Zariski multiplicity conjecture via Floer cohomology

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  - $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ holomorphic germ; $f \in \mathbb{C}[\![z_1, \ldots, z_n]\!]$
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  - **False:** $f = z_1^2 + \cdots + z_n^2$, $\Lambda(\phi^2) = \Lambda(\mathrm{id}) = 0$ if $2|n$.

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- $n > 3$; $\mu < 2^n - 1$; or $n = 3$ and $\mu \leqslant 26$; or $n = 3$, $p_g \leqslant 3$ [Yau–Zhuo '18]

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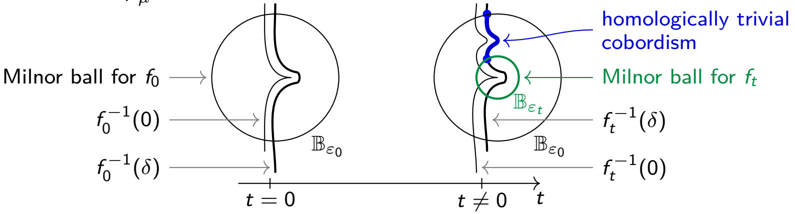

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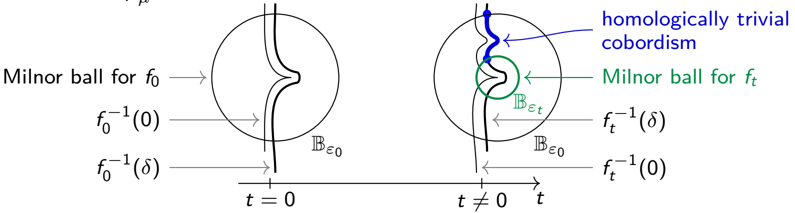

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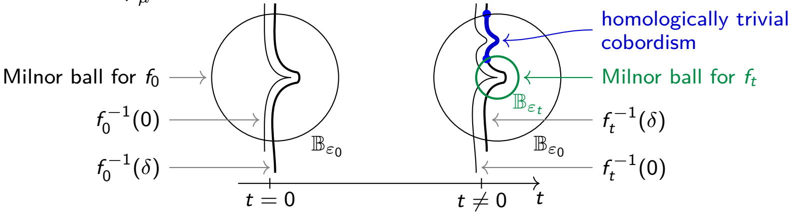

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Problem: cannot isotope ϕ_0 to ϕ_t, because the Milnor radius can shrink! $\square$

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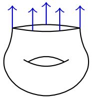

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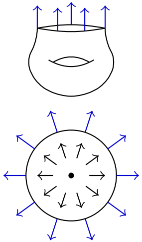

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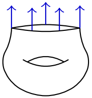

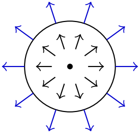

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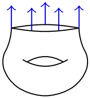

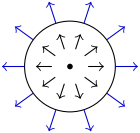

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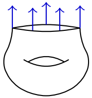

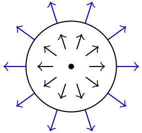

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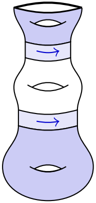

- $B\subseteq M$ is a codimension zero family of fixed points if:
  - $\phi|_B=\mathrm{id}$
  - B - submanifold with boundary (and corners), $\operatorname{codim}_M B=0$,
  - $\exists$ N - neighborhood of B, $H\colon N\to(-\infty,0]$ smooth:
    - $B=H^{-1}(0)$
    - $\phi|_N$ is the time one Hamiltonian flow of H.

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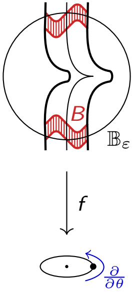

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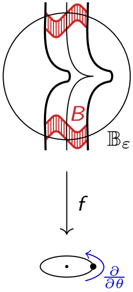

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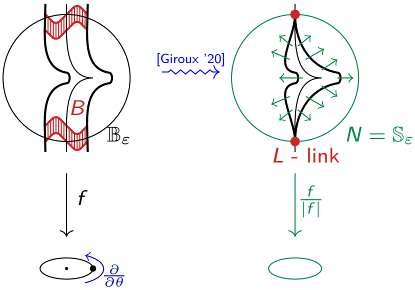

- ε - Milnor radius $\implies$ associated contact pair $(N,L)=(\mathbb{S}_\varepsilon,\mathbb{S}_\varepsilon\cap f^{-1}(0))$.

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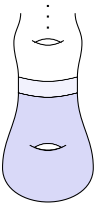
Property 2: (Morse–Bott-type spectral sequence)

- Assume $\mathrm{Fix}(\phi) = \bigsqcup_{i \in I} B_i$, B_i - codimension zero family



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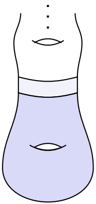

(M, λ, ϕ) - acob $\rightsquigarrow$ $\mathrm{HF}^*(\phi)$ - fixed point Floer cohomology

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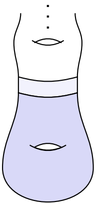

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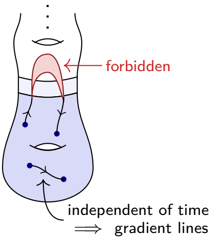

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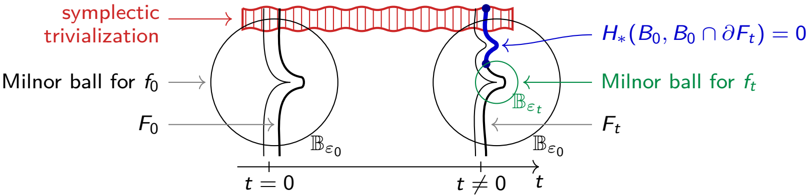

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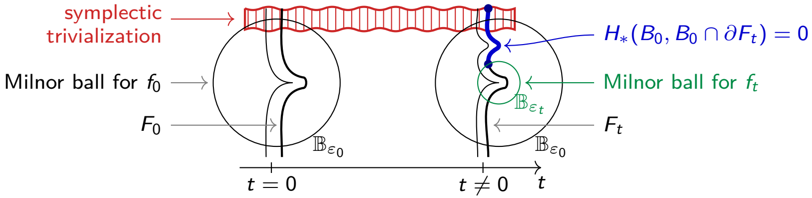

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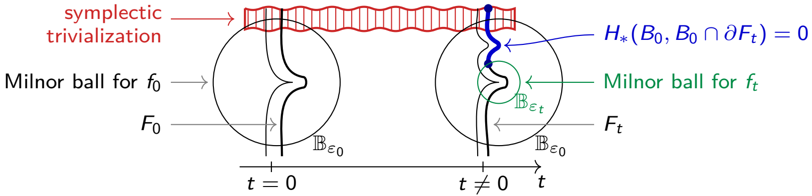

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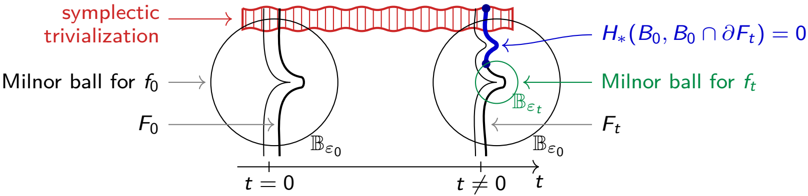

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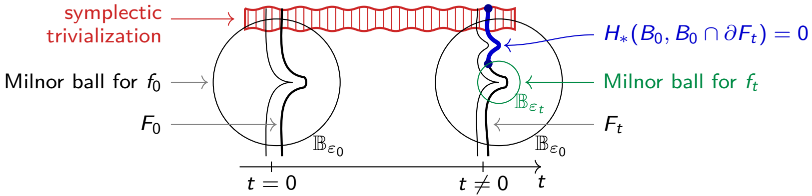

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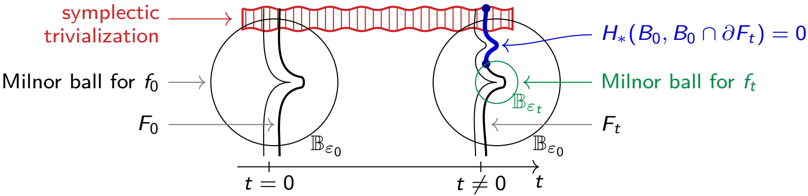

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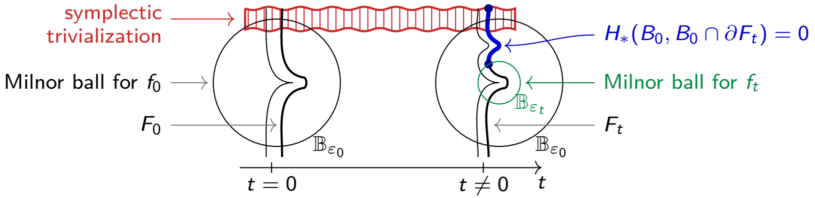

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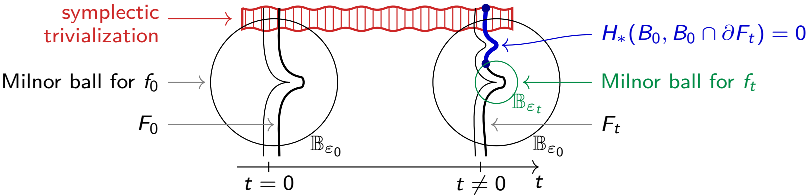

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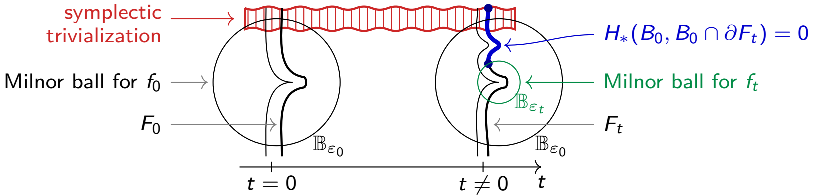

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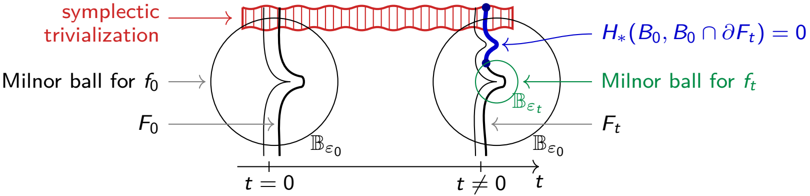

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\Delta_D & \partial A \arrow[l, two heads] \arrow[r, hook] \arrow[d, "radius zero"', "f_{\mathrm{AC}}|_{\partial A}"] & A \arrow[r, "\pi"] \arrow[d, "f_{\mathrm{AC}}"] & X \arrow[d, "f"] \\
& \mathbb{S}^1 \arrow[r, hook] & \mathbb{C}_{\log} \arrow[r] & \mathbb{C}
\end{tikzcd}$$

- $\mu\colon \partial A \to \Delta^N$, $\mu(\partial A)$ - dual complex of D, i.e. $\Delta_D = \bigcup_{X_I \neq \emptyset} \Delta_I$
- $(\pi, \mu)|\colon X_I^\circ \to X_I^\circ \times \Delta_I$ is an $(\mathbb{S}^1)^{\#I}$-bundle.

Principle: passing to radius zero makes the choices irrelevant.

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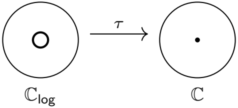

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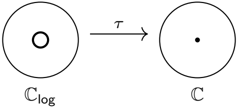

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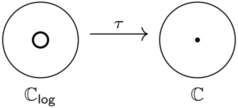

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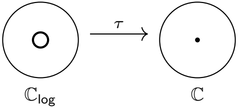

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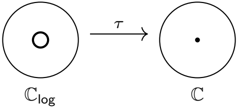

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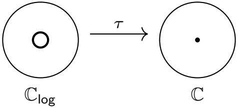

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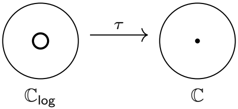

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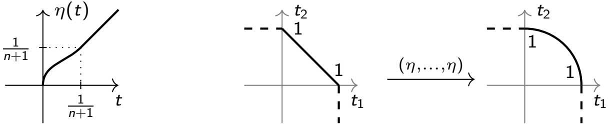

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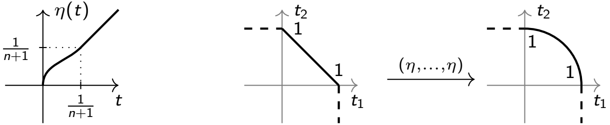

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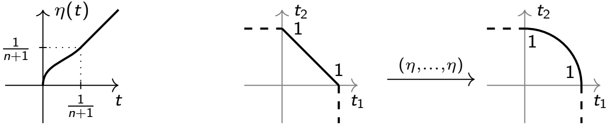

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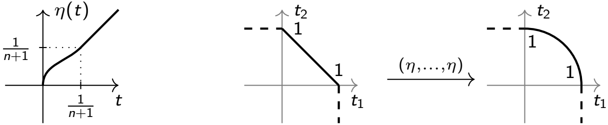

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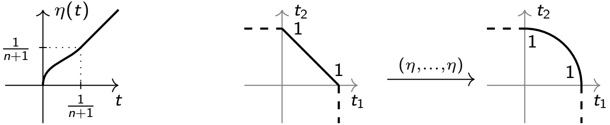

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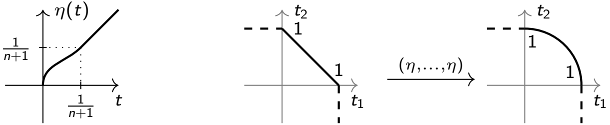

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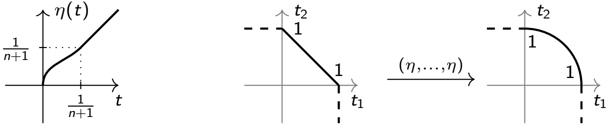

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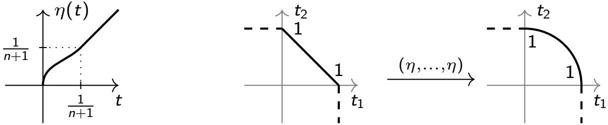

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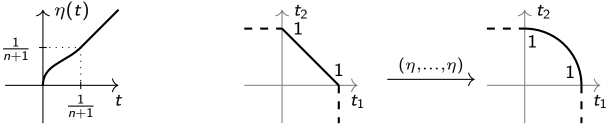

- Fix an atlas $\{(z_1^p,\dots,z_n^p)\colon U^p \to \mathbb{D}^n\}_p$, $f|_{U^p} = (z_1^p)^{m_1}\cdot\ldots\cdot(z_k)^{m_k}$
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  - tropical coordinates $t_j^p = (-m_j \log r_j^p)^{-1}$; $t = (-\log|f|)^{-1} = (\sum_j (t_j^p)^{-1})^{-1}$.
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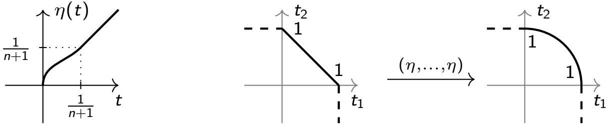

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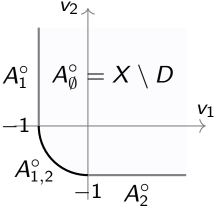

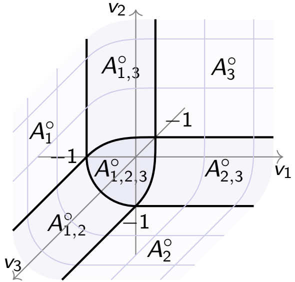

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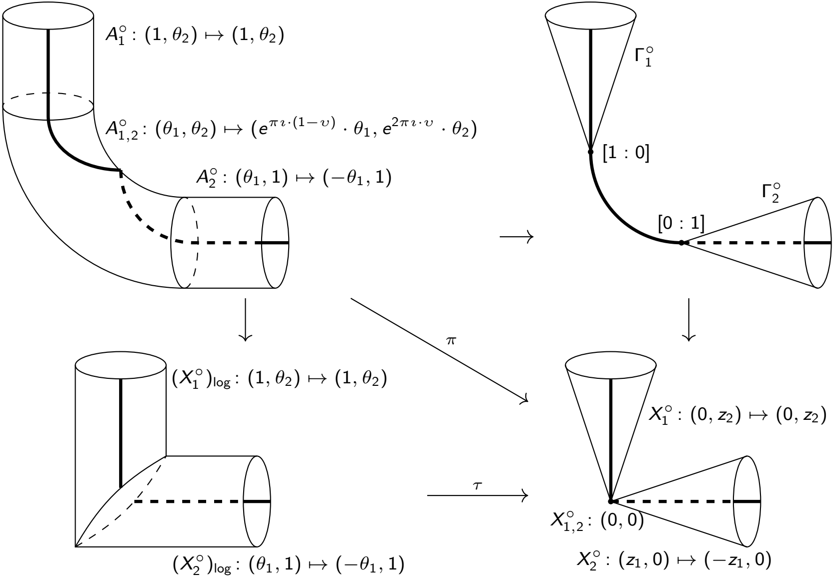
  - m-separatedness: ϕ^m has no fixed points on A_I°, $\#I \geqslant 2$.

Example: $f = z_1^2 z_2 \colon \mathbb{C}^2 \to \mathbb{C}$.



Thank you!