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ON METRIC TYPES THAT ARE DEFINABLE IN AN O-MINIMAL STRUCTURE

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Abstract. In this paper we study the metric spaces that are definable in a polynomially bounded o-minimal structure. We prove that the family of metric spaces definable in a given polynomially bounded o-minimal structure is characterized by the valuation field Λ of the structure. In the last section we prove that the cardinality of this family is that of Λ. In particular these two results answer a conjecture given in [SS] about the countability of the metric types of analytic germs. The proof is a mixture of geometry and model theory.

§0. Introduction. Given a subset of $\mathbb{R}^n$, definable in an o-minimal structure, we may consider it as a metric space if we endow it with the induced metric of $\mathbb{R}^n$. The classification of such subspaces up to bi-Lipschitz homeomorphisms goes back to T. Mostowski's work on complex analytic sets [M]. He was motivated to find a notion of stability more precise than topological stability.

Few years before L. Siebenmann and D. Sullivan had also studied subsets of $\mathbb{R}^n$ that are bi-Lipschitz homeomorphic. In [SS] was asked whether the number of metric types of real analytic germs is countable. By the metric type of a set is the class of all the subsets of $\mathbb{R}^n$ which are bi-Lipschitz homeomorphic to this set.

In [P1, P2] A. Parusiński has generalized T. Mostowski's results to the real case. Recently L. Van den Dries and P. Speissegger have generalized the so called "preparation Theorem" to polynomially bounded o-minimal structures [vD-S]. In [V2], we prove a bi-Lipschitz isotopy theorem in the context of polynomially bounded o-minimal structure using Speissegger and Van den Dries' result. This was new even for semi-algebraic or subanalytic sets since it yields the existence of definable isotopies. Moreover the author introduced a new object, called Lipschitz triangulations, useful to investigate metric properties of metric subspaces of $\mathbb{R}^n$.

In this paper we deal with o-minimal structures and investigate some questions closely related to the L. Siebenmann and D. Sullivan's conjecture which we answer positively.

In section 1 we prove that two polynomially bounded o-minimal structures expanding the same real closed field R define the same metric types if and only if they have the same valuation field. In other words, every set definable in the first one

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is bi-Lipschitz homeomorphic to a set definable in the second one. For instance every global subanalytic set is bi-Lipschitz homeomorphic to a semi-algebraic set. The proof mixes some arguments of model theory with some results proved in [V2] about polynomially bounded o-minimal structures.

In the last section we study the cardinality of the family of definable metric types in a polynomially bounded o-minimal structure. prove that the family of metric types of definable sets in a polynomially bounded o-minimal structure has the cardinality of the set its valuation field. We use the notion of "Lipschitz triangulations" introduced in [V2]. These triangulations, involve a finite number of combinatoric data which totally capture the metric type of the singularity.

Both of these results answer the D. Sullivan and L. Siebenmann's conjecture (see Corollary 2.2.2 and Remark 3).

Notations and conventions. In all this paper R is a real closed field. Given two definable functions, $f, g\colon A \to R$ we will write $f \sim_L g$ (and say that f is *equivalent* to g) if there exist two positive constants C_1 and C_2 in L (where $L \subseteq R$ is a subfield) such that $C_1 f \leq g \leq C_2 f$. We also will write $\sim$ between element of ordered field considering them as constant functions. We will say that a function is a L *Lipschitz function* if it is a Lipschitz function and if the Lipschitz constant can be chosen in L.

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§1. On definable metric types. Let R be a real closed field. In this section $\mathscr{M} = (\mathscr{M}_n)_{n \in N}$ will denote a polynomially bounded o-minimal structure expanding R (see [vD] or [vD-M]). We will denote by Λ the field of exponents of $\mathscr{M}$, that is the set of real numbers $\lambda \in R$ such that $x \to x^\lambda\colon]0; +\infty[\mapsto R$ is definable. We shall need some basic facts on the Stone space of complete types (see [Ma] for basic definitions).

Given an $(n+m)$ type p, it is possible to decompose it as a n type q over R^n (just consider the formulae involving the n first variables) and a type r over K^m where K is an elementary extension realizing the type q (substituting the point $x(q)$ realizing the type q in the formulae of p). Now $x(r)$ realizes r in an elementary extension if and only if $(x(q); x(r))$ realizes p. We then write $p \simeq q \times r$.

The set of complete types, with its topology, may be identified with the Stone space of the set of ultrafilters of the boolean algebra of definable sets. This is convenient to do geometry. We recall some basic facts about ultrafilters of [C] or [BCR1, BCR2].

An ultrafilter of $\mathscr{M}_n$ is a collection of definable subsets satisfying:

(1) $\emptyset \notin p$,
(2) A and B belong to p iff $A \cap B \in p$,
(3) $A \in p$ iff $R^n \setminus A \notin p$.

We denote by $\widetilde{R^n}$ the set of definable ultrafiltres together with the topology that makes $\widetilde{U} = \{p / U \in p\}$, for $U \in \mathscr{M}_n$ a basis of closed-open sets. This topology is of course quasi-compact.

Usually we call the *dimension of the ultrafilter* p the least integer a such that we can find an a-dimensional definable set V in p.

Warning. As we said in the introduction, given a subset of R^n, by *the metric type of a set* we will mean all the subsets of R^n which are bi-Lipschitz homeomorphic to this set (i. e. which are isomorphic as metric subspaces of R^n). This should not be confused with the notion of type above mentioned.

1.1. Some results about o-minimal structures. We shall need a result stating existence of "a good projection" for a given ultrafilter of definable sets.

We recall some results which have been proved in [V2] and give some consequences which will be useful for our present purpose. The first proposition we are going to state was the keystone in the proof of existence of Lipschitz triangulations but which will be used in the next section in another way. We write S^n for the unit sphere in R^{n+1}.

Definition 1.1.1. Let A be a definable set of R^{n+1}. An element λ of S^n is said to be *regular for A* if there exists $\alpha \in \mathbb{Q}^+$ such that:

$$d(\lambda; T) \geq \alpha$$

for any vector subspace T tangent to A at a smooth point of A.

Regular lines do not always exist for a given set. Nevertheless we can prove:

Proposition 1.1.2. [V2] *Let A be a definable subset of R^n of empty interior. Then there exists a definable bi-Lipschitz homeomorphism $h\colon R^n \to R^n$ such that $h(A)$ has a regular line $\lambda \in S^n$.*

This result is true even in the case where the o-minimal structure is not polynomially bounded. We shall need another result of [V2] about L regular decompositions.

Proposition 1.1.3. *There exists $\{\lambda_1, \dots, \lambda_N\} \subseteq S^n$ such that for any $A_1, \dots A_m$ in $\mathscr{M}_{n+1}$ there exists a cellular decomposition $(C_i)_{i \in I}$ of R^{n+1} adapted to all the sets A_k, $1 \leq k \leq m$ such that for each open cell C_i, we may find $\lambda_{j(i)}$, $1 \leq j(i) \leq N$, regular for the boundary of C_i.*

A consequence of this proposition is the following lemma useful for us:

Lemma 1.1.4. *Let $q \in \widetilde{R^n}$. There exists $\lambda_q \in S^n$ such that for any U in q there exists a cell C with $q \in \widetilde{C}$ and λ_q regular for the boundary of C.*

Proof. Suppose that for the vectors $\lambda_1, \dots, \lambda_N$ given by Proposition 1.1.3 the conclusion of the lemma fails. Thus we can find $U_1, \dots, U_n$ in q containing no regular cell for $\lambda_1, \dots, \lambda_N$ respectively. Applying Proposition 1.1.3 to $\cap_{i=1}^N U_i$ we can find a cellular decomposition (C_i). As $q \in \cap_{i=1}^N \widetilde{U_i}$, at least one set C_i has to be in q. But as at least one vector $\lambda_{j(i)}$ is regular with respect to the set C_i, we get a contradiction. ⊣

Remark 1. A more sophisticated argument due to W. Pawłucki proves that the set of lines $\{\lambda_1, \dots, \lambda_N\}$ of Proposition 1.1.3 can be chosen as the canonical basis of R^n. As a consequence we see that the line λ_q in the above proposition can also be chosen among the vectors of the canonical basis.

1.2. Definable metric types. It is difficult to work simultaneously with sets that are definable in two different o-minimal structures. Definability in the first structure might be lost through a homeomorphism that is just definable in the second. So we will first change all the data for definable ones in the first structure and then try to recover the property of lipschitzianity. The following lemma says that it is possible over a generic point p.

As we have just explained, in this section we work with two o-minimal structures $\mathscr{M}^i = (\mathscr{M}^i_n)_{n\in\mathbb{N}}$ expanding R, $i = 1, 2$. We fix two languages $\mathscr{L}_i$, $i = 1, 2$ adding symbols for the elements of $\mathscr{M}^i$ with the interpretation making the elements of $\mathscr{M}^i_n$ the $\mathscr{L}_i$ definable sets of R^n. We write $\mathscr{L}$ for the (disjoint) union of these two languages and keep their respective interpretation in the structure. We write $\widetilde{R^n}$ for the set of complete types of the $\mathscr{L}_1$ structure $\mathscr{M}^1$.

Lemma 1.2.1. *Let $\xi\colon R^n \to R$ be a $\mathscr{L}_1$ definable function and $p \in \widetilde{R^n}$. Assume that ξ is $\sim_L$ to a $\mathscr{L}$ definable L Lipschitz function (with L subfield of R). Then there exists $V \in p$ such that $\xi_{|V}$ is $\sim_L$ to a $\mathscr{L}_1$ definable L Lipschitz function.*

Proof. We prove the result by induction on n. The case $n = 0$ is clear. We may assume that p is of dimension n since otherwise the result follows from the induction hypothesis. Up to a linear change of coordinates we may assume that the line λ_p given by Lemma 1.1.4 is the first vector of the canonical basis. Let ζ be the $\mathscr{L}$ definable L Lipschitz function equivalent to ξ. Decompose $p \simeq q \times r$ with $q \in \widetilde{R}$ and $r \in \widetilde{K^{(n-1)}}$, where K is an $\mathscr{L}$ elementary extension of the $\mathscr{L}$ structure R, realizing the type q (regarding q as a non necessarily complete $\mathscr{L}$ type over R). Then ξ and ζ extend respectively to $\mathscr{L}_1$ and $\mathscr{L}$ definable functions ξ_K and ζ_K defined on K^n. Now let $\zeta_q\colon K^{(n-1)} \to K$ given by $x \mapsto \zeta_K(x(q); x)$ (resp. $\xi_q(x) = \xi_K(x(q); x)$) where $x(q)$ is the element in K realizing the type q.

Apply the induction hypothesis to ξ_q and ζ_q with the type r. Note that as all the Lipschitz constants are in R the Lipschitzness may be expressed by a $\mathscr{L}$ formula with parameters in L. Thus we get a $\mathscr{L}_1$ function defined over an element of p and which has bounded derivatives with respect to the $(n-1)$ last variables. We still denote it by ξ. We now have to replace this function by an equivalent function with a derivative with respect to the first variable bounded as well. Now as above we decompose $p \simeq q' \times r'$ with $q' \in \widetilde{R^{n-1}}$ and $r' \in \widetilde{K'}$ (where K' is an $\mathscr{L}$ elementary extension of R realizing q), but here $x_1(r')$ corresponds to the first variable in the type p. As above we may extend ξ and ζ to ξ' and ζ' defined on K' and set $\zeta_{q'}(x) = \zeta'(x(q'), x)$ and $\xi_{q'}(x) = \xi'(x(q'), x)$ for $x \in K$. If $|\frac{d\xi_{q'}}{dx}(x(r'))| \le C$ for some $C \in L$ the result is clear. Otherwise we may suppose that $|\frac{d\xi_{q'}}{dx}(x(r'))| \ge C$ on $]a; b[$ for some $C \in L$.

Note that if the ultrafilter r' contains no finite interval then as $|\frac{d\xi_{q'}}{dx}(x(r'))| \ge C$ for any C the function ξ cannot be equivalent at infinity to a Lipschitz function.

So we may suppose $-\infty < a < b < +\infty$. By definability the function $\xi_{q'}$, we may assume without loss of generality that $\xi_{q'}$ is positive and increasing over a sufficiently small interval $]a; b[$ into K' belonging to r' with $a \in k(q')$ and $b \in k(q')$. Then we claim that

$$\xi_{q'}(b) \sim_L \xi_{q'}(a). \tag{1.1}$$

If not then

$$\xi_{q'}(b) \sim_L (\xi_{q'}(b) - \xi_{q'}(a)) \ge C(b-a)$$

(since $\frac{d\xi_{q'}}{dx_1} \geq C$) for any $C \in L$. Note that since $\xi_{q'} \sim_L \zeta_{q'}$, if (1.1) fails we have $\zeta_{q'}(b) \geq 2\zeta_{q'}(a)$. But we must have for C large enough $|\zeta_{q'}(b) - \zeta_{q'}(a)| \leq C(b-a)$, so $\zeta_{q'}(b) \leq C(b-a)$ for some $C \in R$. This contradicts $\zeta_{q'} \sim_L \xi_{q'}$. Thus $\xi_{q'}(b) \sim_L \xi_{q'}(a)$.

But this means that $\xi_{q'}(x_1(r')) \sim_L \xi_{q'}(a)$ since $\xi_{q'}$ is increasing on $[a;b]$. By Lemma 1.1.4 and choice of coordinate λ_p we know that there exists a L Lipschitz function, say $\theta\colon V \to R$ defined over an element $V \in q'$ and satisfying $\theta(x(q')) = a$. This implies that $\xi(\theta(x'); x') \sim_L \xi(x)$ where $x = (x_1; x') \in R^{n-1} \times R$ and as $\xi(\theta(x'); x')$ is equivalent to a $\mathscr{L}$ definable L Lipschitz function the result follows from the induction hypothesis. ⊣

We also will need a result about polynomially bounded o-minimal structure which is closely related to the preparation theorem [vD-S] (see also [N], [P2]). The reader can again find the proof in [V2].

Proposition 1.2.2. *Let $\xi\colon R^n \to R$ be a positive definable function. Then there exists a partition of R^n such that over each element of this partition the function ξ is $\sim_{\mathbb{Q}}$ to a product of powers of distances to definable subsets of R^n.*

Now we are ready to prove:

Theorem 1.2.3. *Let $\mathscr{M}^1$ and $\mathscr{M}^2$ be two polynomially bounded o-minimal structures over R having the same valuation field. Then any element of $\mathscr{M}^1_n$ is bi-Lipschitz homeomorphic to an element of $\mathscr{M}^2_n$.*

In other words the two languages $\mathscr{L}_1$ and $\mathscr{L}_2$ define the same metric types.

Proof. We prove by induction the following facts:

(H_n) Let $\xi_1, \dots, \xi_\mu\colon R^n \to R$ be positive $\mathscr{L}_2$ definable and Lipschitz functions and let $A_1, \dots, A_m$ be $\mathscr{L}_2$ definable subsets of R^n. Then there exists a positive $\mathscr{L}_2$ definable bi-Lipschitz homeomorphism $h\colon R^n \to R^n$ such that each $h(A_i)$ is a $\mathscr{L}_1$ definable subset and such that each $\xi_i \circ h^{-1}$ is equivalent to a $\mathscr{L}_1$ definable Lipschitz function.

By Proposition 1.1.2 we may assume that the boundaries of the subsets $A_1, \dots, A_m$ are included into a finite number of graphs of $\mathscr{L}_2$ definable Lipschitz functions $\zeta_1, \dots, \zeta_\nu\colon R^n \to R$ satisfying $\zeta_1 \leq \dots \leq \zeta_\nu$. By Proposition 1.2.2 there exists a $\mathscr{L}_2$ definable partition (V_i) such that over each V_i each function ξ_i is equivalent to a product of powers of distance functions to some subsets (W_{ij}). Consider a cellular definable decomposition of the elements of the families (A_i) and (W_{ij}). This provides a partition of R^n. Apply the induction hypothesis to the family constituted by this partition. Denote by h the obtained homeomorphism. Moreover again by (H_n) we may assume that $\zeta_1 \circ h^{-1}$ and $(\zeta_i - \zeta_{i-1}) \circ h^{-1}$, for $i \geq 2$ are equivalent to some $\mathscr{L}_1$ definable Lipschitz functions ζ'_1 and ζ'_i, $i \geq 2$ respectively. We are going to "lift" the homeomorphism h. It suffices to set

$$\widehat{h}(x; \zeta_i(x)) = (h(x); \zeta'_1(h(x)) + \sum_{j=2}^{i} \zeta'_j \circ h(x)).$$

Then we extend the homeomorphism $\widehat{h}$ over the segments $[\zeta_i(x); \zeta_{i+1}(x)]$, $[\zeta_\nu(x); +\infty[$ and $]-\infty; \zeta_1(x)]$ linearly. This defines a $\mathscr{L}_1$ definable function since all the data are $\mathscr{L}_1$ definable. As the functions ζ_i and ζ'_i all are Lipschitz functions the mapping $\widehat{h}$ is clearly bi-Lipschitz over the graph of each function ζ_i. Moreover as the functions $(\zeta_i - \zeta_{i-1})$ are equivalent to $\zeta_i' \circ h$, $\widehat{h}$ is bi-Lipschitz on the whole of R^n. Let us check that the image of the A_i are $\mathscr{L}_1$ definable. By construction the set A_i is an union of subsets of the graphs ζ_i and some bands of type

$$\{(x; x_n) \in R^{n-1} \times R/\zeta_j(x) < x_n < \zeta_{j+1}(x)\}.$$

These bands are sent onto the corresponding ones defined from the functions $\zeta_1' + \sum_{j=2}^{i} \zeta_j'$ which are $\mathscr{L}_1$ definable subsets. As $\widehat{h}$ lifts h and as the induction hypothesis has been applied to a cellular decomposition compatible with the A_i's each subset $A_i \cap \Gamma_{\zeta_i'}$ (where $\Gamma_{\zeta_i'}$ is the graph of ζ_i') is mapped onto a $\mathscr{L}_1$ definable subset.

Now we have to check that the functions $\xi_s \circ \widehat{h}^{-1}$ are equivalent to $\mathscr{L}_1$ definable Lipschitz functions. Fix $s \leq \mu$. Over each $\widehat{h}(V_i)$ each function $\xi_s \circ \widehat{h}^{-1}$ is equivalent to a function of type

$$d(x; h(W_{s1}))^{\alpha_{s1}} \cdots (x; h(W_{sr}))^{\alpha_{sr}}$$

where the α_{sj} are rational numbers. Since the images of the W_{ij} are $\mathscr{L}_1$ definable each function $\xi_s \circ \widehat{h}^{-1}$ is equivalent to a $\mathscr{L}_1$ definable function over each $\widehat{h}(V_i)$.

By Lemma 1.2.1 for every $p \in \widetilde{R^n}$ we may find a $\mathscr{L}_1$ definable and Lipschitz function $\theta^{s,p}\colon U^p \to R$ which is equivalent to $\xi_s \circ \widehat{h}$ and defined over an element U^p of p. By compactness of $\widetilde{R^n}$ we may extract a finite covering (U_l) of R^n together with corresponding functions $\theta_l^s\colon U_l \to R$ having the same properties as the functions $\theta^{s,p}$.

To complete the induction step we are going to paste all these functions $\theta_l^s\colon U_l \to R$ into a global one $\mathscr{L}_1$ definable, Lipschitz and still equivalent to $\xi_s \circ \widehat{h}^{-1}$. By Proposition 1.1.2, up to a $\mathscr{L}_1$ definable bi-Lipschitz homeomorphism we may assume that the sets $Int(U_l)$ are of the form

$$\{(x'; x_n) \in R^{n-1} \times R/\eta_l(x') < x_n < \eta_{l+1}(x')\}$$

where $l \in \{0, \dots, p+1\}$ and $\eta_l\colon R^{n+1} \to R$ are $\mathscr{L}_1$ definable Lipschitz functions satisfying $-\infty = \eta_0 \leq \cdots \leq \eta_{p+1} = +\infty$.

We paste the functions $\theta_1^s, \dots, \theta_k^s$, for $k \leq p$ into a positive function $\theta_k'^s$, Lipschitz and still equivalent to $\xi_s \circ h^{-1}$, defined on $\cup_{l=1}^k U_l$ by induction on k. Assume this is done until k. Choose a constant $C \in R$ large enough such that $C\theta_{k+1}^s \geq 2\theta_k'^s$ on the graph of η_k (recall that both functions are equivalent to $\xi_k \circ h^{-1}$). Let

$$g(x) = C\theta_{k+1}^s(x'; \eta_k(x')) - \theta_k'^s(x'; \eta_k(x'))$$

and

$$\theta_{k+1}'^s(x) = C\theta_{k+1}^s(x) - \min(C'\theta_{k+1}^s(x); g(x))$$

where $C' < C$ is a constant sufficiently big to have $g(x) \leq C'\theta_{k+1}^s(x)$ over the graph of η_k. It is easy to check that $\theta_{k+1}'^s$ is Lipschitz $\mathscr{L}_1$ definable and coincides with $\theta_k'^s$ over the graph of η_k. Hence it extends $\theta_k'^s$ to a Lipschitz function over U_{k+1}. ⊣

Remark 2. If there is an inclusion between the two structures. That is for instance $\mathscr{M}_n^1 \subseteq \mathscr{M}_n^2$, for all $n \in \mathbb{N}$, then the proof establishes that the homeomorphism may be chosen definable into the largest structure. In particular we have proved that every global analytic set is bi-Lipschitz homeomorphic to a semi-algebraic set by a global subanalytic homeomorphism.

§2. Polynomially bounded o-minimal structures. We end this paper by giving a theorem estimating the cardinality of the family of metric types definable in a polynomially bounded o-minimal structure. We prove that the set of definable metric types in a given polynomially bounded o-minimal structure has the cardinality of its valuation field Λ. In particular the number of metric types of analytic sets is countable. This answers positively a conjecture given by L. Siebenmann and D. Sullivan in [SS].

2.1. Lipschitz triangulations. In this section we recall the notion of Lipschitz triangulations introduced by the author in [V2]. It will be the main tool involved in the proof of Theorem 2.2.1. Existence of such an object for a definable subset in an arbitrary polynomially bounded o-minimal structure is also proved in [V2]. The definition involves two objects that we need to define preliminary. The first one is what is called *standard simplicial functions*. That is a finite sum of functions of type:

$$d(q;\sigma_1)^{\alpha_1}.\ \dots\ .d(q;\sigma_k)^{\alpha_k} \tag{2.2}$$

where $\sigma_1,\dots,\sigma_k$ are simplices of R^n and $\alpha_1,\dots,\alpha_k$ are real numbers. Indeed Definition 2.1.2 will involve standard simplicial functions over $\sigma\times\sigma$ that is such sums of distances involving q or another point q'.

All the standard simplicial functions that we will use will be definable. In other words all the exponents involved will be elements of Λ. Indeed they come from the Preparation Theorem of L. Van Den Dries and P. Speissegger [vD-S].

The second thing we shall need is what we call the tame systems of coordinates, to express the directions along which operate contractions:

DEFINITION 2.1.1. A *tame system of coordinates on* R^n is a family of functions $(\psi_1;\dots;\psi_n)$ of the following form:

$$\psi_i(q)=\frac{q_i-\theta_i(\pi_{i-1}(q))}{|\theta_i(\pi_{i-1}(q))-\theta_i'(\pi_{i-1}(q))|} \tag{2.3}$$

(and 0 whenever $\theta_i\circ\pi_{i-1}(q)=\theta_i'\circ\pi_{i-1}(q)$) where θ_i and θ_i' are piecewise linear functions on R^{i-1} satisfying $\theta_i<\theta_i'$.

Now we can state the definition:

DEFINITION 2.1.2. A *Lipschitz triangulation of* R^n is the data of a finite simplicial complex K together with a homeomorphism $h\colon |L|\to R^n$, where L is a union of open simplices of K, such that for every $\sigma\in L$ there exist $\varphi_{\sigma,1},\dots,\varphi_{\sigma,k}$, standard simplicial functions over $\sigma\times\sigma$ satisfying for any q and q' in σ:

$$|h(q)-h(q')|\sim\sum_{i=1}^{n}\varphi_{\sigma,i}(q;q')\cdot|q_{i,\sigma}-q'_{i,\sigma}| \tag{2.4}$$

where $(q_{1,\sigma},\dots,q_{n,\sigma})$ is a piecewise linear system of coordinates of R^n. Let $A_1,\dots,A_k$ be subsets of R^n. A Lipschitz triangulation of $A_1,\dots,A_k$ is a Lipschitz triangulation of R^n such that each $h^{-1}(A_i)$ is a union of open simplices.

Definable sets in polynomially bounded o-minimal structures turn out to present the significant advantage to be triangulable in this meaning:

THEOREM 2.1.3. [V2] *Let A be definable subset of R^n. Then there exists a definable Lipschitz triangulation of A.*

Moreover, from the construction we can see that the vertices of the simplicial complex K can be chosen from $\mathbb{Q}^n$.

2.2. How many metric types are definable. Lipschitz triangulations allows us to count the number of definable metric types. We have just said that the vertices of the simplicial complexes can be chosen in $\mathbb{Q}^n$. But such finite simplicial complexes are countable. Therefore it is enough to count the simplicial complexes giving different metric types.

In this section $\mathcal{M}$ denotes a polynomially bounded o-minimal structure expanding the real field (see [vD] or [vD-M]). We also denote by Λ the field of exponent of $\mathcal{M}$, that is the set of real numbers $\lambda \in R$ such that $x \to x^\lambda :]0; +\infty[\mapsto R$ is definable.

THEOREM 2.2.1. *Given a polynomially bounded o-minimal structure the number of definable metric types has the cardinality of the set of definable exponents.*

PROOF. First we remark that the family of sets $\{(x; y) \in R^2/y^2 = |x|^\lambda\}$, for $\lambda \in \Lambda$, is of cardinality $|\Lambda|$. As two elements are not bi-Lipschitz homeomorphic it suffices to prove that the cardinality of definable metric type is less than $|\Lambda|$.

According to Theorem 2.1.3 it suffices to bound the number of Lipschitz triangulations with vertices in $\mathbb{Q}^n$. The number of finite such complexes is countable. All the semi-algebraic functions are definable. This implies that Λ contains $\mathbb{Q}$ and so Λ is at least countable. Given a simplicial complex the tame systems of coordinates are determined by some subcomplexes. Each contraction $\varphi_{i,\sigma}$ is characterized by the family of exponents it involves and the family of faces involved in its expansion. The number of possible exponents is $|\Lambda|$ and the number of finite families of subcomplexes is countable. Therefore given a simplicial complex these data browse a range of cardinality not greater than $|\Lambda|$. $\dashv$

This result has the following immediate corollary answering positively a conjecture of L. Siebenmann and D. Sullivan in [SS].

COROLLARY 2.2.2. *The number of metric types of analytic germs is countable.*

REMARK 3. (1) Note that as the triangulations can be chosen among definable mappings we have also proved that analytic germs are countable up to subanalytic bi-Lipschitz homeomorphisms.

(2) Theorem 2.1.3 states triangulability not only for the germs but for any definable sets. One may deduce that the metric types of global subanalytic sets are countable. But all the analytic sets are not in an o-minimal structure.

(3) Given a semi-algebraic family the number of metric types of the elements of the family is finite. Therefore it is easy to prove that the number of metric types of semi-algebraic sets is countable since we may bound the complexity of the formula defining the family. Hence Proposition 1.2.3 gives a second proof of the countability of metric types of global subanalytic sets. But for an arbitrary o-minimal structure this proposition is not sufficient to get a bound.

REFERENCES

[BCR1] J. Bochnak, M. Coste, and M-F. Roy, ***Géométrie Algébrique Réelle***, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 12, Springer-Verlag, 1987.

[BCR2] ———, ***Real algebraic geometry***, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 36, Springer-Verlag, 1998.

[C] M. Coste, ***An introduction to o-minimal geometry***, Dottorato di Ricerca in Matematica, Dip. Mat. Univ. Pisa, Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000.

[vD] L. van den Dries, ***Tame topology and o-minimal structures***, London Mathematical Society Lecture Note Series, 248, Cambridge University Press, Cambridge, 1998.

[vD-M] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, ***Duke Mathematical Journal***, vol. 84 (1996), no. 2, pp. 497–540.

[vD-S] L. van den Dries and P. Speissegger, *O-minimal preparation theorems*, to appear.

[Ma] D. Marker, ***Model theory. An introduction***, Graduate Texts in Mathematics, 217, Springer-Verlag, New York, 2002.

[M] T. Mostowski, ***Lipschitz equisingularity***, Dissertationes Mathematicae (Rozprawy Matematyczne), vol. 243, 1985, 46 pp.

[N] K. Nowak, *A proof of the valuation property for polynomialy bounded o-minimal strucutres*, preprint.

[P1] A. Parusiński, *Lipschitz properties of semi-analytic sets*, ***Université de Grenoble, Annales de l'Institut Fourier***, vol. 38 (1988), no. 4, pp. 189–213.

[P2] ———, *Lipschitz stratification of subanalytic sets*, ***Annales Scientifiques de l'Ecole Normale Supérieure. Quatrième Série***, vol. 27 (1994), no. 6, pp. 661–696.

[SS] L. Siebenmann and D. Sullivan, *On complexes that are Lipschitz manifolds*, ***Proceedings of the Georgia Topology Conference, Athens, Ga., 1977***, Geometric topology, Academic Press, New York-London, 1979, pp. 503–525.

[V1] G. Valette, *A bi-Lipschitz version of Hardt's theorem*, ***Comptes Rendu de l'Académie des Sciences, Paris***, vol. 340 (2005), no. 12, pp. 895–900.

[V2] ———, *Lipschitz triangulations*, ***Illinois Journal of Mathematics***, to appear.

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