Non-degenerate locally tame complete intersection varieties and geometry of non-isolated hypersurface singularities

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A new criterion to test Whitney equisingularity

\[ f(t, z_1, \ldots, z_n) = f(t, z) \] polynomial function on \( \mathbb{C} \times \mathbb{C}^n \) such that \( f(t, 0) = 0 \);
as usual, we put \( f_t(z) := f(t, z) \)

The aim of this talk is to investigate the local geometry, at singular points, of the hypersurfaces \( V(f_t) := f_t^{-1}(0) \subseteq \mathbb{C}^n \) as the parameter \( t \) varies.

Hereafter, I am only interested in such a local situation, that is, in (arbitrarily small representatives of) germs at the origin.

**Theorem (Briançon)** Suppose that for all small \( t \):
1. \( f_t \) has an isolated singularity at 0
2. Newton diagram of \( f_t \) is independent of \( t \)
3. \( V(f_t) \) is non-degenerate

Then the family \( \{ V(f_t) \} \) is Whitney equisingular, that is, \( \exists \) Whitney (\( b \))-regular stratification of \( V(f) := f^{-1}(0) \subseteq \mathbb{C} \times \mathbb{C}^n \) such that the \( t \)-axis \( \mathbb{C} \times \{0\} \) is a stratum.

In particular, the local, ambient, topological type of \( V(f_t) \) is independent of \( t \).
· Write \( f_t(z) = \sum c_\alpha z^\alpha \), where \( z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \)

· \( \Gamma_+(f_t) = \) Newton polyhedron of \( f_t \), i.e., the convex hull in \( \mathbb{R}_+^n \) of

\[
\bigcup_{\alpha, c_\alpha \neq 0} (\alpha + \mathbb{R}_+^n)
\]

· \( \Gamma(f_t) = \) Newton diagram of \( f_t \), i.e., the union of the compact faces of \( \Gamma_+(f_t) \)

· \( \forall w := (w_1, \ldots, w_n) \in \mathbb{N}^n \setminus \{0\} : \)

\[
\Delta(w; f_t) := \{ \alpha \in \Gamma_+(f_t) \mid \sum_{i=1}^n w_i \alpha_i \text{ is minimal} \}
\]

(possibly non-compact) face of \( \Gamma_+(f_t) \)

Non-degeneracy

\( V(f_t) \) is non-degenerate if for any “positive” weight vector \( w \):

\[
V(f_t, w) \cap \mathbb{C}^{\ast n}
\]

is a reduced non-singular hypersurface in \( \mathbb{C}^{\ast n} \) (i.e., \( f_t, w \) has no critical point in \( V(f_t, w) \cap \mathbb{C}^{\ast n} \))
**Whitney \((b)\)-regular stratification of \(V(f)\)**

This is a complex analytic stratification of \(V(f)\) such that for any pair of strata \(S_1, S_2\) and any point \(p \in S_1 \cap \bar{S}_2\), we have \(S_2\) is **Whitney \((b)\)-regular** over \(S_1\) at \(p\), that is,

\[
\begin{align*}
\{p_k\} & \text{ in } S_1, \ p_k \to p \\
\{q_k\} & \text{ in } S_2, \ q_k \to p \\
\text{line}(p_k, q_k) & \to \ell \\
T_{q_k}S_2 & \to T
\end{align*}
\Rightarrow \ell \subseteq T
\]
Local tameness

- \( \forall I \subseteq \{1, \ldots, n\} \), let \( C^I := \{ z \in \mathbb{C}^n ; z_i = 0 \text{ if } i \notin I \} \)
- \( \mathcal{V}_{f_t} := \{ I ; f_t|_{C^I} \equiv 0 \} \)

in this case \( C^I \) is called a vanishing coordinate subspace for \( f_t \)

\[ \Rightarrow \quad \mathcal{V}(f_t) \text{ is locally tame if } \exists R(f_t) > 0 \text{ such that:} \]

- \( \forall I := \{ i_1, \ldots, i_m \} \neq \emptyset \text{ in } \mathcal{V}_{f_t} ; \)
- \( \forall w = (w_1, \ldots, w_n) \in \mathbb{N}^n \setminus \{0\} \) with \( \{ i ; w_i = 0 \} = I ; \)
- \( \forall u_{i_1}, \ldots, u_{i_m} \in \mathbb{C}^* \) with \( \sum_{j=1}^m |u_{i_j}|^2 < R(f_t) ; \)

the intersection

\[ \mathcal{V}(f_{t,w}) \cap \{ z \in \mathbb{C}^{\ast n} | z_{i_1} = u_{i_1}, \ldots, z_{i_m} = u_{i_m} \} \]

is a reduced non-singular hypersurface in \( \{ z \in \mathbb{C}^{\ast n} | z_{i_1} = u_{i_1}, \ldots, z_{i_m} = u_{i_m} \} \), that is, \( f_{t,w} \) has no critical point in

\[ \mathcal{V}(f_{t,w}) \cap \{ z \in \mathbb{C}^{\ast n} | z_{i_1} = u_{i_1}, \ldots, z_{i_m} = u_{i_m} \} \]

as a function of the \( n-m \) variables \( (z_i)_{i \notin I} \)

\( \triangleright R(f_t) = \text{radius of local tameness for } f_t \)
Example \( f_t = z_1^2z_2^3 + z_1^3z_2^2 + tz_1^2z_2^4 \)

\[ l = \{1\}, \{2\} \text{ or } \{1, 2\} \]

\[ f_t|_{C^*\{1\}}(z_1, 0) = 0 \Rightarrow \{1\} \in \mathcal{V}_{f_t} \]
\[ f_t|_{C^*\{2\}}(0, z_2) = 0 \Rightarrow \{2\} \in \mathcal{V}_{f_t} \]
\[ \{1, 2\} \notin \mathcal{V}_{f_t} \]

\[ w = (w_1, w_2) \] with \( \{i ; w_i = 0\} = \{1\} \), i.e, with \( w_1 = 0 \)
\[ \Rightarrow \Delta(w; f_t) = \{\alpha \in \Gamma_+(f_t) \mid w_2\alpha_2 \text{ minimal}\} \]
\[ \forall u_1 \in \mathbb{C}^* \text{ the function } z_2 \mapsto f_t,w(u_1, z_2) = u_1^3z_2^2 \text{ has no critical points on} \]
\[ \mathcal{V}(f_t,w) \cap \{z \in \mathbb{C}^{*2} \mid z_1 = u_1\} \]

\[ w' = (w'_1, w'_2) \] with \( \{i ; w_i' = 0\} = \{2\} \), i.e, with \( w'_2 = 0 \)
\[ \Rightarrow \Delta(w'; f_t) = \{\alpha \in \Gamma_+(f_t) \mid w'_1\alpha_1 \text{ minimal}\} \]
\[ \forall u_2 \in \mathbb{C}^* \text{ the function } z_1 \mapsto f_t,w'(z_1, u_2) = z_1^2u_2^3 + tz_1^2u_2^4 \text{ has no critical points on} \]
\[ \mathcal{V}(f_t,w') \cap \{z \in \mathbb{C}^{*2} \mid z_2 = u_2\} \]

provided that \( |u_2| < 1/|t| \) \( t \neq 0 \) [because the derivative is \( 2z_1u_2^3(1 + tu_2) \)]

Therefore, \( \mathcal{V}(f_t) \) is locally tame and \( R(f_t) = 1/|t| \) is a radius of local tameness for \( f_t \) if \( t \neq 0 \) while \( R(f_0) = \infty \) is a radius of local tameness for \( f_0 \).
Theorem (M. Oka and CE, 2017) Suppose that for all small $t$:

1. Newton diagram of $f_t$ is independent of $t$
2. $V(f_t)$ is non-degenerate
3. $V(f_t)$ is locally tame and there is a radius of local tameness for $f_t$ which is $> R$ for some $R > 0$ independent of $t$

Then the family $\{ V(f_t) \}$ is Whitney equisingular

This is a generalization of Briançon’s theorem to non-isolated singularities

Example $f_t = z_1^2 z_2^3 + z_1^3 z_2^2 + tz_1^2 z_2^4$

It is locally tame ($R(f_0) = \infty$ and $R(f_t) = 1/|t|$ if $t \neq 0$); we have $R(f_t) > R := 1$ for all $|t| < 1$. It is also non-degenerate, and has constant Newton diagram. Thus the family $\{ V(f_t) \}$ is Whitney equisingular.

Remark

In general, if $h = h_1 \cdot h_2$ and if $\dim_0 (V(h_1) \cap V(h_2)) \geq 1$, then $h$ is never non-degenerate if its Newton diagram intersect each coordinate axis.
Non-degenerate, locally tame, complete intersection variety

$h^1(z), \ldots, h^{k_0}(z)$ polynomial functions on $\mathbb{C}^n$ such that $h^k(0) = 0$

$V(h^1, \ldots, h^{k_0}) = \{ z ; h^1(z) = \cdots = h^{k_0}(z) = 0 \}$ is a non-degenerate, locally tame, complete intersection variety if

1. for any “positive” weight vector $w$:

$$V(h_w^1, \ldots, h_w^{k_0}) \cap \mathbb{C}^*n$$

is a reduced, non-singular, complete intersection variety in $\mathbb{C}^*n$, that is, the $k_0$-form $dh_w^1 \wedge \cdots \wedge dh_w^{k_0}$ is nowhere vanishing in $V(h_w^1, \ldots, h_w^{k_0}) \cap \mathbb{C}^*n$

2. $\exists R(h^1, \ldots, h^{k_0}) > 0$ such that:

- $\forall I := \{ i_1, \ldots, i_m \} \neq \emptyset$ in $\mathcal{V}_h^1 \cap \cdots \cap \mathcal{V}_h^{k_0}$;
- $\forall w = (w_1, \ldots, w_n) \in \mathbb{N}^n \setminus \{0\}$ with $\{ i ; w_i = 0 \} = I$;
- $\forall u_{i_1}, \ldots, u_{i_m} \in \mathbb{C}^*$ with $\sum_{j=1}^m |u_{i_j}|^2 < R(h^1, \ldots, h^{k_0})$;

the intersection

$$V(h_w^1, \ldots, h_w^{k_0}) \cap \{ z \in \mathbb{C}^*n \mid z_{i_1} = u_{i_1}, \ldots, z_{i_m} = u_{i_m} \}$$

is a reduced, non-singular, complete intersection variety in

$\{ z \in \mathbb{C}^*n \mid z_{i_1} = u_{i_1}, \ldots, z_{i_m} = u_{i_m} \}$

$R(h^1, \ldots, h^{k_0}) = \text{radius of local tameness for } \{ h^1, \ldots, h^{k_0} \}$

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Non-degenerate, locally tame complete intersection varieties...
Let $f(t, z) := f^1(t, z) \cdots f^{k_0}(t, z)$, where $f^k(t, z)$ is a polynomial function on $\mathbb{C} \times \mathbb{C}^n$ such that $f^k(t, 0) = 0$; as usual, put $f_t(z) := f(t, z)$ and $f^k_t(z) := f^k(t, z)$.

**Theorem (M. Oka and CE, 2021)** Suppose that for all small $t$:

1. $\forall k = 1, \ldots, k_0$, the Newton diagram of $f^k_t$ is independent of $t$.
2. $\forall \{k_1, \ldots, k_p\} \subseteq \{1, \ldots, k_0\}$:
   - $V(f^k_1, \ldots, f^k_p)$ non-degenerate, locally tame, complete intersection variety
   - there is a radius of local tameness for $\{f^k_1, \ldots, f^k_p\}$ which is $> R$ for some $R > 0$ independent of $t$ and of $\{k_1, \ldots, k_p\}$ (families $\{f_t\}$ satisfying (1) and (2) above are called **Newton-admissible**)

Then $\{V(f_t)\}$ is Whitney equisingular

✍️ This extends all previous theorems

**Example** $f^k(t, z) := \sum_{i=1}^n (a_i(k) + t b_i(k)) z_i^{p_i} z_{i+1}^{q_i} (1 \leq k \leq k_0)$

- $p_i, q_i \in \mathbb{N}^*$ are so that there exists a positive weight vector $w = (w_1, \ldots, w_n)$ so that $f^k_t(z), \ldots, f^{k_0}_t(z)$ are weighted homogeneous polynomials of $w$-degree $d$
- For generic $a_i(k), b_i(k) \in \mathbb{C}^*$, the family $\{f_t\}$ is Newton-admissible, and hence $\{V(f_t)\}$ is Whitney equisingular
Sketch of the proof

A Whitney \((b)\)-regular stratification of \(V(f)\) answering the theorem (i.e., having \(C \times \{0\}\) as a stratum) is given by the canonical toric stratification of \(V(f)\)

- For any \(I \subseteq \{1, \ldots, n\}\) and \(K \subseteq \{1, \ldots, k_0\}\) let

\[
S^I(K) := \bigcap_{k \in K} \left( V(f^k) \cap (C \times C^*^I) \right) \setminus \bigcup_{k \in K^c} \left( V(f^k) \cap (C \times C^*^I) \right)
\]

where \(C^*^I := \{ z \in C^n ; z_i = 0 \iff i \notin I \}\); note that \(S^\emptyset(\{1, \ldots, k_0\}) = C \times \{0\}\).

- Non-degeneracy condition \(\Rightarrow S^I(K)\) is smooth (near 0)

It follows that the collection of all sets \(S^I(K)\) is a complex analytic stratification of \(V(f)\); it is called the canonical toric stratification of \(V(f)\).

- We must show that for any

\[
I \subseteq J \subseteq \{1, \ldots, n\} \quad \text{and} \quad L \subseteq K \subseteq \{1, \ldots, k_0\} \quad \text{with} \quad S^I(K) \cap S^J(L) \neq \emptyset,
\]

\(S^J(L)\) is Whitney \((b)\)-regular over \(S^I(K)\) at any point \((t^0, z^0) \in S^I(K) \cap S^J(L)\).
Pick real analytic paths
\[ \rho(s) := (t(s), z(s)) \quad \text{and} \quad \rho'(s) := (t'(s), z'(s)) \]
in \( \mathbb{C} \times \mathbb{C}^n \) such that:

1. \( \rho(0) = \rho'(0) = (t^0, z^0) \)
2. \( \rho'(s) \in S^I(K) \) and \( \rho(s) \in S^J(L) \) for \( s \neq 0 \)

Put \( \ell(s) := \rho(s) - \rho'(s) \). By the curve selection lemma, it suffices to prove that

\[ \ell_\infty := \lim_{s \to 0} \frac{\ell(s)}{\| \ell(s) \|} \in T_\infty := \lim_{s \to 0} T_{\rho(s)} S^J(L). \]

To simplify, assume \( J = \{1, \ldots, n\} \) and \( L = \{1, \ldots, k_L\} \). Note that

\[ T_{\rho(s)} S^J(L) = (df^1(\rho(s)) \wedge \cdots \wedge df^{k_L}(\rho(s)))^\perp \]
\[ = \{ v \in T_{\rho(s)}(\mathbb{C} \times \mathbb{C}^n) \mid \langle v, (df^1(\rho(s)) \wedge \cdots \wedge df^{k_L}(\rho(s))) \rangle = 0 \}. \]

Write \( o_{k_L} := \operatorname{ord}_s df^1(\rho(s)) \wedge \cdots \wedge df^{k_L}(\rho(s)) \) and put

\[ \omega_\infty := \lim_{s \to 0} \frac{1}{s^{o_{k_L}}} \cdot df^1(\rho(s)) \wedge \cdots \wedge df^{k_L}(\rho(s)). \]

Then \( T_\infty = \omega_\infty^\perp \), and we must prove that \( \ell_\infty \in \omega_\infty^\perp \).
To do that, we show that \( \exists \) linearly independent 1-forms \( \omega_1, \ldots, \omega_{k_L} \) such that

\[
\omega_\infty = \omega_1 \wedge \cdots \wedge \omega_{k_L}
\]

(then, of course, it will be enough to prove that \( \omega_k(\ell_\infty) = 0 \) for all \( 1 \leq k \leq k_L \))

The main difficulty is that, in general, if \( I \in V_{f_t^k} \) for some of the functions \( f_t^k \), then the limits of the normalized 1-forms

\[
df^1(\rho(s)), \ldots, df^{k_L}(\rho(s))
\]

as \( s \to 0 \) are not linearly independent. The subterfuge used to solve this problem is to replace the corresponding differential \( df^k(\rho(s)) \) with a term of the form

\[
df^k(\rho(s)) + \sum_{k'=1}^{k-1} c_{k,k'}(s) df^{k'}(\rho(s)),
\]

where \( c_{k,k'}(s) \) are suitable polynomials.

The forms \( \omega_k \) corresponding to the functions \( f^k \) such that \( I \in V_{f_t^k} \) can be constructed using the above subterfuge and the uniform local tameness condition.

The forms \( \omega_k \) corresponding to the functions \( f^k \) such that \( I \notin V_{f_t^k} \) are constructed using the non-degeneracy condition.
Application: Thom’s $a_f$ condition

Pick a sufficiently small representative of (the germ at 0 of)

$$f(t, z) := f^1(t, z) \cdots f^{k_0}(t, z)$$

so that 0 is the only possible critical value of $f$. Then we have:

**Theorem (Parusiński and Briançon-Maisonobe-Merle)** If $S$ is a Whitney $(b)$-regular stratification of $V(f)$, then $S$ satisfies Thom’s $a_f$ condition, that is, for any stratum $S$, any point $p \in S$, and any sequence $\{p_q\} \notin V(f)$ such that $p_q \to p$ and $T_p V(f - f(p_q)) \to T$, we have $T_p S \subseteq T$

**Theorem (M. Oka and CE, 2021)** If $\{f_t\}$ is Newton-admissible, then the canonical toric stratification $S$ of $V(f)$ satisfies Thom’s $a_f$ condition

**Proof** By our theorem, $S$ is Whitney $(b)$-regular. Then we apply the theorem of Parusiński and Briançon-Maisonobe-Merle.
Application to the Milnor fibration

Theorem (M. Oka and CE, 2021) If \( \{f_t\} \) is Newton-admissible, then the Milnor fibrations of \( f_0 \) and \( f_t \) at 0 are isomorphic for all small \( t \)

Ingredients of the proof

1. The canonical toric stratification \( S \) of \( V(f) \) satisfies Thom’s \( a_f \) condition
2. For any \( t \) and any \( \varepsilon > 0 \) sufficiently small

\[
\{t\} \times S_{\varepsilon} \cap ((\{t\} \times \mathbb{C}^n) \cap S)
\]

for any \( S \in S \) (this follows from the Whitney \((b)\)-regularity condition)

3. Using (1) and (2) we show that \( \{f_t\} \) has a uniform stable radius for the Milnor fibrations of the \( f_t \)'s at 0

4. Theorem (Oka) \( \{f_t\} \) has a uniform stable radius \( \Rightarrow \) the Milnor fibrations of \( f_0 \) and \( f_t \) at 0 are isomorphic for all small \( t \)
Thank you for your attention!