (in particular $e_{|U_N} = Id$) so that $e(U_k^l) \subset Z$. Define $g = (h \circ e)^{-1}$. It is easy to see $\mu(g(U_N)) > s - \frac{(1-s)}{2N} + \frac{(1-s)}{2N} = s$. This is a contradiction to the definition of s. We conclude $\sup_{g \in G} \mu(g(U_1)) > 1 - \epsilon$. $\square$

From Theorem 9.2 and Theorem 9.3 we have:

Theorem 9.4. *Let the Peano continuum X be SRI and let (X, G) have the boundary shrinking property. If (X, G) is base-wise shrinkable then (M, G) is strongly proximal.*

10. On the Structure of $V(M(S^2))$

Let $X = S^2$, where S^2 is be the two-dimensional sphere. S^2 is strongly arcwise-inseparable but not strongly $\mathbb{R}$-inseparable. One may ask if Theorem 8.6 still holds in this setting. The following theorem answers this question negatively.

Theorem 10.1. *Let $X = S^2$. There exist $F \subseteq V(M)$ which is not point-like, nor space-like, neither fiber-like.*

Proof. To facilitate notation assume $X = S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ Let $p = (0, 0, -1), n = (0, 0, 1)$ be the "south" and "north" poles of X. Let T_p be the tangent space of X at p. Let $\hat{x} \in T_p$ be the unit tangent vector in the direction of the x-axis. We are going to define a family of curves $q_{(x,\theta)\to n}$, where $x \in X$ and $\theta = \theta(x) \in [0, 2\pi]$ represents an angle. We start by defining for $\theta \in [0, 2\pi]$ the curve $q_{(p,\theta)\to n} : [0,1] \to X$ as the unique geodesic of X with $q_{(p,\theta)\to n}(0) = p$, $q_{(p,\theta)\to n}(1) = n$ and $\angle(\hat{x}, \dot{q}_{(p,\theta)\to n}(0)) = \theta$. Fix $t \in (0, 1]$ and $\theta \in [0, 2\pi]$. Let $x = q_{(p,\theta)\to n}(t)$ and denote by $q_{(x,\theta)\to n}$: $[0, 1] \to X$ the unique geodesic of X with $q_{(x,\theta)\to n}(0) = x$, $q_{(x,\theta)\to n}(1) = n$ and $q_{(x,\theta)\to n}([0, 1]) \subset q_{(p,\theta)\to n}([0, 1])$. Given $c = \{c_\alpha\}_{\alpha\in A} \in M$ and $l \in C([0, h], X)$ with $\{l(h)\} = r(c)$ define the "concatenated maximal chain": $s(l, c) = \{l([0, t]\}_{t\in[0,h]} \cup \{l([0, h]) \cup c_\alpha\}_{\alpha\in A}$. Notice $s(l, c) \in M_{l(0)}$. Let

$c = \{\overline{B}(n,\epsilon\}_{\epsilon\geq 0} \in M$. Define:

$$F = \{s(q_{(q_{(p,\theta)\to n}(t),\theta)\to n}, c)\}_{(t,\theta)\in[0,1]\times[0,2\pi]}$$

It is easy to see $F \in V(M)$. Indeed given $\{s(q_{(x_i,\theta_i)\to n}, c)\}_{i=1}^{\infty} \subseteq F$ a converging sequence, there is $x^* \in X$ and $\theta^* \in [0,2\pi)$ so that by passing to a subsequence $x_i \to_{i\to\infty} x^*$ and $\theta_i \to_{i\to\infty} \theta^* \, mod\, 2\pi$. Clearly $s(q_{(x_i,\theta_i)\to n}, c) \to_{i\to\infty} s(q_{(x_*,\theta_*)\to n}, c)$ for the original sequence. Notice $r(F) = X$, conclude F is not point-like, nor fiber-like. We will now show F is neither space-like. Let $e : [0,2\pi] \to X$ be the "equatorial" great circle $e(t) = (cos(\pi + t), sin(\pi + t), 0)$. Let $w : [0,2\pi] \to X$ be the "Greenwich" great circle $w(t) = (0, sin(-t), cos(-t))$. Let $m_1 \in M_{(-1,0,0)}$, $m_2 \in M_{(0,0,1)}$ be arbitrary elements. Define $c_1 = s(e, m_1) \in M_{e(0)}$ and $c_2 = s(w, m_2) \in M_{w(0)}$. We will show that for any $A \in \mathrm{Cl}_{V(M)}(GF)$ one has $\{c_1, c_2\} \notin A$. In particular $M \notin \mathrm{Cl}_{V(M)}(GF)$. Our proof will be based on the following observation: if $E, D \in C(X)$, then by the Jordan Separation Theorem there exists $\epsilon_0 > 0$ so that $D \subset B(w([0,\pi]), \epsilon_0)$ and $E \subset B(e([0,\pi]), \epsilon_0)$ imply that $E \cap D \neq \emptyset$ and in fact $E \cap D \subset I$ where $I = B(w([0,\pi]), \epsilon_0) \cap B(e([0,\pi]), \epsilon_0)$. We choose $0 < \epsilon < \min\{\epsilon_0, \frac{1}{2} d(e(0), w(0))\}$.

Assume for a contradiction that there exist $g \in G$ and $f_1, f_2 \in F$ so that $d(g(f_i), c_i) < \epsilon$ for $i = 1, 2$. In particular there exist $Y_i \in f_i$ for $i = 1, 2$ so that $d(g(Y_1), e([0,\pi])) < \epsilon$ and $d(g(Y_2), w([0,\pi])) < \epsilon$. We also have $d(r(g(Y_1)), e(0)) < \epsilon$ and $d(r(g(Y_2)), w(0)) < \epsilon$, which implies that $r(g(Y_1) \neq r(g(Y_2)$. As $g(Y_1)$ and $g(Y_2)$ intersect, i.e. $\emptyset \neq g(Y_1) \cap g(Y_2) \subset I$, we conclude that $Y_1 = q_{(x_1,\theta_1)\to n}([0,1)) \cup B(n, \epsilon_1)$ and $Y_1 = q_{(x_2,\theta_2)\to n}([0,1)) \cup B(n, \epsilon_2)$ for $x_1 \neq x_2$ and w.l.o.g $\epsilon_1 \geq \epsilon_2 \geq 0$. Notice $B(e([0,\pi]), \epsilon_0) \setminus I$ has two components. Let J_1 be the component with $r(g(Y_1)) \subset J_1$. Similarly let J_2 be the component of $B(w([0,\pi]), \epsilon_0) \setminus I$ with $r(g(Y_2)) \subset J_2$. We conclude that:

$$g(f_1) = s(\{g \circ q_{(x_1,\theta_1)\to n}([0,t])\}_{t\in[0,1]}, g(c)),$$
$$g(f_2) = s(\{g \circ q_{(x_2,\theta_2)\to n}([0,t])\}_{t\in[0,1]}, g(c)),$$

where $g(q_{(x_i,\theta_i)\to n}([0,1])) \subset J_i \cup I$, $i = 1, 2$. In other words until $g(f_1)$ and $g(f_2)$ "meet" they are confined to $J_1 \cup I$ and $J_2 \cup I$ respectively. After they "meet" they develop identically (which corresponds to the $g(c)$ part of the concatenation). This is a clear contradiction to $d(g(f_i), c_i) < \epsilon$ for $i = 1, 2$ for ϵ small enough. □

11. Manifolds and the Hilbert Cube

In this section we present classes of examples to which one can apply the results of the article.

Theorem 11.1. *Let X be a two-dimensional closed topological manifold and G a locally transitive group acting on X, then $(M(X), G)$ is minimal and* *the only minimal subspace of* $(\Phi(X), G)$*. Moreover the universal minimal space* (U_G, G) *is not transitive.*

Proof. As X is a closed topological manifold, X is a Peano continuum. By assumption G acts transitively on X. By Lemma A.1 X is strongly arcwise-inseparable. These facts enable us to conclude by Theorem 6.5 and Corollary 6.6 that $(M(X), G)$ is minimal and (U_G, G) is not transitive. Moreover as X is a closed topological manifold, it is easy to see X has the telescoping annuli property. This implies by Theorem 7.2 that $(M(X), G)$ is the only minimal subspace of $(\Phi(X), G)$. □

Theorem 11.2. *Let* X *be a closed topological manifold of dimension* $n \geq 3$ *and* G *a subgroup of the homeomorphism group of* X*. If* $G \supset Homeo_0(X)$ *or* X *has a smooth structure such that* $G \supset Diffeo_0(X)$*, then* $(M(X), G)$ *is minimal and strongly proximal and the only minimal subspace of* $(\Phi(X), G)$*. The only minimal subspaces of* $(V(M(X)), G)$ *are* $\{M(X)\}$*,* $\{M(X)_x\}_{x\in X}$ *and* $\big\{\{c\} \,|\, c \in M(X)\big\}$*. The universal minimal space* (U_G, G) *is not transitive.*

Proof. Notice that by the discussion of locally transitive group actions in the end of Section 3, $G \supset Homeo_0(X)$ or $G \supset Diffeo_0(X)$ imply that G acts locally transitively on X. Using Theorem 11.1 we conclude (U_G, G) is not transitive and $(M(X), G)$ is minimal and the only minimal subspace of $(\Phi(X), G)$. By Lemma A.3 X is strongly $\mathbb{R}$-inseparable. By Lemma A.5 (X, G) has the boundary shrinking property. The last two facts enable us to conclude that the only minimal subspaces of $(V(M(X)), G)$ are $\{M(X)\}$, $\{M(X)_x\}_{x\in X}$ and $\big\{\{c\} \,|\, c \in M(X)\big\}$. Finally it is easy to verify that (X, G) is base-wise shrinkable which implies $(M(X), G)$ is strongly proximal □

Recall that the Hilbert cube is defined to be $Q = [-1, 1]^{\mathbb{N}}$, equipped with the metric $d((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}) = \max\{\frac{|x_n - y_n|}{n} \,|\, n = 1, 2, \ldots\}$.

Theorem 11.3. *Let* $G = Homeo(Q)$*. The* G*-space* $(M(Q), G)$ *is minimal and strongly proximal and the only minimal subspace of* $(\Phi(Q), G)$*. The only minimal subspaces of* $(V(M(Q)), G)$ *are* $\{M(Q)\}$*,* $\{M(Q)_x\}_{x\in X}$ *and* $\big\{\{c\} \,|\, c \in M(Q)\big\}$*. The universal minimal space* (U_G, G) *is not* 1*-transitive.*

Proof. The Hilbert cube Q is metric, compact, connected and locally connected. G acts locally transitively by Lemma A.2. By Lemma A.4 Q has the telescoping annuli property. By Lemma A.3 Q is strongly $\mathbb{R}$-inseparable. Using Theorem 6.5, Theorem 7.2 and Corollary 6.6 conclude $(M(Q), G)$ is minimal, $(M(Q), G)$ is the only minimal subspace of $(\Phi(Q), G)$ and (U_G, G) is not transitive. We now proceed to prove that the only minimal subspaces of $(V(M(Q)), G)$ are $\{M(Q)\}$, $\{M(Q)_x\}_{x\in X}$ and $\big\{\{c\} \,|\, c \in M(Q)\big\}$. The natural approach would be to use Theorem 8.6. However in order to use it one has to show (Q, G) has the boundary shrinking property. Unfortunately we were not able to do that (see Question 12.2). A careful reading shows the boundary shrinking property is used in the proof of Theorem 8.6 only via the use of Lemma 8.4. We give in Lemma A.7 a replace proof for (Q, G) and thus achieve the above mentioned result. By Lemma A.8 $(M(Q), G)$ is strongly proximal. □

12. Open Questions

We are ideally interested in finding all minimal subspaces of $V^n(X)$ for $n \in \mathbb{N}$, unfortunately this turned out to be too difficult and we leave it as a question to the reader:

Question 12.1. *Can one characterize all minimal subspaces of $V^n(X)$, $n \geq 2$?*

A natural way to prove the results of section 8 for $(Q, Homeo(Q))$ is to show that $(Q, Homeo(Q))$ has the (BSP) property. Unfortunately we are unable to settle the following question:

Question 12.2. *Does $(Q, Homeo(Q))$ have the boundary shrinking property?.*

In the Introduction we mentioned Open Questions 28 & 29 of [Pes05] which ask for an explicit description of the universal minimal space of the group of homeomorphisms $Homeo(X)$, X being a closed manifold of dimension 2 or higher or the Hilbert cube. In view of our results we reformulate these question to the following question:

Question 12.3. *Is the universal minimal space for the group Homeo(X), X being a closed manifold of dimension* 3 *or higher or the Hilbert cube, equal to the space M(X)?*

Appendix

The appendix contains various topological results used in Sections 3 and 11. The first three are reformulations of facts which are well known, sometimes in a greater generality. For the reader's convenience we provide however detailed arguments or bibliographical hints.

Lemma A.1. *If X is a two-dimensional closed topological manifold then X is strongly arcwise-inseparable.*

Proof. Let $U \subset X$ be open and connected and $J \subset U$ be an arc. The connectivity of $U \setminus J$ is known in a greater generality when J is a **cell-like** compact subset of X, that is, one which can be contracted to a point in each of its neighborhoods. See corollary 4B on p. 121 of [Dav86]. □

Lemma A.2. *$Homeo(Q)$ is locally transitive and is n-transitive for all $n \in \mathbb{N}$.*

Proof. In [BP75] p. 145 in the proof of Proposition 8.1 Bessaga and Pełczyński show $Homeo(Q)$ is **strongly locally homogeneous**, i.e for each $x \in Q$ there exists a basis of open neighborhoods $\{V_i\}$ so that $Hoemeo(Q)_{V_i}$ acts transitively on V_i, for each i. Thus $Homeo(Q)$ is locally transitive and, by Theorem 4.1, also n–transitive. □

Lemma A.3. *If X is a closed topological manifold of dimension $n \geq 3$ or X is the Hilbert cube, then X is strongly $\mathbb{R}$-inseparable.*

Proof. Let J be an arc in X and $U \subset X$ be a connected open set. To establish that $U \setminus J$ is connected we consider 3 cases:

1) X is an n–manifold and U is homeomorphic to $\mathbb{R}^n$, where $n \geq 3$. Then $U \setminus S$ is connected for any closed set $S \subset X$ of dimension not greater than $n-2$; see Theorem 1.8.13 in [Eng78]. In particular, this applies to $S = J$.

2) $X = Q$. By the definition of the product topology of Q there exists $n \geq 3$ and a chart $V \subset \prod_{k=1}^{n}(-1,1)_k$, $V \cong \mathbb{R}^n$ such that $V \times \prod_{k>n}[-1,1]_k \subset U$. Given $x, y \in U$ there exists by the lemma above an $f \in Homeo(Q)_U$ such that $f(x)$ and $f(y)$ belong to $V \times \{0\}$. By 1) above, the set $V \times \{0\} \setminus f(J)$ is connected and hence there is an arc K in it connecting $f(x)$ to $f(y)$. Clearly, $f^{-1}(K)$ is an arc in $U \setminus J$ connecting x to y. Since x, y are arbitrary points of $U \setminus J$, this set is connected.

3) The case where X is an n–manifold but U is not homeomorphic to $\mathbb{R}^n$ follows from 1) in precisely the same manner, using the 2-transitivity of $Homeo(X)_U$. □

Lemma A.4. *Q has the telescoping annuli property.*

Proof. In order to show that there is a $(1,\epsilon)$ annuli telescope around a given point $x \in Q$ we first note that by Lemma A.2 we can assume without loss of generality that $x = (0, 0, \ldots)$. As in the proof above there exists a set of diameter smaller than ϵ which contains 0 and is of the form $V \times \prod_{k>n}[-1,1]_k$, for some open set $V \subset \prod_{k=1}^{n}(-1,1)$. Let $\{B, U\}$ be an $(1,\delta)$ annuli telescope in V around $(0,\ldots,0)$, for some $\delta > 0$ and the Euclidean metric of V. Define $B' = B \times \prod_{k>n}[-1,1]_k$ and $U' = U \times \prod_{k>n}[-1,1]_k$. It is clear one can choose δ small enough so that $\{B', U'\}$ is an $(1,\epsilon)$ annuli telescope around $x = 0$. □

Lemma A.5. *Let X be a closed topological manifold of dimension $n \geq 2$ and G be a subgroup of $Homeo(X)$. If $G \supseteq Homeo_0(X)$ or X is a smooth manifold and $G \supseteq Diffeo_0(X)$ then (X, G) has the boundary shrinking property.*

Proof. Let $x \in X$ and $\epsilon > 0$. Since X is a manifold one can find a chart $C \cong \mathbb{R}^n$ so that $x \in C \subset B(x, \frac{\epsilon}{2})$. Let A and E be open balls (in the Euclidean metric of C) with center x, such that $\overline{A} \subset E \subset C$. It is easy to see that A has the (G, ϵ, x)-boundary shrinking property, with $E \setminus A$ being a set ϵ–encircling $\overline{A}$. □

Lemma A.6. *For $n \geq 3$, let $x \in Int(I^n)$ and $F \in V(M_x)$. If F is not point-like then F is fiber-like.*

Proof. This is a simple generalization of the techniques used in this article. One uses strongly the fact that $x \in Int(I^n)$. The idea is that all constructions in the first 9 sections can be done inside $Int(I^n)$ and therefore the statement of the lemma follows from same statement for S^n (proven in Lemma 8.4 applied to $X = S^n$). □

For $n \in \mathbb{N}$ it will be convenient to denote the product $\prod_{i=1}^{n}[-1,1]_i$ by I^n, $\prod_{i=n}^{\infty}[-1,1]_i$ by I_n^{∞} and the standard projection of Q onto I^n by π_n. Given a homeomorphism g of I^n we write $\widetilde{g}$ for the homeomorphism of Q which composed with π_n is equal to g and composed with the projection onto I_{n+1}^{∞} equals to this projection. In the following lemmas it would be convenient to use the metric $d((x_m)_{m=1}^n, (y_m)_{m=1}^n) = \max\{\frac{|x_m - y_m|}{m} \,|\, m = 1, 2, \ldots, n\}$ on I^n and the metric $d((x_m)_{m=1}^{\infty}, (y_m)_{m=1}^{\infty}) = \max\{\frac{|x_m - y_m|}{m} \,|\, m = 1, 2, \ldots\}$ on Q.

Lemma A.7. *Let $x = (0, 0, \ldots) \in Q$. Let $F \in V(Q_x)$, then F is either point-like or fiber-like.*

Proof. According to Lemma A.6 for each $n \in \mathbb{N}$, $\pi_n(F) \in V(M_{\pi_n(x)}(I^n))$ is either point-like or fiber-like. We either have that (1) there exist an increasing sequence of integers $n_1 < n_2 < \ldots$ so that $\pi_{n_i}(F)$ is fiber-like, or (2) there exist an increasing sequence of integers $n_1 < n_2 < \ldots$ so that $\pi_{n_i}(F)$ is point-like. Assume case (1). We claim F is fiber-like. Let $\epsilon > 0$ be given. Let $\{c_i\}_{i=1}^L \subset M_x(Q)$ be an ϵ-net of $M_x(Q)$. Choose $i \in \mathbb{N}$ such that $\frac{1}{n_i} < \epsilon$. Find $g \in \mathrm{Homeo}(I^{n_i})$ so that $d(g(\pi_{n_i}(F)), M_{\pi_{n_i}(x)}(I^{n_i})) < \epsilon$. In particular there exists $\{f_k\}_{k=1}^L \subset F$ so that $d(g(\pi_{n_i}(f_k)), \pi_{n_i}(c_k)) < \epsilon$ for $k = 1, \ldots, L$. As $\frac{1}{n_i} < \epsilon$, one concludes $d(\tilde{g}(f_k), c_k) < \epsilon$. This implies $d(\tilde{g}(F), M_x(Q)) < 2\epsilon$. Now assume case (2). We claim F is point-like. Similarly to the proof of the previous case fix $i \in \mathbb{N}$ such that $\frac{1}{n_i} < \epsilon$. Find $g \in \mathrm{Homeo}(I^{n_i})$ and $c = \{c_\alpha\}_{\alpha \in A} \in M_{\pi_{n_i}(x)}(I^{n_i})$ so that $d(g(\pi_{n_i}(F)), \{c\}) < \epsilon$. Let $p \in M_x(Q)$ so that $\{c_\alpha \times I_{n_i+1}^{\infty}\}_{\alpha \in A} \subset p$ (this corresponds to finding $\{e_\beta\}_{\beta \in B} \in M_x(\pi_{n_i}(x) \times I_{n_i+1}^{\infty})$ and defining $P = \{e_\beta\}_{\beta \in B} \cup \{c_\alpha \times I_{n_i+1}^{\infty}\}_{\alpha \in A}$). As $\frac{1}{n_i} < \epsilon$, one concludes $d(\tilde{g}(F), \{p\}) < \epsilon$. Indeed fix $f \in F$. Let $R \in f \in F$. Find $C \in c$ so that $d(\pi_{n_i}(R), C) < \epsilon$, which implies $d(R, C \times I_{n_i+1}^{\infty}) < \epsilon$. Let $P \in p$. If $P = C \times I_{n_i+1}^{\infty}$ for some $C \in c$ then one can find $R \in f$ so that

$d(R, C \times I^{\infty}_{n_i+1}) < \epsilon$. If this is not the case we must have $P \subseteq r(c) \times I^{\infty}_{n_i+1}$. As $d\Big(r(c), r(\pi_{n_i}(f))\Big) = d(r(c), \pi_{n_i}(r(f))) < \epsilon$, we conclude $d(P, r(f)) < \epsilon$. □

In order to show that $(M(Q), Homeo(Q))$ is strongly proximal, we first prove the following lemma:

Lemma A.8. *$(Q, Homeo(Q))$ is strongly proximal.*

Proof. Clearly it is enough to show that $(Homeo(I^n), I^n)$ is strongly proximal for each $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Notice that ∂I^n and $Homeo(I^n)$ have a property which is very similar to (albeit weaker than) the boundary shrinking property. Indeed for any closed $W \subsetneq \partial I^n$ and any $y \in \partial I^n$ and $\delta > 0$ there is $h \in Homeo(I^n)$ so that $h(W) \subset B(y, \delta)$. Let us call this property the **Weak Boundary Shrinking Property**. Let $\mu \in \mathfrak{M}(I^n)$. Let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$. Denote $b = \mu(\partial I^n)$. By choosing N disjoint open subsets of ∂I^n, considering their complements and using the weak boundary shrinking property we find $h_1 \in Homeo(I^n)$ and $y \in \partial I^n$ so that $\mu(h_1(B(y, \epsilon)) \cap \partial I^n) > (1 - \epsilon)b$. Let $P_k = [-1 + \frac{1}{k}, 1 - \frac{1}{k}]^n$, $k \in \mathbb{N}$. Notice $I^n = \bigcup_{k=1}^{\infty} P_k$. As $h_1(Int(I^n)) = Int(I^n)$, there is $q \in \mathbb{N}$ so that $\mu(h_1(P_q)) > (1-\epsilon)(1-b)$. Again relying on the structure of $Homeo(I^n)$ we can find $h_2 \in Homeo(I^n)$ so that $[h_2]_{|\partial I^n} = Id$ and $h_2(h_1(P_q)) \subset B(y, \epsilon)$. Let $h = h_2 \circ h_1$. Conclude $\mu(h(B(y, \epsilon))) > (1-\epsilon)b + (1-\epsilon)(1-b) = (1-\epsilon)$. □

We cannot use Theorem 9.2 directly as we have not shown $(Q, Homeo(Q))$ has the (BSP) property. Instead we prove directly:

Theorem A.9. *$(M(Q), Homeo(Q))$ is strongly proximal.*

Proof. Let $\mu \in \mathfrak{M}(M(Q))$. Let $r_*(\mu)$ be the projection of μ under the map $r : M \to Q$. Using the strong proximality of $(Q, Homeo(Q))$ one can assume without loss of generality that $r_*(\mu)(\{x\}) = 1$ for $x = (0, 0, \ldots)$. Let $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$. Using the ideas appearing in the proof of Lemma A.6 and Theorem 9.2, one can find $g_N \in Homeo(I^N)$ and $c_N \in M(I^N)$ so that $[g_N \circ \pi_N]_*(\mu))(B(c_N, 4\epsilon)) > 1 - \epsilon$. Using the same ideas appearing in Lemma A.7 one finds $c \in M(Q)$ so that $\tilde{g}_*(\mu)(B(c, 5\epsilon)) > 1 - \epsilon$. □

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