Irregular Hodge theory: Applications to arithmetic and mirror symmetry

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Origins and motivations of irreg. Hodge theory

Deligne, 1984.

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- ***Griffiths' regularity theorem:***
  - (V,∇): alg. vect. bdle with connect. on a quasi-proj. curve.
  - (V,∇) underlies a PVHS $\implies$ ∇ has reg. sing. at ∞.
- E.g., regularity of the Gauss-Manin connection.
- Complex analogues of exponential sums over finite fields: (V,∇) with ***irreg. sing.*** at ∞.
- Is there a Hodge realization for such objects?
- Typical example: "e^x" on $\mathbb{A}^1 \xhookrightarrow{j} \mathbb{P}^1$, i.e., $(j_*\mathscr{O}_{\mathbb{A}^1}, \mathrm{d}+\mathrm{d}x)$.

- ***Deligne*** defines a $\searrow$ filtration $F^\bullet(j_*V)$ in many examples.
- $\rightsquigarrow$ Filtration of the de Rham complex

$$\boxed{F^p\,\mathrm{DR}(j_*V,\nabla) := \{0 \to F^p(j_*V) \xrightarrow{\nabla} \Omega^1_{\mathbb{P}^1} \otimes F^{p-1}(j_*V) \to 0\}}$$

- In these examples, ***degeneration at*** E^1, i.e.,

$$\boldsymbol{H}^1(\mathbb{P}^1, F^p\,\mathrm{DR}(j_*V,\nabla)) \hookrightarrow \boldsymbol{H}^1(\mathbb{P}^1, \mathrm{DR}(j_*V,\nabla)).$$

- Filtration indexed by $p \in A+\mathbb{N}$, $A \subset [0,1)$ finite.
- *What could be the use of a "Hodge filtration" which does not lead to Hodge theory? A hope it that it imposes bounds to p-adic valuations of eigenvalues of Frobenius.*

Adolphson-Sperber, 1987–89.

- Lower bound of the p-adic Newton polygon of the L-function attached to a nondeg. Laurent pol. $f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ given by a Newton polygon attached to f.
- $\rightsquigarrow$ Answers Deligne's hope, but no Hodge filtration.
- (Would like to interpret this as "Newton above Hodge".)

Simpson, 1990.

- Non abelian Hodge theory on curves. Correspondence between (V, ∇) with reg. sing. (tame) at ∞ and stable tame parabolic Higgs bdles.
- Simpson suggests it would be possible to extend this correspondence to (V, ∇) ***wild*** (i.e., with irreg. sing.).
- $\rightsquigarrow$ Positive answer on curves by CS and Biquard-Boalch (2000 $\pm\varepsilon$).
- Positive answer (any dimension) by T. Mochizuki (2011).
- ***Drawback:*** no Hodge filtration.

Mirror symmetry for Fano's.

- Need to consider a pair (X, f), $f : X \to \mathbb{A}^1$, X smooth quasi-proj., as possible mirror of a Fano mfld.
- $\rightsquigarrow$ Various cohomologies $H^\bullet(X, f)$ attached to (X, f), e.g.
  - dual of Betti homology (Lefschetz thimbles),
  - de Rham cohomology: hypercohom of $(\Omega_X^\bullet, \mathrm{d} + \mathrm{d}f)$,
  - Periodic cyclic homology,
  - Exponential motives.

Questions on the Hodge theory of Landau-Ginzburg models.

- If (X, f) is mirror of a Fano mfld Y, what is the Hodge filtration on $H^\bullet(X, f)$ corresponding to that of $H^\bullet(Y)$?
- If Y is a Fano orbifold (e.g. toric, like $\mathbb{P}(w_0, \dots, w_n)$), $H^\bullet_{\mathrm{orb}}(Y)$ (Chen-Ruan) has rational exponents (corresponding to "twisted sectors"). Natural to expect that $F^\bullet$ for (X, f) is indexed by $A + \mathbb{N}$, $A \subset [0, 1) \cap \mathbb{Q}$.
- If Y is a Fano mfld, how to translate to $F^\bullet H^n(X, f)$ Hard Lefschetz for $c_1(TY)$?

E_1-degeneration

Hodge realization for a pair (X,f).

- X smooth quasi-proj.
- Choose a compact. $f:\overline{X}\to\mathbb{P}^1$ of f s.t. $D=\overline{X}\smallsetminus X$ ncd.
- $P:=f^*(\infty)$, $\quad |P|\subset D$.

$$\boxed{H^k_{\mathrm{dR}}(X,f)\simeq\begin{cases}\boldsymbol{H}^k(\overline{X},(\Omega^\bullet_{\overline{X}}(*D),\mathrm{d}+\mathrm{d}f)),\\ \boldsymbol{H}^k(\overline{X},(\Omega^\bullet_{\overline{X}}(\log D,f),\mathrm{d}+\mathrm{d}f))\end{cases}}$$

$$\begin{aligned}\Omega^k_{\overline{X}}(\log D,f)&:=\Big\{\omega\in\Omega^k_{\overline{X}}(\log D)\mid \mathrm{d}f\wedge\omega\in\Omega^{k+1}_{\overline{X}}(\log D)\Big\}\\ &=\Big\{\omega\in\Omega^k_{\overline{X}}(\log D)\mid (\mathrm{d}+\mathrm{d}f\wedge)\omega\in\Omega^{k+1}_{\overline{X}}(\log D)\Big\}\end{aligned}$$

- Quasi-isomorphic filtered complexes:
  - Yu: $\quad F^\bullet(\Omega^\bullet_{\overline{X}}(*D),\mathrm{d}+\mathrm{d}f)$,
  - K-K-P: $F^\bullet(\Omega^\bullet_{\overline{X}}(\log D,f),\mathrm{d}+\mathrm{d}f))$.

$$F^p(\Omega^\bullet_{\overline{X}}(\log D,f),\mathrm{d}):=\{0\to\underset{p}{\Omega^p(\log D,f)}\to\cdots\to\Omega^n(\log D,f)\to 0\}$$

- Recall: for X quasi-projective (and $f\equiv 0$)

***Theorem* (Degeneration at E_1, Deligne (Hodge II, 1972)).**

$$\boldsymbol{H}^\bullet(\overline{X},F^p(\Omega^\bullet_{\overline{X}}(\log D),\mathrm{d}))\hookrightarrow\boldsymbol{H}^\bullet(\overline{X},(\Omega^\bullet_{\overline{X}}(\log D),\mathrm{d}))\simeq H^\bullet(X,\mathbb{C}).$$

***Theorem* (Esnault-S.-Yu, Katzarkov-Kontsevich-Pantev, M. Saito, T. Mochizuki).**

- *The spectral seq. for $F^\bullet(\Omega^\bullet_{\overline{X}}(*D),\mathrm{d}+\mathrm{d}f)$, equivalently for $F^\bullet(\Omega^\bullet_{\overline{X}}(\log D,f),\mathrm{d}+\mathrm{d}f))$, degenerates at E_1.*
- $\rightsquigarrow$ ***Irreg. Hodge filtr.*** $F^\bullet H^k_{\mathrm{dR}}(X,f)$.

- Four different proofs:
  - M. Saito uses a comparison with nearby cycles of f along $f^*(\infty)$ and Steenbrink/Schmid limit theorems.
  - K-K-P use reduction to char. p à la Deligne-Illusie. But need assumption that $f^*(\infty)$ is reduced.
  - E-S-Y use reduction to $X=\mathbb{A}^1$ by pushing forward by f and previous results on CS extending the original construction of Deligne on curves by means of ***twistor D-modules***.
  - T. Mochizuki uses the full strength of twistor D-modules in arbitrary dimensions.
- Can take into account multiplicities of $f^*(\infty)$ to refine $F^\bullet$ and index it by $A+\mathbb{N}$,

$$A=\Big\{\ell/m_i\mid 0\leqslant\ell<m_i,\ m_i=\text{mult. of a component of }f^*(\infty)\Big\}.$$

Irregular Hodge-Tate structures

- $H := H^k(X, f)$, monodromy induced by $H^k(X, e^{i\theta} f)_{\theta\in[0,2\pi]}$
- $F^\bullet_{\mathrm{irr}} H$: irreg. Hodge filtr.
- if ***unipotent*** monodromy $\rightsquigarrow$ Jakobson-Morosov filtr. $M_\bullet H$ associated to its nilpotent part.
- Define $W_\ell H = M_{\ell-k} H$
- ***unipotent*** monodromy $\Longrightarrow$ jumps of $F^\bullet_{\mathrm{irr}} H$ are integers.

Definition. $H^k(X, f)$ is ***irreg. Hodge-Tate*** if ***unipotent*** monodr. and

$$\forall p, \quad \boxed{\dim \mathrm{gr}^W_{2p} H = \dim \mathrm{gr}^p_{F_{\mathrm{irr}}} H \quad \text{and} \quad \mathrm{gr}^W_{2p+1} H = 0}$$

***Conjecture* (K-K-P, 2017).** *If (X, f) is the Landau-Ginzburg model mirror to a projective Fano mfld Y, then $H^n(X, f)$ ($n = \dim X$) is irregular Hodge-Tate.*

Many works on the conjecture.

- Lunts, Przyjalkowski, Harder
- Shamoto

The toric case.

- Lattices $M \subset \mathbb{R}^n$, $N = M^\vee$.
- $\Delta \subset \mathbb{R}^n$: reflexive simplicial polyhedron with vertices in M, s.t. 0 is the only integral point in the interior of Δ.
- Δ^*: dual polyhedron (vertices in N and of the same kind as Δ).
- Σ: fan dual to Δ, $=$ cone $(0, \Delta^*)$.
- $Y = \mathbb{P}_\Sigma$ assumed smooth, hence toric Fano (Batyrev).
- Chow ring $A^*(Y) \simeq H^{2\star}(Y, \mathbb{Z})$ generated by div. classes D_v, $v \in \mathrm{Vertices}(\Delta^*) =: V(\Delta^*)$.
- $c_1(K_Y^\vee) = \sum_{v\in V(\Delta^*)} D_v$ satisfies Hard Lefschetz on $H^{2\star}(Y, \mathbb{Q})$.

- Coordinates $x_1, \dots, x_n$ s.t. $\mathbb{C}[N] = \mathbb{C}[x, x^{-1}]$.

$$X := \operatorname{Spec} \mathbb{C}[x, x^{-1}],$$

$$f : X \longrightarrow \mathbb{A}^1, \qquad f(x) = \sum_{v \in V(\Delta^*)} x^v$$

$$H^n_{\mathrm{dR}}(X, f) = \Omega^n_X / (\mathrm{d} + \mathrm{d}f \wedge) \Omega^{n-1}_X \simeq \big[\mathbb{C}[x, x^{-1}]/(\partial f)\big] \cdot \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}$$

- Newton filtration $\mathcal{N}_\bullet$ on the Jacobian ring $\mathbb{Q}[x, x^{-1}]/(\partial f)$
- Borisov-Chen-Smith: $\boxed{H^{2\star}(Y, \mathbb{C}) \simeq \mathrm{gr}^{\mathcal{N}}_{\star}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big)}$
- ***Hard Lefschetz*** $\implies$ $\forall k$ s.t. $0 \leqslant k \leqslant n/2$,

$$\boxed{f^{n-2k} : \mathrm{gr}^{\mathcal{N}}_{k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big) \xrightarrow{\sim} \mathrm{gr}^{\mathcal{N}}_{n-k}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big)}$$

- Idea of Varchenko from Singularity theory (Doklady, 1981): interpret multipl. by f as the nilpotent part of a monodromy operator.
- Adapt and apply this idea to $H^n_{\mathrm{dR}}(X, f)$
- One shows that

$$\dim F^p_{\mathrm{irr}} H^n_{\mathrm{dR}}(X, F) = \dim \mathcal{N}_{n-p}\big(\mathbb{C}[x, x^{-1}]/(\partial f)\big).$$

- $\implies$ irreg. Hodge-Tate property. $\square$

Computation of Hodge numbers by means of irregular Hodge theory

- Standard course of calculus: often easier to compute convolution $f \star g$ by applying ***Fourier transformation***.
- Same idea for Hodge nbrs.
- Arithmetic motivation: Functional equation for the L-function attached to symmetric power moments of Kloosterman sums.
- Complex analogue of the Kloosterman sums: modified Bessel differential equation on $\mathbb{G}_{\mathrm{m}}$.
- $\mathrm{Kl}_2 : (\mathcal{O}^2_{\mathbb{G}_{\mathrm{m}}}, \nabla), \quad \nabla(v_0, v_1) = (v_0, v_1) \cdot \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \cdot \dfrac{\mathrm{d}z}{z}.$
- For $k \geqslant 1$, want to consider $\operatorname{Sym}^k \mathrm{Kl}_2$:
  - free $\mathbb{C}[z, z^{-1}]$-mod. rk $k+1$ with connection,

  and its de Rham cohomology

$$H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2) = \operatorname{coker}\Big[\nabla : \operatorname{Sym}^k \mathrm{Kl}_2 \longrightarrow \operatorname{Sym}^k \mathrm{Kl}_2 \otimes \frac{\mathrm{d}z}{z}\Big]$$

***Theorem* (Fresán-S-Yu).** *Assume k odd for simplicity.*

- *$H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ canonically endowed with a MHS of weights $k+1$ & $2k+2$.*
- *$\dim H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)^{p,q} = 1$ if $p+q = k+1$ and $p = 2, \dots, k-1$ or $p = q = k+1$, and 0 otherwise.*

Synopsis.

- ***Motivations.*** Series of papers by Broadhurst-Roberts: some Feynman integrals expressed as period integrals

  $$\int_0^\infty I_0(t)^a K_0(t)^b t^c \, \mathrm{d}t \qquad (I_0, K_0 : \text{“modified Bessel functions”}).$$

  $\rightsquigarrow$ various conjectures on L fns of Kloosterman moments.
- On $\mathrm{Sym}^k \mathrm{Kl}_2$, ∇ has a regular sing. at $z=0$, but an ***irregular*** one at ∞, hence ***does not*** underlie a PVHS (Griffiths th.).
- $H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ has a ***motivic*** interpretation: this explains the MHS.
- $\mathrm{Sym}^k \mathrm{Kl}_2$ underlies a ***variation of irregular Hodge structure*** (i.e., an irregular mixed Hodge module on $\mathbb{P}^1 \supset \mathbb{G}_{\mathrm{m}}$).
- $\implies$ $H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ endowed with an ***irregular Hodge filtration***.
- We prove that this irreg. Hodge filtr. ***coincides*** with the Hodge filtr. of the MHS.
- We compute this irreg. Hodge filtration by toric methods of Adolphson-Sperber & Yu. (Irreg. analogue of Danilov-Khovanski computation for toric hypersurfaces).