Algebraic geometry of commuting matrices

arXiv:2106.13137

Joachim Jelisiejew, partly joint with K. Šivic, J.M. Landsberg & A. Pal, . . .

IMPANGA, 19 Nov 2021

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For simplicity, everything over $\mathbb{C}$. $\mathbb{M}_d$ denotes $d \times d$ matrices.

Definition

The variety* $C_n(\mathbb{M}_d)$ of n-tuples of commuting $d \times d$ is defined as

$$C_n(\mathbb{M}_d) = \{(x_1, \ldots, x_n) \in (\mathbb{M}_d)^n \mid \forall i, j : x_i x_j = x_j x_i\}.$$

$(\mathbb{P}\mathbb{M}_d)^n$ quadratic equations.

$Gr(n, \mathbb{M}_d)$

principal component

Motzkin, Taussky'55: $n = 2$ $C_2(\mathbb{M}_d) \ni (x_1, x_2)$ irreducible (singular)

...& Stark 2021 $d = 2$ $C_n(\mathbb{M}_2)$ irreducible toric. trace matrices.

The good open (not necessarily dense!) locus

$$\{(x_1, \ldots, x_n) \in \mathbb{M}_d^n \mid \text{ simultaneously diagonalizable}\}$$

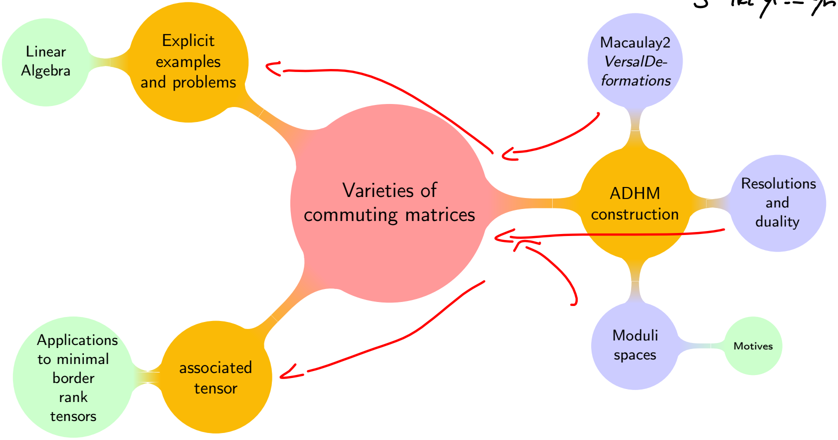
„nice"

It closure is an irreducible component, *the principal component*.

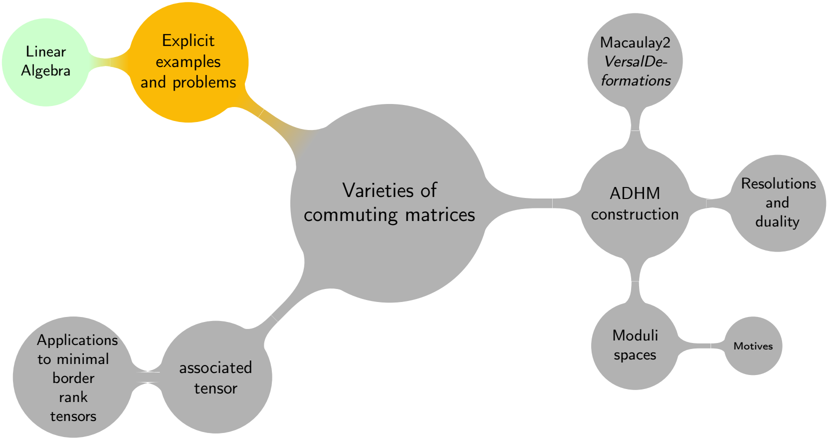
Subjective big picture

ADHM $(x_1, \ldots, x_n) \in M_d \rightsquigarrow M \in S\text{-mod}$

$S = \Bbbk[y_1 \ldots y_n]$



Linear algebra



Key open questions, small number of matrices

Classify points of $C_n(\mathbb{M}_d)$ up to $\mathrm{GL}_n \times \mathrm{GL}_d$-action for small n, d.

$GL_d \curvearrowright M_d$ by conj. Answer known: $d \leq 4$

$GL_n \curvearrowright (M)^n$ by linear comb.

Find equations for the principal component inside $C_n(\mathbb{M}_d)$.

$\overline{\{(x_1 \dots x_n) \mid x_i \text{ diagonalizable}\}}$

Is the scheme $C_2(\mathbb{M}_d)$ reduced? Is it Cohen-Macaulay?
(Charbonnel arXiv:2006.12942)

claims yes. ~97 pages

Key open questions II, small number of matrices

What is the smallest d such that $C_3(\mathbb{M}_d)$ is reducible?

Known: $12 \leq d \leq 29$, lower bound Šivic, upper bound Holbrook, Omladič+εŠivic.

Obtained: $\dim C_3(M_d) > \dim$ principal

$C_2(M_d)$ irreducible $\forall d \geqslant 29$.

For any d describe (a general point of) any component of $C_3(\mathbb{M}_d)$ other than the principal one. Describe any explicit triple (x_1, x_2, x_3) outside the principal component.

For $(x_1, x_2, x_3) \in C_3(\mathbb{M}_d)$ is it true that $\dim_{\mathbb{C}}(\mathbb{C}[x_1, x_2, x_3] \subset \mathbb{M}_d) \leq d$? (Gerstenhaben's question)

True for the principal component.

Larger number of matrices: nothing (was) known

Classical swindle (e.g. Guralnick):

$$x_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, x_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The algebra $\mathbb{C}[x_1, x_2, x_3, x_4]$ is

$\forall_{i,j} \; \underbrace{x_i x_j}_{} = 0 = \underbrace{x_j x_i}_{}$

$$\lambda \cdot Id_4 + \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \lambda \in \mathbb{C}.$$

$(x_1, \ldots, x_n) \not\ni$ principal

$\longleftarrow$ 5-dimensional

Violates Gerstenhaber's bound. $C_n(\mathbb{M}_d)$ is reducible for $n, d \geq 4$.

What are the components of $C_n(\mathbb{M}_d)$ in general?

Theorem (J-Šivic)

The number of irreducible components of $C_n(\mathbb{M}_d)$ for $d \leq 7$ is as shown in Table 1; we also have explicit descriptions of general points of each component (and general points are smooth).

	$d \leq 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$	$d \gg 0$
$n \leq 2$	1	1	1	1	1	1	1
$n = 3$	1	1	1 princ. example	1	1	1	$\gg 0$
$n = 4$	1	1	2	2	2	2	$\gg 0$
$n = 5$	1	1	2	4	4	8	$\gg 0$
$n = 6$	1	1	2	4	7	11	$\gg 0$
$n \geq 7$	1	1	2	4	7	13	$\gg 0$

Table: *Number of components of $C_n(\mathbb{M}_d)$*

Results in matrix flavour I

Typically (not always), the elementary components have the form, up to GL_d action and adding scalar matrices,

$$x_1, \ldots, x_n \in \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$$

where $*$ is $m \times (d-m)$ matrix for some fixed m and for n large enough.

Results in matrix flavour II

Theorem (J-Šivic)

The variety $C_n(\mathbb{M}_d)$ has generically nonreduced components for all $n \geq 4$ and $d \geq 8$. For example, the locus of quadruples of the form (up to GL_8 action and adding scalar matrices):

$$\begin{bmatrix} 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a generically nonreduced component.

Locus $\mathcal{L}$ of quadruples

$C_4(M_8)$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{(-)^T}.$$

$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} = x$

$y: \ xy = yx \Rightarrow y \in \mathbb{C}[x]$

$\mathrm{rank} \sim \frac{d}{2}$

$\mathcal{L} \subseteq M_{4\times 4}^{4} = \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, $\ S_3 \circlearrowleft$

$\cup$

D divisor

$\widehat{Seg} \subseteq \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$

$\sigma_6 \widehat{Seg} \not\subseteq D$

Computer: $\mathcal{L} \cap$ (principal component) $\subset \mathcal{L}$ is a divisor. Get a divisor in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ invariant under S_3 and $GL_4 \times GL_4 \times GL_4$.

Which divisor is it?

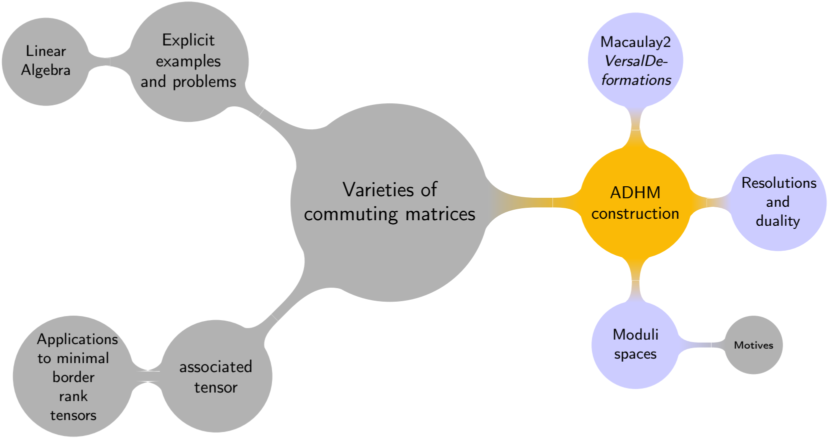
$(x_2, x_3) \in \{y : yx_1 = x_1 y\} = C_2(\mathrm{cent}\, x_1)$

$C_3(M_d) \ni (x_1, x_2, x_3)$

$(x_1 \ldots x_n)$ $\quad \mathbb{C}[x_1 \ldots x_n] = \mathbb{C}[y_1 \ldots y_n]$ in M_d

$(y_1 \ldots y_n)$

ADHM construction



ADHM construction

$x_i \circlearrowright \mathbb{C}^d$

Definition

Let $S = \mathbb{C}[y_1, \ldots, y_n]$. For $(x_1, \ldots, x_n) \in C_n(\mathbb{M}_d)$ we define an S-module structure on $\mathbb{C}^d$ by

$$y_i \cdot v := x_i(v) \qquad \text{for all } v \in \mathbb{C}^d.$$

We will denote the resulting *module associated to* $(x_1, \ldots, x_n)$ by M.

The module has tons of invariants: number of generators, Hilbert function, resolution etc. which we employ to get a better grasp on the matrices themselves.

ADHM construction – example

$$x_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad x_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In the associated module $M = \langle e_1, e_2, e_3, e_4 \rangle$ we have $x_1(e_3) = e_1$ and so on. The module M is graded, generated by e_3, e_4, with Hilbert series $2 + 2T$ and resolution:

$$\begin{bmatrix} 2 & 6 & 4 & - & - \\ - & - & 4 & 6 & 2 \end{bmatrix}$$

ADHM construction – abstractly

space	objects
$\mathrm{Mod}^d(\mathbb{A}^n)$	modules
$\mathrm{Quot}_r^d(\mathbb{A}^n)$	modules with fixed r generators
$C_n(\mathbb{M}_d)$	modules with fixed basis
$\mathcal{U}^{\mathrm{st}}$	modules with fixed basis and fixed r generators

variety of stable data

$$\begin{tikzcd}
\mathcal{U}^{\mathrm{st}} \arrow[rr, "\text{smooth fib.dim. } rd", "\text{surjective if } n \geq d"'] \arrow[d, "/\,\mathrm{GL}_d"] \arrow[drr, phantom, "\lrcorner", very near start] & & C_n(\mathbb{M}_d) \arrow[d, "/\,\mathrm{GL}_d"] \\
\mathrm{Quot}_r^d(\mathbb{A}^n) \arrow[rr] & & \mathrm{Mod}^d(\mathbb{A}^n)
\end{tikzcd}$$

$(x_1 \dots x_n)$

$\downarrow$

[illegible]

Table: Moduli spaces

Results in Quot flavour

Theorem (J-Šivic)

	$d \leq 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$	$d \gg 0$
$n \leq 2$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$
$n = 3$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$\gg 0$
$n = 4$	$1, \ldots$	$1, \ldots$	$1, 2, \ldots$	$1, 2, \ldots$	$1, 2, \ldots$	$1, 2, \ldots$	$\gg 0$
$n = 5$	$1, \ldots$	$1, \ldots$	$1, 2, \ldots$	$1, 3, 4, \ldots$	$1, 3, 4, \ldots$	$1, 4, 7, 8, \ldots$	$\gg 0$
$n = 6$	$1, \ldots$	$1, \ldots$	$1, 2, \ldots$	$1, 3, 4, \ldots$	$1, 4, 6, 7, \ldots$	$1, 5, 9, 11, \ldots$	$\gg 0$
$n \geq 7$	$1, \ldots$	$1, \ldots$	$1, 2, \ldots$	$1, 3, 4, \ldots$	$1, 4, 6, 7, \ldots$	$1, 6, 10, 12, 13, \ldots$	$\gg 0$

Table: *Number of components of $\mathrm{Quot}_r^d(\mathbb{A}^n)$. In each entry, consecutive numbers correspond to the number of components for $r = 1, 2, \ldots$ and "$\ldots$" means that the numbers stabilize at the value of the last entry. In particular, we see that for $r \geq 5$ we already have all the components (for $d \leq 7$).*

Key open questions, in Quot flavour

Is the scheme $\mathrm{Quot}^d_{\gg 0}(\mathbb{A}^2)$ reduced? Is it Cohen-Macaulay? One can take d instead of $\gg 0$.

known to be singular

principal component of $C_n(M_d)$ $\longleftrightarrow$ modules limits of semisimple $(= \oplus S/\mathfrak{m})$

What is the smallest d such that there exists a zero-dimensional $\mathbb{C}[x_1, x_2, x_3]$-module M with $\dim_{\mathbb{C}} M = d$ which is not a limit of semisimple modules? What is **any** explicit example (for any d, not necessarily small) of such a module?

For a zero-dimensional module M over $S = \mathbb{C}[x_1, x_2, x_3]$ is it true that $\dim_{\mathbb{C}} \mathrm{im}(S \to \mathrm{End}(M)) \leq \dim_{\mathbb{C}} M$?

Key open questions, in Quot flavour

J. Weyman:

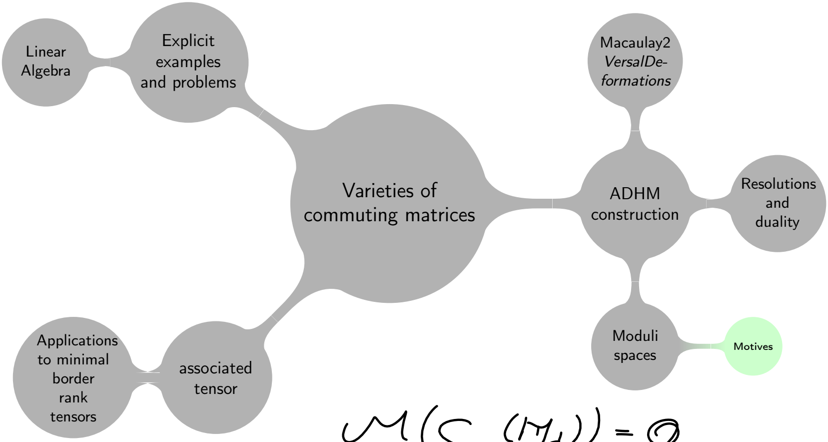
What can be said about deformations of zero-dimensional modules over $\mathbb{C}[x_1, x_2, x_3]$?

What about self-dual modules, not necessarily in three variables?

Proposition (Wojtala)

Structural results on Hilbert functions (e.g. Iarrobino's symmetric decomposition) extend from algebras to modules.

Motives



$$\mathcal{M}(C_n(M_d)) = 0$$

cone over $(0 \ldots 0)$

$S^{\oplus r} \twoheadrightarrow M$ $\longmapsto$ $M \in \mathrm{Vect}_d$, $\mathbb{C}^{\oplus r} \xrightarrow{} S^{\oplus r} \twoheadrightarrow M$, $f \neq 0$

Let $\mathrm{Quot}_r^d(\mathbb{A}^\infty) = \mathrm{colim}_n\, \mathrm{Quot}_r^d(\mathbb{A}^n)$ and let

$\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+1}$

$$\mathrm{Vect}_r^d = \left\{ f \colon \mathbb{C}^{\oplus r} \to V \mid \dim_{\mathbb{C}} V = d,\ f \text{ is non-zero} \right\}$$

this is formally an open subset of the tautological bundle over the stack Vect^d. We have a forgetful map

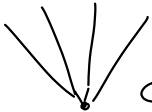
$d=2$ answer $=$?

$\mathbb{C}^* \ltimes \mathbb{C}^+$

$\mathbb{C}^*$-eq motives

$$i \colon \mathrm{Quot}_r^d(\mathbb{A}^\infty) \to \mathrm{Vect}_r^d$$

iso on H^*

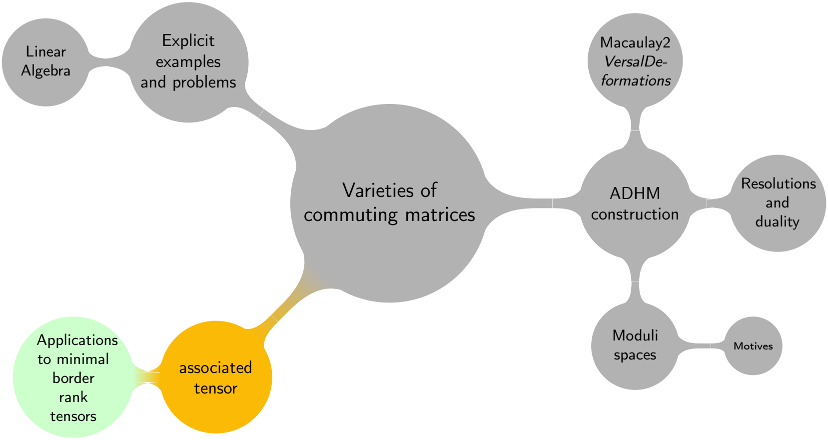


$C_n(M_d) \ni (x_1 \dots x_n)$

$(\lambda Id, \dots, \lambda Id) \longleftarrow (x_1 + \lambda Id, \dots, x_n + \lambda Id)$

Theorem (J-Nardin-Yakerson)

The map i is an $\mathbb{A}^1$-equivalence on affines. For example, the ring $H^*(\mathrm{Quot}_r^d(\mathbb{A}^\infty)(\mathbb{C}), \mathbb{Z})$ is isomorphic to $\mathbb{Z}[c_1, \dots, c_d]/c_d^r$.


Linear Algebra
Explicit examples and problems
Macaulay2 *VersalDeformations*
Varieties of commuting matrices
ADHM construction
Resolutions and duality
Moduli spaces
Motives
Applications to minimal border rank tensors
associated tensor

Connection to tensors

s, $\mathbb{C}^d$ brank tensors $= \overline{(GL(A)\times GL(B)\times GL(C))\,T}$

Let A, B, C have dimension d, let $T \in A \otimes B \otimes C$. A tensor has rank at most d if it has the form $T = \sum_{i=1}^{d} a_i \otimes b_i \otimes c_i$. It has *border rank* at most d if it is a limit of such. A tensor is *concise* if it does not live in any $A' \otimes B' \otimes C'$ where $(-)'$ are subspaces, at least one of them proper. Concise tensors have border rank at least d.

Problem (Nightmare)

Classify concise tensors of border rank d.

Problem (Open problem, reasonable)

Do the same for small d. ($d \leq 4$ done although not written down, but already $d = 5$ seems very open)

Find equations of minimal border rank tensors among concise tensors. ($\Leftrightarrow$ describe $\sigma_d(\text{Segre}) \cap \{\text{concise}\} \subset \mathbb{P}(A \otimes B \otimes C)$.)

Proposition (known)

T of minimal border rank $\Longrightarrow$ T satisfies Strassen's equations.

For $(x_1, \ldots, x_{d-1}) \in C_{d-1}(\mathbb{M}_d)$ form a *naive* tensor $\sum_{i=1}^{d} a_i \otimes x_i \in A \otimes B \otimes C$, where $x_d = \mathrm{Id}_d$.
$T \in A \otimes B \otimes C$ is 1_A*-generic* if for some $\alpha \in A^*$ the element $T(\alpha) \in B \otimes C$ is a matrix of full rank.

concise $T \in A \otimes B \otimes C$ satisfying Strassen's equations	is isomorphic to
general	??
1_A-generic	naive tensor from $(x_1, \ldots, x_{d-1})$ ($\Leftrightarrow$ structure tensor of module)
1_A- and 1_B-generic	structure tensor of algebra
1_A- and 1_B- and 1_C-generic	structure tensor of Gorenstein algebra

Proposition (Landsberg-Michałek, *Abelian tensors*)

The naive tensor is of minimal border rank iff $(x_1, \ldots, x_{d-1})$ is in the principal component.

Corollary

1_A-generic, 1_B-generic tensors satisfying Strassen's equations are of minimal border rank for all $d \leq 7$.

$\mathrm{Hilb}_{\leq 7}(\mathbb{A}^n)$ irreducible

What about 1_A-generic?

1_A : Quot & components.

Connection to tensors cd.

Theorem (J)

For $d \leq 6$, a tuple $(x_1, \dots, x_{d-1}) \in C_{d-1}(\mathbb{M}_d)$ is in the principal component iff $\dim(\mathbb{C}[x_1, \dots, x_{d-1}]) \leq d$.

End-closed condition: The condition $\dim(\mathbb{C}[x_1, \dots, x_{d-1}]) \leq d$ is conveniently rewritten as: the space $V = \langle x_1, \dots, x_{d-1}, \mathrm{Id}_d \rangle$ satisfies $V \cdot V \subset V$.

Corollary

1_A-generic, ~~1_B-generic~~ tensors satisfying Strassen's equations and End-closed condition are of minimal border rank. $d \leq 6$

Nasty example

There is (up to degeneration and taking into account J-Šivic) one nontrivial example.

$$\begin{bmatrix} 0 & 0 & e_1 & 0 & e_2 & e_3 \\ 0 & 0 & 0 & e_1 & e_4 & e_5 \\ 0 & 0 & 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

/Π

def $\underbrace{\text{th}}$ of /Π

The deformation of this tuple, parameterized by $\lambda \in \mathbb{C}$ is given by

$$\begin{bmatrix} 0 & \lambda^2 e_4 & e_1 & -\lambda e_5 & e_2 & e_3 \\ -\lambda e_1 & 0 & -\lambda e_4 & e_1 & e_5 & e_4 \\ -\lambda^3 e_4 & \lambda^2 e_1 & 0 & \lambda^2 e_4 & e_1 & -\lambda e_5 \\ 0 & 0 & 0 & -\lambda^2 e_5 & \lambda e_2 - \lambda e_4 & \lambda e_3 + e_1 \\ 0 & 0 & 0 & 0 & -\lambda^2 e_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda^2 e_5 \end{bmatrix}$$

Theorem (J-Landsberg-Pal)

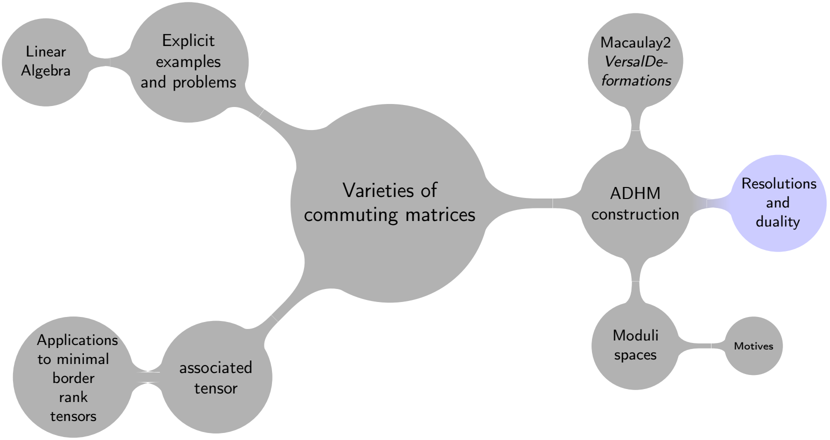
For $d = 5$, concise minimal border rank tensors are cut out of concise tensors by

1. *Strassen's equations (actually $p = 1$ Koszul flattenings),*
2. *End-closed condition,*
3. $(1, 1, 1)$*-equations (coming from border apolarity by Buczyńska-Buczyński).*

& $d=6$, 1_A-generic 1-2 suffices.

Thanks for attention!

A word about methods



Methods I: Macaulay's inverse systems

Macaulay's inverse systems / apolarity for modules.
$S = \mathbb{C}[y_1, \ldots, y_n] \quad T = \mathbb{C}[z_1, \ldots, z_n]$
$F = Se_1 \oplus Se_2 \oplus \ldots \oplus Se_r$
$F^* := Te_1^* \oplus \ldots \oplus Te_r^*$
Is an S-module via $y_i y_j \circ (z_i^2 z_j)e_k^* = z_i e_k^*$. Admits a pairing $F \times F^* \to \mathbb{C}$ defined usually on dual bases.

Theorem (J-Šivic)

For every $M = F/K$ annihilated by $S_{\gg 0}$ there exist $\sigma_1, \ldots, \sigma_r \in F^*$ such that $K = (S\sigma_1 + \ldots + S\sigma_r)^\perp$. Say: M apolar to $\sigma_1, \ldots, \sigma_r$.

Example

The module coming from $x_1, \ldots, x_4$ is the apolar module of $z_1 e_3^* + z_2 e_4^*, z_3 e_3^* + z_4 e_4^*$.

Example

The module coming from $x_1, \ldots, x_4$ is the apolar module of $z_1 e_3^* + z_2 e_4^*, z_3 e_3^* + z_4 e_4^*$.

$$x_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad x_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In the associated module $M = \langle e_1, e_2, e_3, e_4 \rangle$ we have $x_1(e_3) = e_1$ and so on. So $M = (Se_3 \oplus Se_4)/K$ with
$K = (y_1 e_4,\ y_2 e_3,\ y_2 e_4 - y_1 e_3,\ y_3 e_4,\ y_4 e_3,\ y_4 e_4 - y_3 e_3)S$.

Methods II: Białynicki-Birula decomposition

$\mathbb{G}_{\mathrm{m}} \circlearrowleft \mathrm{Quot}_r^d(\mathbb{A}^n)$.

geometry	algebra
$[F/K]$ is $\mathbb{G}_{\mathrm{m}}$-fixed	$K \subset F$ homogeneous
$\mathrm{Hom}(K, F/K)_i$	$\varphi\colon K \to F/K$ shifting degree by i

Proposition

If $\mathrm{Hom}(K, F/K)_{>0} = 0$ *or* $\mathrm{Hom}(K, F/K)_{<0} = 0$ *locally there exists a retraction onto fixed points.*

Methods II: Białynicki-Birula decomposition – scribbling