Non-degenerate locally tame complete intersection varieties and geometry of non-isolated hypersurface singularities

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A new criterion to test Whitney equisingularity

$f(t,z_1,\ldots,z_n)=f(t,\mathrm{z})$ polynomial function on $\mathbb{C}\times\mathbb{C}^n$ such that $f(t,0)=0$; as usual, we put $f_t(\mathrm{z}):=f(t,\mathrm{z})$

The aim of this talk is to investigate the local geometry, at singular points, of the hypersurfaces $V(f_t):=f_t^{-1}(0)\subseteq\mathbb{C}^n$ as the parameter t varies

☞ Hereafter, I am only interested in such a local situation, that is, in (arbitrarily small representatives of) germs at the origin

Theorem (Briançon) Suppose that for all small t:

1. f_t has an isolated singularity at 0
2. Newton diagram of f_t is independent of t
3. $V(f_t)$ is non-degenerate

Then the family $\{V(f_t)\}$ is Whitney equisingular, that is, $\exists$ Whitney (b)-regular stratification of $V(f):=f^{-1}(0)\subseteq\mathbb{C}\times\mathbb{C}^n$ such that the t-axis $\mathbb{C}\times\{0\}$ is a stratum

☞ In particular, the local, ambient, topological type of $V(f_t)$ is independent of t

· Write $f_t(z) = \sum c_\alpha z^\alpha$, where $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$

· $\Gamma_+(f_t)$ = Newton polyhedron of f_t, i.e., the convex hull in $\mathbb{R}_+^n$ of

$$\bigcup_{\alpha,\, c_\alpha \neq 0} (\alpha + \mathbb{R}_+^n)$$

· $\Gamma(f_t)$ = Newton diagram of f_t, i.e., the union of the compact faces of $\Gamma_+(f_t)$

· $\forall w := (w_1, \ldots, w_n) \in \mathbb{N}^n \smallsetminus \{0\}$:

$$\underbrace{\Delta(w; f_t)}_{\text{(possibly non-compact) face of } \Gamma_+(f_t)} := \{\alpha \in \Gamma_+(f_t) \mid \sum_{i=1}^n w_i \alpha_i \text{ is minimal}\} \quad \text{and} \quad \underbrace{f_{t,w}(z)}_{\text{face function of } f_t \text{ w.r.t. weight } w} := \sum_{\alpha \in \Delta(w; f_t)} c_\alpha z^\alpha$$

Non-degeneracy

$V(f_t)$ is non-degenerate if for any "positive" weight vector w:

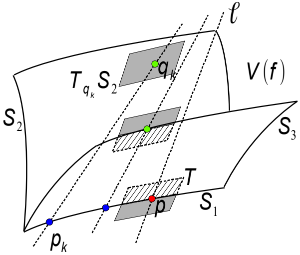
$$V(f_{t,w}) \cap \mathbb{C}^{*n}$$

is a reduced non-singular hypersurface in $\mathbb{C}^{*n}$ (i.e., $f_{t,w}$ has no critical point in $V(f_{t,w}) \cap \mathbb{C}^{*n}$)

Whitney (b)-regular stratification of $V(f)$

This is a complex analytic stratification of $V(f)$ such that for any pair of strata S_1, S_2 and any point $\mathrm{p} \in S_1 \cap \bar{S}_2$, we have S_2 is Whitney (b)-regular over S_1 at p, that is,

$$\left.\begin{array}{l} \{\mathrm{p}_k\} \text{ in } S_1,\ \mathrm{p}_k \to \mathrm{p} \\ \{\mathrm{q}_k\} \text{ in } S_2,\ \mathrm{q}_k \to \mathrm{p} \\ \mathrm{line}(\mathrm{p}_k, \mathrm{q}_k) \to \ell \\ T_{\mathrm{q}_k} S_2 \to T \end{array} \right| \Rightarrow \ell \subseteq T$$



Local tameness

- $\forall I \subseteq \{1, \dots, n\}$, let $\mathbb{C}^I := \{z \in \mathbb{C}^n \,;\, z_i = 0 \text{ if } i \notin I\}$
- $\mathcal{V}_{f_t} := \{I \,;\, \underbrace{f_t|_{\mathbb{C}^I} \equiv 0}_{\substack{\text{in this case } \mathbb{C}^I \text{ is called a} \\ \text{vanishing coordinate subspace for } f_t}}\}$

- $V(f_t)$ is locally tame if $\exists R(f_t) > 0$ such that:
  - $\forall I := \{i_1, \dots, i_m\} \neq \varnothing$ in $\mathcal{V}_{f_t}$;
  - $\forall w = (w_1, \dots, w_n) \in \mathbb{N}^n \smallsetminus \{0\}$ with $\{i \,;\, w_i = 0\} = I$;
  - $\forall u_{i_1}, \dots, u_{i_m} \in \mathbb{C}^*$ with $\sum_{j=1}^{m} |u_{i_j}|^2 < R(f_t)$;

the intersection

$$V(f_{t,w}) \cap \{z \in \mathbb{C}^{*n} \mid z_{i_1} = u_{i_1}, \dots, z_{i_m} = u_{i_m}\}$$

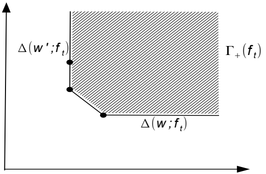
is a reduced non-singular hypersurface in $\{z \in \mathbb{C}^{*n} \mid z_{i_1} = u_{i_1}, \dots, z_{i_m} = u_{i_m}\}$, that is, $f_{t,w}$ has no critical point in

$$V(f_{t,w}) \cap \{z \in \mathbb{C}^{*n} \mid z_{i_1} = u_{i_1}, \dots, z_{i_m} = u_{i_m}\}$$

as a function of the $n - m$ variables $(z_i)_{i \notin I}$

☞ $R(f_t)$ = radius of local tameness for f_t

Example $f_t = z_1^2 z_2^3 + z_1^3 z_2^2 + t z_1^2 z_2^4$



- $I = \{1\}$, $\{2\}$ or $\{1,2\}$
  - $f_t|_{\mathbb{C}^{\{1\}}}(z_1, 0) \equiv 0 \Rightarrow \{1\} \in \mathcal{V}_{f_t}$
  - $f_t|_{\mathbb{C}^{\{2\}}}(0, z_2) \equiv 0 \Rightarrow \{2\} \in \mathcal{V}_{f_t}$
  - $\{1,2\} \notin \mathcal{V}_{f_t}$
- $w = (w_1, w_2)$ with $\{i\,;\, w_i = 0\} = \{1\}$, i.e, with $w_1 = 0$
  $\Rightarrow \Delta(w; f_t) = \{\alpha \in \Gamma_+(f_t) \mid w_2\alpha_2 \text{ minimal}\}$
  $\forall\, u_1 \in \mathbb{C}^*$ the function $z_2 \mapsto f_{t,w}(u_1, z_2) = u_1^3 z_2^2$ has no critical points on
  $$V(f_{t,w}) \cap \{z \in \mathbb{C}^{*2} \mid z_1 = u_1\}$$
- $w' = (w_1', w_2')$ with $\{i\,;\, w_i' = 0\} = \{2\}$, i.e, with $w_2' = 0$
  $\Rightarrow \Delta(w'; f_t) = \{\alpha \in \Gamma_+(f_t) \mid w_1'\alpha_1 \text{ minimal}\}$
  $\forall\, u_2 \in \mathbb{C}^*$ the function $z_1 \mapsto f_{t,w'}(z_1, u_2) = z_1^2 u_2^3 + t z_1^2 u_2^4$ has no critical points on
  $$V(f_{t,w'}) \cap \{z \in \mathbb{C}^{*2} \mid z_2 = u_2\}$$
  provided that $|u_2| < 1/|t|$ $(t \neq 0)$ [because the derivative is $2z_1 u_2^3(1 + tu_2)$]

Therefore, $V(f_t)$ is locally tame and $R(f_t) = 1/|t|$ is a radius of local tameness for f_t if $t \neq 0$ while $R(f_0) = \infty$ is a radius of local tameness for f_0

Theorem (M. Oka and CE, 2017) Suppose that for all small t:

1. Newton diagram of f_t is independent of t
2. $V(f_t)$ is non-degenerate
3. $V(f_t)$ is locally tame and there is a radius of local tameness for f_t which is $> R$ for some $R > 0$ independent of t

Then the family $\{V(f_t)\}$ is Whitney equisingular

☞ This is a generalization of Briançon's theorem to non-isolated singularities

Example $f_t = z_1^2 z_2^3 + z_1^3 z_2^2 + t z_1^2 z_2^4$

It is locally tame ($R(f_0) = \infty$ and $R(f_t) = 1/|t|$ if $t \neq 0$); we have $R(f_t) > R := 1$ for all $|t| < 1$. It is also non-degenerate, and has constant Newton diagram. Thus the family $\{V(f_t)\}$ is Whitney equisingular.

Remark

In general, if $h = h_1 \cdot h_2$ and if $\dim_0(V(h_1) \cap V(h_2)) \geq 1$, then h is never non-degenerate if its Newton diagram intersect each coordinate axis

Non-degenerate, locally tame, complete intersection variety

$h^1(z), \dots, h^{k_0}(z)$ polynomial functions on $\mathbb{C}^n$ such that $h^k(0) = 0$

- $V(h^1, \dots, h^{k_0}) = \{z\,;\, h^1(z) = \cdots = h^{k_0}(z) = 0\}$ is a non-degenerate, locally tame, complete intersection variety if

1. for any "positive" weight vector w:

$$V(h^1_{\mathrm{w}}, \dots, h^{k_0}_{\mathrm{w}}) \cap \mathbb{C}^{*n}$$

is a reduced, non-singular, complete intersection variety in $\mathbb{C}^{*n}$, that is, the k_0-form $dh^1_{\mathrm{w}} \wedge \cdots \wedge dh^{k_0}_{\mathrm{w}}$ is nowhere vanishing in $V(h^1_{\mathrm{w}}, \dots, h^{k_0}_{\mathrm{w}}) \cap \mathbb{C}^{*n}$

2. $\exists R(h^1, \dots, h^{k_0}) > 0$ such that:
   - $\forall I := \{i_1, \dots, i_m\} \neq \varnothing$ in $\mathcal{V}_{h^1} \cap \cdots \cap \mathcal{V}_{h^{k_0}}$;
   - $\forall \mathrm{w} = (w_1, \dots, w_n) \in \mathbb{N}^n \setminus \{0\}$ with $\{i\,;\, w_i = 0\} = I$;
   - $\forall u_{i_1}, \dots, u_{i_m} \in \mathbb{C}^*$ with $\sum_{j=1}^m |u_{i_j}|^2 < R(h^1, \dots, h^{k_0})$;

   the intersection

$$V(h^1_{\mathrm{w}}, \dots, h^{k_0}_{\mathrm{w}}) \cap \{z \in \mathbb{C}^{*n} \mid z_{i_1} = u_{i_1}, \dots, z_{i_m} = u_{i_m}\}$$

   is a reduced, non-singular, complete intersection variety in $\{z \in \mathbb{C}^{*n} \mid z_{i_1} = u_{i_1}, \dots, z_{i_m} = u_{i_m}\}$

☞ $R(h^1, \dots, h^{k_0})$ = radius of local tameness for $\{h^1, \dots, h^{k_0}\}$

Let $f(t,z) := f^1(t,z)\cdots f^{k_0}(t,z)$, where $f^k(t,z)$ is a polynomial function on $\mathbb{C}\times\mathbb{C}^n$ such that $f^k(t,0)=0$; as usual, put $f_t(z) := f(t,z)$ and $f_t^k(z) := f^k(t,z)$

Theorem (M. Oka and CE, 2021) Suppose that for all small t:

1. $\forall k = 1,\dots,k_0$, the Newton diagram of f_t^k is independent of t
2. $\forall\{k_1,\dots,k_p\}\subseteq\{1,\dots,k_0\}$:
   - $V(f_t^{k_1},\dots,f_t^{k_p})$ non-degenerate, locally tame, complete intersection variety
   - there is a radius of local tameness for $\{f_t^{k_1},\dots,f_t^{k_p}\}$ which is $> R$ for some $R>0$ independent of t and of $\{k_1,\dots,k_p\}$

(families $\{f_t\}$ satisfying (1) and (2) above are called Newton-admissible)

Then $\{V(f_t)\}$ is Whitney equisingular

☞ This extends all previous theorems

Example $f^k(t,z) := \sum_{i=1}^n (a_i(k)+t\,b_i(k))\,z_i^{p_i}z_{i+1}^{q_i}$ $(1\le k\le k_0)$

· $p_i, q_i\in\mathbb{N}^*$ are so that there exists a positive weight vector $\mathrm{w}=(w_1,\dots,w_n)$ so that $f_t^1(z),\dots,f_t^{k_0}(z)$ are weighted homogeneous polynomials of w-degree d

· For generic $a_i(k), b_i(k)\in\mathbb{C}^*$, the family $\{f_t\}$ is Newton-admissible, and hence $\{V(f_t)\}$ is Whitney equisingular

Sketch of the proof

A Whitney (b)-regular stratification of $V(f)$ answering the theorem (i.e., having $\mathbb{C} \times \{0\}$ as a stratum) is given by the canonical toric stratification of $V(f)$

- For any $I \subseteq \{1, \ldots, n\}$ and $K \subseteq \{1, \ldots, k_0\}$ let

$$S^I(K) := \bigcap_{k \in K} \Big(V(f^k) \cap (\mathbb{C} \times \mathbb{C}^{*I}) \Big) \setminus \bigcup_{k \in K^c} \Big(V(f^k) \cap (\mathbb{C} \times \mathbb{C}^{*I}) \Big)$$

where $\mathbb{C}^{*I} := \{z \in \mathbb{C}^n ; z_i = 0 \Leftrightarrow i \notin I\}$; note that $S^{\varnothing}(\{1, \ldots, k_0\}) = \mathbb{C} \times \{0\}$.

- Non-degeneracy condition $\Rightarrow S^I(K)$ is smooth (near 0)

It follows that the collection of all sets $S^I(K)$ is a complex analytic stratification of $V(f)$; it is called the canonical toric stratification of $V(f)$.

- We must show that for any

$$I \subseteq J \subseteq \{1, \ldots, n\} \quad \text{and} \quad L \subseteq K \subseteq \{1, \ldots, k_0\} \quad \text{with} \quad S^I(K) \cap \overline{S^J(L)} \neq \varnothing,$$

$S^J(L)$ is Whitney (b)-regular over $S^I(K)$ at any point $(t^0, z^0) \in S^I(K) \cap \overline{S^J(L)}$.

· Pick real analytic paths

$$\rho(s) := (t(s), z(s)) \quad \text{and} \quad \rho'(s) := (t'(s), z'(s))$$

in $\mathbb{C} \times \mathbb{C}^n$ such that:

1. $\rho(0) = \rho'(0) = (t^0, z^0)$
2. $\rho'(s) \in S^I(K)$ and $\rho(s) \in S^J(L)$ for $s \neq 0$

Put $\ell(s) := \rho(s) - \rho'(s)$. By the curve selection lemma, it suffices to prove that

$$\ell_\infty := \lim_{s \to 0} \frac{\ell(s)}{\|\ell(s)\|} \in T_\infty := \lim_{s \to 0} T_{\rho(s)} S^J(L).$$

· To simplify, assume $J = \{1, \ldots, n\}$ and $L = \{1, \ldots, k_L\}$. Note that

$$\begin{aligned} T_{\rho(s)} S^J(L) &= (df^1(\rho(s)) \wedge \cdots \wedge df^{k_L}(\rho(s)))^\perp \\ &= \{\mathrm{v} \in T_{\rho(s)}(\mathbb{C} \times \mathbb{C}^n) \mid \imath_{\mathrm{v}}(df^1(\rho(s)) \wedge \cdots \wedge df^{k_L}(\rho(s))) = 0\}. \end{aligned}$$

Write $o_{k_L} := \mathrm{ord}_s \; df^1(\rho(s)) \wedge \cdots \wedge df^{k_L}(\rho(s))$ and put

$$\omega_\infty := \lim_{s \to 0} \frac{1}{s^{o_{k_L}}} \cdot df^1(\rho(s)) \wedge \cdots \wedge df^{k_L}(\rho(s)).$$

Then $T_\infty = \omega_\infty^\perp$, and we must prove that $\ell_\infty \in \omega_\infty^\perp$.

· To do that, we show that $\exists$ linearly independent 1-forms $\omega_1, \dots, \omega_{k_L}$ such that

$$\omega_\infty = \omega_1 \wedge \cdots \wedge \omega_{k_L}$$

(then, of course, it will be enough to prove that $\omega_k(\ell_\infty) = 0$ for all $1 \le k \le k_L$)

· The main difficulty is that, in general, if $I \in \mathcal{V}_{f_t^k}$ for some of the functions f_t^k, then the limits of the normalized 1-forms

$$df^1(\rho(s)), \dots, df^{k_L}(\rho(s))$$

as $s \to 0$ are not linearly independent. The subterfuge used to solve this problem is to replace the corresponding differential $df^k(\rho(s))$ with a term of the form

$$df^k(\rho(s)) + \sum_{k'=1}^{k-1} c_{k,k'}(s)\, df^{k'}(\rho(s)),$$

where $c_{k,k'}(s)$ are suitable polynomials.

· The forms ω_k corresponding to the functions f^k such that $I \in \mathcal{V}_{f_t^k}$ can be constructed using the above subterfuge and the uniform local tameness condition.

· The forms ω_k corresponding to the functions f^k such that $I \notin \mathcal{V}_{f_t^k}$ are constructed using the non-degeneracy condition.

Application: Thom's a_f condition

Pick a sufficiently small representative of (the germ at 0 of)

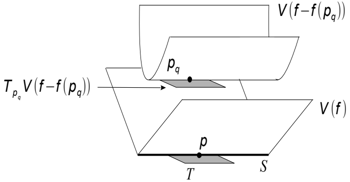
$$f(t,z) := f^1(t,z)\cdots f^{k_0}(t,z)$$

so that 0 is the only possible critical value of f. Then we have:

Theorem (Parusiński and Briançon-Maisonobe-Merle) If $\mathcal{S}$ is a Whitney (b)-regular stratification of $V(f)$, then $\mathcal{S}$ satisfies Thom's a_f condition, that is, for any stratum S, any point $p \in S$, and any sequence $\{p_q\} \notin V(f)$ such that

$$p_q \to p \quad \text{and} \quad T_{p_q}V(f - f(p_q)) \to T,$$

we have $T_pS \subseteq T$



Theorem (M. Oka and CE, 2021) If $\{f_t\}$ is Newton-admissible, then the canonical toric stratification $\mathcal{S}$ of $V(f)$ satisfies Thom's a_f condition

Proof By our theorem, $\mathcal{S}$ is Whitney (b)-regular. Then we apply the theorem of Parusiński and Briançon-Maisonobe-Merle.

Application to the Milnor fibration

Theorem (M. Oka and CE, 2021) If $\{f_t\}$ is Newton-admissible, then the Milnor fibrations of f_0 and f_t at 0 are isomorphic for all small t

Ingredients of the proof

1. The canonical toric stratification $\mathcal{S}$ of $V(f)$ satisfies Thom's a_f condition
2. For any t and any $\varepsilon > 0$ sufficiently small
$$\{t\} \times S_\varepsilon \pitchfork ((\{t\} \times \mathbb{C}^n) \cap S)$$
for any $S \in \mathcal{S}$ (this follows from the Whitney (b)-regularity condition)
3. Using (1) and (2) we show that $\{f_t\}$ has a uniform stable radius for the Milnor fibrations of the f_t's at 0
4. **Theorem (Oka)** $\{f_t\}$ has a uniform stable radius $\Rightarrow$ the Milnor fibrations of f_0 and f_t at 0 are isomorphic for all small t

Thank you for your attention!