Minimality and Uniqueness for Decompositions of Symmetric Tensors

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Based on the following joint works with Luca Chiantini:

"Minimality and uniqueness for dec. of specific ternary forms" (2020) to appear on Math.Comp.

"On the description of identifiable quartics" (2021) arXiv:2106.06730 [math.AG]

IMPANGA seminar 2021/2022

December 17, 2021

PRELIMINARIES

- Notation
- Kruskal's ranks and criterion
- The Hilbert function

BEYOND KRUSKAL'S BOUND

- The case of ternary forms
- Focus on ternary nonics
- The special case of quartics in 5 variables

REFERENCES

- $d, n \in \mathbb{N}$
- $\mathbb{C}^{n+1}$: {linear forms in $x_0, \dots, x_n$ / $\mathbb{C}$}
- $S^d\mathbb{C}^{n+1}$: {forms of degree d in $x_0, \dots, x_n$ / $\mathbb{C}$}
- $T \in S^d\mathbb{C}^{n+1} \rightsquigarrow [T] \in \mathbb{P}(S^d\mathbb{C}^{n+1}) \cong \mathbb{P}^N$, $N = \binom{n+d}{d} - 1$
- $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ d^{th}-*Veronese embedding* of $\mathbb{P}^n$
  $\nu_d([a_0x_0 + \ldots + a_nx_n]) = [(a_0x_0 + \ldots + a_nx_n)^d]$
- $A = \{L_1, \dots, L_{\ell(A)}\} \subset \mathbb{P}^n \rightsquigarrow \langle \nu_d(A) \rangle = \langle L_1^d, \dots, L_{\ell(A)}^d \rangle$

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Let $A \subset \mathbb{P}^n$ be a finite set and let $T \in \mathbb{P}^N$

Definition

- A is a *computes* $T \in \mathbb{P}^N$ if $T \in \langle \nu_d(A) \rangle$, i.e.
$$T = \lambda_1 L_1^d + \ldots + \lambda_{\ell(A)} L_{\ell(A)}^d, \ \exists\ \lambda_1, \ldots, \lambda_{\ell(A)} \in \mathbb{C}$$
- A is *non-redundant* if $T \in \langle \nu_d(A) \rangle \wedge \forall\ A' \subsetneq A\ T \notin \langle \nu_d(A') \rangle$
- A is *minimal* if $T \in \langle \nu_d(A) \rangle \wedge \ell(A) = \min \{\ell(B) \,|\, T \in \langle \nu_d(B) \rangle\}$
$\Longrightarrow r = \ell(A)$ is the *rank* of T
$\Longrightarrow$ *A computes the rank of T*
- T is *identifiable* if $\exists\,!\ A$ computing the rank of T

Applications: engineering [AGHKT], chemistry [AD],...

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The *j-th Kruskal's rank* of A is

$$k_j(A) = \max \left\{ k \;\middle|\; \forall \textit{ submatrix of } \begin{pmatrix} L_1^j \\ \vdots \\ L_{\ell(A)}^j \end{pmatrix} \textit{ with k rows has rank } k \right\}$$

Remarks

- $k_j(A) \leq \min\{\ell(A), N+1\}$
- A general $\Longrightarrow k_j(A)$ maximal

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Theorem: Kruskal's criterion for forms [COV17]

Let $d \geq 3$, let $A \subset \mathbb{P}^n$ be a non-redundant set computing $T \in \mathbb{P}^N$. Assume that $d = d_1 + d_2 + d_3$, with $d_1 \geq d_2 \geq d_3 \geq 1$. If

$$\ell(A) \leq \frac{k_{d_1}(A) + k_{d_2}(A) + k_{d_3}(A) - 2}{2} \quad (1)$$

then T has rank $\ell(A)$ and it is identifiable.

Kruskal's bound for $S^9\mathbb{C}^3$ and $S^4\mathbb{C}^5$

- ✓ $n = 2, d = 9 = 4 + 4 + 1$:
  $k_4(A) = \min\{\ell(A), 15\},\ k_1(A) = \min\{\ell(A), 3\} \Longrightarrow \ell(A) \leq 15$
- ✓ $n = 4, d = 4 = 2 + 1 + 1$:
  $k_2(A) = \min\{\ell(A), 15\},\ k_1(A) = \min\{\ell(A), 5\} \Longrightarrow \ell(A) \leq 8$

• [COV17]: L. Chiantini, G. Ottaviani, N. Vannieuwenhoven *"Effective criteria for specific identifiability of tensors and forms"* SIAM J. Matrix Anal. Appl. 38 (2017) 656-681

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- *Hilbert function* of A

$$h_A : \mathbb{N} \longrightarrow \mathbb{N}, \quad h_A(j) = rank \begin{pmatrix} L_1^j \\ \vdots \\ L_{\ell(A)}^j \end{pmatrix}$$

- *First difference of the Hilbert function* of A

$$Dh_A(j) = h_A(j) - h_A(j-1)$$

r general points in $\mathbb{P}^4$, with $9 \leq r \leq 13$

j	0	1	2	3	...
$h_A(j)$	1	5	r	r	...
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Theorem: beyond Kruskal's bound for ternary forms [AC20]

Let $T \in \mathbb{P}(S^d\mathbb{C}^3)$ and let $A \subset \mathbb{P}^2$ be a non-redundant finite set computing T. The form T is identifiable of rank $\ell(A)$ if one of the following occurs:

- $d = 2m$
  - $k_{m-1}(A) = \min\{\ell(A), \binom{m+1}{2}\}$
  - $h_A(m) = \ell(A) \leq \binom{m+2}{2} - 2$
- $d = 2m+1$
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Beyond Kruskal's bound for $S^9\mathbb{C}^3$

✓ $n = 2, d = 9 = 2 \cdot 4 + 1 \Longrightarrow m = 4 \Longrightarrow \ell(A) \leq 17$

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Assume that $n = 2$ and $d = 9$. Consider:

- $\star$ $T \in \mathbb{P}(S^9\mathbb{C}^3) \cong \mathbb{P}^{54}$
- $\star$ $A = \{L_1, \ldots, L_{\ell(A)}\} \subset \mathbb{P}^2$ non-redundant set computing T
  - ✓ $k_4(A) = \min\{\ell(A), 15\}$
  - ✓ $h_A(5) = \ell(A)$

Classically known case

▷ T general $\Rightarrow$ T has rank 19 [AH] $\Rightarrow$ T is not identifiable [GM]

$$\ell(A) = 18?$$

• [AH]: J. Alexander, A. Hirschowitz *"Polynomial interpolation in several variables"* J. Algebraic Geom. 4 (2) (1995) 201-222

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Hilbert function, first difference and ideal of A

j	0	1	2	3	4	5	6	...
$h_A(j)$	1	3	6	10	15	18	18	...
$Dh_A(j)$	1	2	3	4	5	3	0	...

▷ $0 \longrightarrow R(-7)^{\oplus 3} \xrightarrow{M} R(-5)^{\oplus 3} \oplus R(-6) \longrightarrow I_A \longrightarrow 0$

$$R = \mathbb{C}[x_0, x_1, x_2],\ M = \begin{pmatrix} q_1 & q_2 & q_3 \\ q_4 & q_5 & q_6 \\ q_7 & q_8 & q_9 \\ \ell_1 & \ell_2 & \ell_3 \end{pmatrix},\ I_A = (Q_1, Q_2, Q_3, S)$$

- $B = \{L'_1, \dots, L'_{\ell(B)}\} \subset \mathbb{P}^2$ another non-redundant dec. of T with $\ell(B) \leq 18$ and $A \cap B = \emptyset$

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⋆ $B = \{L'_1, \dots, L'_{\ell(B)}\} \subset \mathbb{P}^2$ another non-redundant dec. of T with $\ell(B) \leq 18$ and $A \cap B = \emptyset$

⋆ $A = \{L_1, \dots, L_{18}\} \subset \mathbb{P}^2$ non-redundant set computing T

✓ $k_4(A) = 15$

✓ $h_A(5) = 18$

Hilbert function, first difference and ideal of A

▷

j	0	1	2	3	4	5	6	...
$h_A(j)$	1	3	6	10	15	18	18	...
$Dh_A(j)$	1	2	3	4	5	3	0	...

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Let A and B be as above and $Z = A \cup B$. Then:

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Let A and B be as above and $Z = A \cup B$. Then:

- ✓ $Dh_Z(10) > 0$ [AC20];
- ✓ $Dh_Z(0) + \ldots + Dh_Z(j) \leq Dh_Z(10-j) + \ldots + Dh_Z(10)$ $\forall j \in \{0, \ldots, 10\}$, [ACV].

- [AC20]: E. Angelini, L. Chiantini *"On the identifiability of ternary forms"* Lin. Alg. Appl. 599 (2020) 36-65
- [ACV]: E. Angelini, L. Chiantini, N. Vannieuwenhoven *"Identifiability beyond Kruskal's bound for symmetric tensors of degree 4"* Rend. Lincei Mat. Applic. 29 (2018), 465-485

Proposition: possible cases for Dh_Z [AC21]

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j	0	1	2	3	4	5	6	7	8	9	10	11	...
$Dh_Z(j)$	1	2	3	4	5	6	5	4	3	2	1	0	...

and $\ell(B) = \mathbf{18}$, $\ell(Z) = 36$

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Theorem: description of (T,A) in Case 1 [AC21]

The map $f : \mathbb{G}(1,9) \dashrightarrow \langle \nu_9(A) \rangle$ defined by

$$f(W) = \langle \nu_9(A) \rangle \cap \langle \nu_9(B) \rangle = \mathbb{P}((I_A)_9 + (I_B)_9)^\vee$$

is a linear projection of $\mathbb{G}(1,9)$ and is birational onto the image.

Corollaries

- ⋆ $\langle \nu_9(A) \rangle$ contains $f(\mathbb{G}(1,9))$, a hypersurface of ternary nonics with 2 dec. computing the rank
- ⋆ $T \notin f(\mathbb{G}(1,9)) \Rightarrow T$ has rank 18 and A is the unique dec.

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Corollary

- $\star$ $\langle \nu_9(A) \rangle$ contains $f'(E)$, a 13-dim'l subvariety of ternary nonics T for which A is a non-redundant but not minimal dec. and the rank of T is 17.

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The algorithm for minimality

INPUT $A = \{L_1, \ldots, L_{18}\} \subset \mathbb{P}^2$

$T = \sum_{i=1}^{18} \lambda_i L_i^9 = [(t_0, \ldots, t_{54})]$

PROCEDURE

- check that:
  1) $\dim\langle L_1^9, \ldots, L_{18}^9\rangle = 18$ ✓
  2) $k_4(A) = 15$ ✓
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- construct the ideal I_A and its Hilbert-Burch matrix M
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- assume $a_1 = 1$ and $q_{10} = 0$
  (resp. $a_2 = 1$ and $q_{11} = 0$, $a_3 = 1$ and $q_{12} = 0$)
- construct the matrix N_1 whose rows yield a set of generators for $(I_A)_9 + (I_B)_9$
  (resp. N_2, N_3)
- compute

$$d_1 = \max_{4 \leq i \leq 15} \dim \left\{ (a_2, \ldots, a_{15}) \in \mathbb{C}^{14} \,|\, N_1 \cdot \begin{pmatrix} t_0 \\ \vdots \\ t_{54} \end{pmatrix} = \underline{0} \cap a_i = 1 \right\}$$

(resp. d_2, d_3)

- check that:

4) $d_1 = d_2 = d_3 = -1$ ✓

OUTPUT A is minimal for T, i.e. T has rank 18

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Example 1

INPUT $A = \begin{pmatrix} 1 & 0 & -1 & 1 & 1 & 2 & 4 & 1 & 5 & 6 & 1 & 1 & 6 & -7 & 3 & 2 & 6 & -7 \\ 1 & 1 & 2 & 2 & -2 & 1 & 2 & 5 & 2 & 2 & 7 & 7 & 5 & 2 & 7 & -5 & 3 & 6 \\ 1 & 2 & 1 & 3 & 0 & 4 & -3 & 1 & 3 & 3 & 7 & 3 & 4 & 3 & 4 & 1 & -4 & 6 \end{pmatrix}$

$$T = \sum_{i=1}^{18} L_i^9 = [4283x_0^9 - 14212x_0^8x_1 + 2365x_0^7x_1^2 - 11335x_0^6x_1^3 + 10354x_0^5x_1^4 +$$
$$-7342x_0^4x_1^5 + 11432x_0^3x_1^6 - 15881x_0^2x_1^7 - 10204x_0x_1^8 - 663x_1^9 - 10837x_0^8x_2 - 6573x_0^7x_1x_2 +$$
$$+6070x_0^6x_1^2x_2 - 12124x_0^5x_1^3x_2 + 8455x_0^4x_1^4x_2 - 9097x_0^3x_1^5x_2 + 200x_0^2x_1^6x_2 + 11563x_0x_1^7x_2 +$$
$$+11173x_1^8x_2 + 2810x_0^7x_2^2 + 5187x_0^6x_1x_2^2 - 1688x_0^5x_1^2x_2^2 - 3089x_0^4x_1^3x_2^2 + 8745x_0^3x_1^4x_2^2 +$$
$$+12508x_0^2x_1^5x_2^2 + 151x_0x_1^6x_2^2 + 11119x_1^7x_2^2 + 11414x_0^6x_2^3 + 2714x_0^5x_1x_2^3 + 11939x_0^4x_1^2x_2^3 +$$
$$+5024x_0^3x_1^3x_2^3 + 10884x_0^2x_1^4x_2^3 + 8404x_0x_1^5x_2^3 + 755x_1^6x_2^3 + 15891x_0^5x_2^4 - 1013x_0^4x_1x_2^4 +$$
$$-11790x_0^3x_1^2x_2^4 + 14982x_0^2x_1^3x_2^4 - 8411x_0x_1^4x_2^4 - 5236x_1^5x_2^4 + 4416x_0^4x_2^5 - 11481x_0^3x_1x_2^5 +$$
$$+14698x_0^2x_1^2x_2^5 + 5309x_0x_1^3x_2^5 + 11614x_1^4x_2^5 - 9777x_0^3x_2^6 - 2702x_0^2x_1x_2^6 - 5846x_0x_1^2x_2^6 +$$
$$-10960x_1^3x_2^6 - 8430x_0^2x_2^7 + 7085x_0x_1x_2^7 + 12763x_1^2x_2^7 - 14136x_0x_2^8 - 9808x_1x_2^8 + 9194x_2^9]$$

PROCEDURE all the 4 tests provide positive answer

OUTPUT T has rank 18

Example 1

INPUT $A = \begin{pmatrix} 1 & 0 & -1 & 1 & 1 & 2 & 4 & 1 & 5 & 6 & 1 & 1 & 6 & -7 & 3 & 2 & 6 & -7 \\ 1 & 1 & 2 & 2 & -2 & 1 & 2 & 5 & 2 & 2 & 7 & 7 & 5 & 2 & 7 & -5 & 3 & 6 \\ 1 & 2 & 1 & 3 & 0 & 4 & -3 & 1 & 3 & 3 & 7 & 3 & 4 & 3 & 4 & 1 & -4 & 6 \end{pmatrix}$

$$T = \sum_{i=1}^{18} L_i^9 = [4283x_0^9 - 14212x_0^8x_1 + 2365x_0^7x_1^2 - 11335x_0^6x_1^3 + 10354x_0^5x_1^4 +$$
$$-7342x_0^4x_1^5 + 11432x_0^3x_1^6 - 15881x_0^2x_1^7 - 10204x_0x_1^8 - 663x_1^9 - 10837x_0^8x_2 - 6573x_0^7x_1x_2 +$$
$$+6070x_0^6x_1^2x_2 - 12124x_0^5x_1^3x_2 + 8455x_0^4x_1^4x_2 - 9097x_0^3x_1^5x_2 + 200x_0^2x_1^6x_2 + 11563x_0x_1^7x_2 +$$
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PROCEDURE all the 4 tests provide positive answer

OUTPUT T has rank 18

Example 2

INPUT A as in Example 1

$$T = \sum_{i=1}^{18} \lambda_i L_i^9 = [9666x_0^9 + 13004x_0^8x_1 + 12463x_0^7x_1^2 - 13235x_0^6x_1^3 - 15442x_0^5x_1^4 +$$
$$+15509x_0^4x_1^5 - 6311x_0^3x_1^6 - 2390x_0^2x_1^7 + 547x_0x_1^8 - 119x_1^9 - 14916x_0^8x_2 + 1822x_0^7x_1x_2 +$$
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$$+4417x_1^8x_2 - 11823x_0^7x_3^2 - 737x_0^6x_1x_1^2 - 7616x_0^5x_1^2x_2^2 + 11293x_0^4x_1^3x_2^2 - 8260x_0^3x_1^4x_2^2 +$$
$$-9332x_0^2x_1^5x_2^2 + 7078x_0x_1^6x_2^2 - 4553x_1^7x_2^2 - 15941x_0^6x_2^3 + 4339x_0^5x_1x_2^3 - 4251x_0^4x_1^2x_2^3 +$$
$$+9854x_0^3x_1^3x_2^3 - 22x_0^2x_1^4x_2^3 + 8408x_0x_1^5x_2^3 + 11858x_1^6x_2^3 - 9161x_0^5x_2^4 - 9854x_0^4x_1x_2^4 +$$
$$-13165x_0^3x_1^2x_2^4 - 2105x_0^2x_1^3x_2^4 - 8715x_0x_1^4x_2^4 + 390x_1^5x_2^4 - 9955x_0^4x_2^5 - 11013x_0^3x_1x_2^5 +$$
$$-10651x_0^2x_1^2x_2^5 - 3850x_0x_1^3x_2^5 + 4029x_1^4x_2^5 - 11735x_0^3x_2^6 - 12427x_0^2x_1x_2^6 + 12255x_0x_1^2x_2^6 +$$
$$-3686x_1^3x_2^6 - 2271x_0^2x_2^7 + 5939x_0x_1x_2^7 - 3402x_1^2x_2^7 + 13298x_0x_2^8 + 6455x_1x_2^8 + x_2^9]$$

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$$-13629, 7308, -8496, 2940, 11348, -12437, -6712, 4086, -823, -2818)$$

PROCEDURE tests $1), 2), 3)$ ✓ but test $4)$ ×

OUTPUT T has rank 17, computed by...

Example 2

INPUT A as in Example 1

$$T = \sum_{i=1}^{18} \lambda_i L_i^9 = [9666x_0^9 + 13004x_0^8x_1 + 12463x_0^7x_1^2 - 13235x_0^6x_1^3 - 15442x_0^5x_1^4 +$$
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PROCEDURE tests $1), 2), 3)$ ✓ but test $4)$ ×

OUTPUT T has rank 17, computed by...

Example 2

INPUT A as in Example 1

$$T=\sum_{i=1}^{18}\lambda_i L_i^9 = [9666x_0^9+13004x_0^8x_1+12463x_0^7x_1^2-13235x_0^6x_1^3-15442x_0^5x_1^4+ +15509x_0^4x_1^5-6311x_0^3x_1^6-2390x_0^2x_1^7+547x_0x_1^8-119x_1^9-14916x_0^8x_2+1822x_0^7x_1x_2+ -8022x_0^6x_1^2x_2-9386x_0^5x_1^3x_2-2742x_0^4x_1^4x_2+10541x_0^3x_1^5x_2+1156x_0^2x_1^6x_2-12023x_0x_1^7x_2+ +4417x_1^8x_2-11823x_0^7x_3^2-737x_0^6x_1x_1^2-7616x_0^5x_1^2x_2^2+11293x_0^4x_1^3x_2^2-8260x_0^3x_1^4x_2^2+ -9332x_0^2x_1^5x_2^2+7078x_0x_1^6x_2^2-4553x_1^7x_2^2-15941x_0^6x_2^3+4339x_0^5x_1x_2^3-4251x_0^4x_1^2x_2^3+ +9854x_0^3x_1^3x_2^3-22x_0^2x_1^4x_2^3+8408x_0x_1^5x_2^3+11858x_1^6x_2^3-9161x_0^5x_2^4-9854x_0^4x_1x_2^4+ -13165x_0^3x_1^2x_2^4-2105x_0^2x_1^3x_2^4-8715x_0x_1^4x_2^4+390x_1^5x_2^4-9955x_0^4x_2^5-11013x_0^3x_1x_2^5+ -10651x_0^2x_1^2x_2^5-3850x_0x_1^3x_2^5+4029x_1^4x_2^5-11735x_0^3x_2^6-12427x_0^2x_1x_2^6+12255x_0x_1^2x_2^6+ -3686x_1^3x_2^6-2271x_0^2x_2^7+5939x_0x_1x_2^7-3402x_1^2x_2^7+13298x_0x_2^8+6455x_1x_2^8+x_2^9]$$

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OUTPUT T has rank 17, computed by...

OUTPUT

$$B = \begin{pmatrix} 1 & 62.6659 & 29.7378 \\ 1 & 13.368 + 38.1825\,i & -19.099 + 7.53788\,i \\ 1 & 13.368 - 38.1825\,i & -19.099 - 7.53788\,i \\ 1 & 35.333 & 40.797 \\ 1 & 14.7061 & 27.8538 \\ 1 & 10.7119 & 4.95399 \\ 1 & -0.796312 & 2.23381 \\ 1 & 1.06064 + 0.13583\,i & 1.62951 - 0.563286\,i \\ 1 & 1.06064 - 0.13583\,i & 1.62951 + 0.563286\,i \\ 1 & 0.737271 & -0.0631582 \\ 1 & -0.245331 & -0.76262 \\ 1 & -0.187307 & 0.100519 \\ 1 & -0.0870499 & -0.126324 \\ 1 & 0.00104432 & 0.00164595 \\ 1 & 0.306581 + 0.0182712\,i & -0.877193 - 0.031211\,i \\ 1 & 0.306581 + 0.0182712\,i & -0.877193 + 0.031211\,i \\ 1 & 0.390447 & 0.585521 \end{pmatrix}$$

Assume that $n = 5$ and $d = 4$. Consider:

- ⋆ $T \in \mathbb{P}(S^4\mathbb{C}^5) \cong \mathbb{P}^{69}$
- ⋆ $A = \{L_1, \dots, L_{\ell(A)}\} \subset \mathbb{P}^4$ non-redundant set computing T
  - ✓ $k_1(A) = 5$
  - ✓ $k_2(A) = \min\{\ell(A), 15\}$

Classically known cases

- ▷ Kruskal's bound: $\ell(A) \leq 8$
- ▷ T general of rank $\geq 14 \Longrightarrow T$ is not identifiable [AH], [CO]

$$9 \leq \ell(A) \leq 13\,?$$

• [AH]: J. Alexander, A. Hirschowitz *"Polynomial interpolation in several variables"* J. Algebraic Geom. 4 (2) (1995) 201-222

• [CO]: L. Chiantini, G. Ottaviani *"On generic identifiability of 3-tensors of small rank"* SIAM J Matrix Anal Appl. 33 (2012), 1018-1037

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Theorem: minimality result [AC]

Let $T \in \mathbb{P}(S^4\mathbb{C}^5)$ be a form with a dec. $A \subset \mathbb{P}^4$ of $\ell(A) \leq 13$ and

1) A is non-redundant
2) $k_1(A) = 5$
3) $k_2(A) = \ell(A)$

Then A is minimal for T, i.e. $\ell(A)$ is the rank of T.

- [AC]: E. Angelini, L. Chiantini *"On the description of identifiable quartics"* (2021) arXiv:2106.06730 [math.AG], submitted for publication

Proposition: minimality and uniqueness result [AC]

Let $T \in \mathbb{P}(S^4\mathbb{C}^5)$ be a form with a dec. $A \subset \mathbb{P}^4$ of $\ell(A) \leq 11$ and

1) A is non-redundant
2) $k_1(A) = 5$
3) $k_2(A) = \ell(A)$
4) the base locus $\mathcal{Q}$ of the system of quadrics $\supset A$ is finite (of length $< 2\ell(A)$)

Then A is the unique minimal decomposition of T.

- [AC]: E. Angelini, L. Chiantini *"On the description of identifiable quartics"* (2021) arXiv:2106.06730 [math.AG], submitted for publication

Proposition: alternative dec. for rank 12 [AC]

Let $T \in \mathbb{P}(S^4\mathbb{C}^5)$ be a form with a dec. $A \subset \mathbb{P}^4$ of $\ell(A) = 12$ and

1) A is non-redundant
2) $k_1(A) = 5$
3) $k_2(A) = 12$
4) $\forall A' \subset A$ of $\ell(A') \leq 11$, the base locus $\mathcal{Q}'$ of the system of quadrics $\supset A'$ is finite (of length $< 2\ell(A')$)
5) the base locus of the system of quadrics $\supset A$ is an irreducible curve C and $0 \to R(-6)^7 \to R(-5)^{24} \to R(-4)^{27} \to R(-3)^8 \oplus R(-2)^3 \to I_A \to 0$

If B is another dec. of T of $\ell(B) = 12$, then $A \cup B$ is a complete intersection (2,2,2,3).

- [AC]: E. Angelini, L. Chiantini *"On the description of identifiable quartics"* (2021) arXiv:2106.06730 [math.AG]

Theorem: description of (T,A) [AC]

The map $f : \mathbb{P}((I_A)_3/(I_C)_3)) \cong \mathbb{P}^7 \dashrightarrow \langle \nu_4(A) \rangle \cong \mathbb{P}^{11}$ defined by

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is birational onto the image.

Corollaries

- $\langle \nu_4(A) \rangle$ contains $f(\mathbb{P}^7)$, a 7-dim'l subvariety of quartics in 5 variables of rank 12 with 2 dec., which is a project. of $\nu_{11}(\mathbb{P}^7)$
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The algorithm for minimality and uniqueness

INPUT $A = \{L_1, \ldots, L_{12}\} \subset \mathbb{P}^4$
$T = \sum_{i=1}^{12} \lambda_i L_i^4 = [(t_0, \ldots, t_{69})]$

PROCEDURE

- check that:

1) $\dim\langle L_1^4, \ldots, L_{12}^4\rangle = 12$ ✓
2) $k_1(A) = 5$ ✓
3) $k_2(A) = 12$ ✓
4) $\forall A' \subset A$ of $\ell(A') = 11$ the base locus $\mathcal{Q}'$ of the system of quadrics $\supset A'$ is finite (of length ≤ 16) ✓
5) the b.l. of the system of quadrics $\supset A$ is an irreducible curve
6) $\dim\langle T_{L_1^4}\nu_4(\mathbb{P}^4), \ldots, T_{L_{12}^4}\nu_4(\mathbb{P}^4)\rangle = 59$ ✓

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- construct the ideal I_A
- construct the ideal of a hypothetical dec. B with $\ell(B) = 12$

$$F = \sum_{i=1}^{8} a_i C_i$$

- construct the matrix N whose rows yield a set of generators for $(I_A)_4 + (I_B)_4$
- compute

$$linsys = \dim \left\{ (a_1, \dots, a_8) \in \mathbb{C}^8 \,|\, N \cdot \begin{pmatrix} t_0 \\ \vdots \\ t_{69} \end{pmatrix} = \underline{0} \right\}$$

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$$T = \sum_{i=1}^{12} L_i^4 = [2454x_0^4 - 14837x_0^3x_1 - 9546x_0^2x_1^2 + 12272x_0x_1^3 + 5779x_1^4 + 9852x_0^3x_2 + \\ +11840x_0^2x_1x_2 + 4479x_0x_1^2x_2 - 6699x_1^3x_2 + 5245x_0^2x_2^2 + 7347x_0x_1x_2^2 + 979x_1^2x_2^2 - 14274x_0x_2^3 + \\ -10753x_1x_2^3 - 15547x_2^4 + 3625x_0^3x_3 + 1511x_0^2x_1x_3 - 7021x_0x_1^2x_3 + 8756x_2^3x_3 - 12116x_0^2x_2x_3 + \\ -11133x_0x_1x_2x_3 - 4526x_1^2x_2x_3 - 8491x_0x_2^2x_3 + 12057x_1x_2^2x_3 - 9401x_2^3x_3 - 10613x_0^2x_3^2 + \\ -6878x_0x_1x_3^2 + 8208x_1^2x_3^2 + 3405x_0x_2x_3^2 + 10766x_1x_2x_3^2 - 13732z^2x_3^2 + 14028x_0x_3^3 - 9572x_1x_3^3 + \\ -11158x_2x_3^3 - 2774x_3^4 - 5103x_0^3x_4 + 5136x_0^2x_1x_4 + 10632x_0x_1^2x_4 - 15393x_1^3x_4 - 4914x_0^2x_2x_4 + \\ +8047x_0x_1x_2x_4 - 4020x_1^2x_2x_4 - 1609x_0x_2^2x_4 + 14390x_1x_2^2x_4 - 5791x_2^3x_4 + 8743x_0^2x_3x_4 + \\ -14600x_0x_1x_3x_4 + 11388x_1^2x_3x_4 + 6681x_0x_2x_3x_4 + 15846x_1x_2x_3x_4 + 9266x_2^2x_3x_4 + 3649x_0x_3^2x_4 + \\ -4887x_1x_3^2x_4 + 12361x_2x_3^2x_4 + 8699x_3^3x_4 + 12211x_0^2x_4^2 - 10563x_0x_1x_4^2 - 13952x_1^2x_4^2 + \\ +2139x_0x_2x_4^2 - 12182x_1x_2x_4^2 - 7237x_2^2x_4^2 - 113x_0x_3x_4^2 - 1224x_1x_3x_4^2 - 2612x_2x_3x_4^2 + \\ +13999x_3^2x_4^2 - 6977x_0x_4^3 - 8368x_1x_4^3 + 1738x_2x_4^3 - 14977x_3x_4^3 + 3637x_4^4]$$

PROCEDURE all the 7 tests ✓

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