Configurations of smooth rational curves on superspecial K3 surfaces in small characteristics

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Happy 60th birthday, Piotr!

Happy 61st birthday, Piotr!

Introduction

Example 1

C : a nonsingular complete curve of genus 2

$J(C)$: the Jacobian variety of C

ι : the inversion of $J(C)$

$J(C)_n$: the group of n-torsion points of $J(C)$

$J(C)/\langle\iota\rangle$: the quotient surface,

$\pi : J(C) \longrightarrow J(C)/\langle\iota\rangle$: the projection

$\mathrm{Km}(J(C))$

: the resolution of singularities of $J(C)/\langle\iota\rangle$

: Kummer surface (K3 surface)

$$\begin{array}{ccc} J(C) & & \mathrm{Km}(J(C)) \\ \pi \downarrow & \swarrow & \\ J(C)/\langle \iota \rangle & & \end{array}$$

Set $\mathcal{N}' = \{T_a C \mid a \in J(C)_2\}$

: 16 smooth curves of genus 2

Set $\mathcal{N} = \{$proper transformes of curves in $\pi(\mathcal{N}')\}$

Set $\mathcal{E} = \{$the exceptinal curves in $\mathrm{Km}(J(C))\}$

$\mathcal{N}, \mathcal{E}$: sets of 16 smooth rational curves which mutually don't intersect

For each $G \in \mathcal{N}$, there exsist 6 curves in $\mathcal{E}$ which intersect G transversely, and vice versa.

Such a configuration is called a $(16)_6$-configuration.

(We call this a Kummer configuration.)

Example 2

$\mathbf{P}^2(\mathbf{F}_q)$: the projective plane over $\mathbf{F}_q$

$\mathcal{A}$: the set of $\mathbf{F}_q$-rational points on $\mathbf{P}^2(\mathbf{F}_q)$
$\mathcal{B}$: the set of lines defined over $\mathbf{F}_q$ on $\mathbf{P}^2(\mathbf{F}_q)$

$$|\mathcal{A}| = |\mathcal{B}| = q^2 + q + 1$$

Take a point $P \in \mathcal{A}$.
There exist $q+1$ lines in $\mathcal{B}$ passing through P.
Take a line $\ell \in \mathcal{B}$.
There exist $q+1$ points in $\mathcal{A}$ contained in ℓ.

Such a configuration is called
$((q^2+q+1)_{q+1}, (q^2+q+1)_{q+1})$-configuration
or $(q^2+q+1)_{q+1}$-(symmetric) configuration.

Problem: What kind of configuration exists on a K3 surface?

A game of

$\mathbf{F}_{p^2}$-rational points and smooth rational curves defined over $\mathbf{F}_{p^2}$

k: algebraically closed field of characteristic $p > 0$

X: K3 surface over k

X is said to be supersingular if $\rho(X) = b_2(X)$.

$\rho(X)$: Picard number ,

$b_2(X)$: the second Betti number

X supersingular

$$\mathrm{disc}\ (\mathrm{NS}(X)) = -p^{2\sigma} \qquad (1 \leq \sigma \leq 10)$$

σ : Artin invariant

char. $k \geq 3$

X:supersingular K3 surface with Artin invariant 1

$\Leftrightarrow X \simeq \mathrm{Km}(E \times E)$ (unique)

$\exists E$: supersingular ellitpic curve

A K3 surface X is said to be superspecial if X is a supersingular K3 surface with Artin invariant 1.

Theorem 1.1 (**S.Kondo, I. Dolgachev, I. Shimada, T.K., T. Shioda, etc.**) *In characteristic 2, let X be the superspecial K3 surface. Then, there exists a $(21)_5$-configuration on X.*

Theorem 1.2 (**S.Kondo-T.K.**)

In characteristic 3, let X be the superspecial K3 surface. Then, there exists a $(16)_{10}$*-configuration on X.*

Remark

16_{10}-configuration $/\mathbf{C}$

M. Traynard (1907)

W. Barth and I. Nieto (1994)

Existence of 16_{10}-configuration on some simple abelian surfaces.

Theorem 1.3 (**T.K.-S.Kondo-I.Shimada**)

k : an algebraically closed field of characteristic 5

X : the superspecial K3 surface over k.

Then, there exists a set $\mathcal{S}$ of 96 smooth rational curves on X which are divided into six sets

$$\mathcal{S}_{00}, \mathcal{S}_{01}, \mathcal{S}_{02}, \mathcal{S}_{10}, \mathcal{S}_{11}, \mathcal{S}_{12}$$

of disjoint 16 smooth rational curves with the following properties:

(i) *If $i \neq j$, then $\mathcal{S}_{\nu i}$ and $\mathcal{S}_{\nu j}$ form a $(16)_6$-configuration for $\nu = 1$ and 1.*

(ii) *For $i = 0, 1, 2$, the sets $\mathcal{S}_{0i}$ and $\mathcal{S}_{1i}$ form a $(16)_{12}$-confguration*

(iii) *If $i \neq j$, then $\mathcal{S}_{0i}$ and $\mathcal{S}_{1j}$ form a $(16)_4$-confguration.*

$$\begin{array}{cc} \mathcal{S}_{00} & \mathcal{S}_{10} \\ \mathcal{S}_{01} & \mathcal{S}_{11} \\ \mathcal{S}_{02} & \mathcal{S}_{12} \end{array}$$

(i) *If* $i \neq j$, *then* $\mathcal{S}_{\nu i}$ *and* $\mathcal{S}_{\nu j}$ *form a* $(16)_6$-*configuration for* $\nu = 1$ *and* 1.

(ii) *For* $i = 0, 1, 2$, *the sets* $\mathcal{S}_{0i}$ *and* $\mathcal{S}_{1i}$ *form a* $(16)_{12}$-*confguration*

(iii) *If* $i \neq j$, *then* $\mathcal{S}_{0i}$ *and* $\mathcal{S}_{1j}$ *form a* $(16)_4$-*confguration.*

Theorem 1.4 *There exists a model of* X *over* $\mathbf{F}_{25}$ *such that all rational curves in* $\mathcal{S}$ *and all intersection points of curves in* $\mathcal{S}$ *are defined over* $\mathbf{F}_{25}$.

Construction

(i) Geometric construction (Using an abelian surface)

(ii) Lattice theoretic construction (Using Leech lattice)

Geometric construction

k: an algebraically closed field of characteristic $p > 0$

E: a supersingular elliptic curve

$$A = E \times E$$

$$\begin{array}{ccc} A & & \mathrm{Km}(A) \\ \pi \downarrow & \swarrow & \\ A/\langle \iota \rangle & & \end{array}$$

Theory of superspecial abelian surface

E: a supersingular elliptic curve

$A = E \times E$

$\mathrm{NS}(A)$: Néron Severi group of A

$Y = E \times \{P_\infty\} + \{P_\infty\} \times E$

a principal divisor on A

$\mathcal{O} = \mathrm{End}(E)$, $B = \mathrm{End}(E) \otimes \mathbf{Q}$

B : a quaternion division algebra over $\mathbf{Q}$ with discriminant p

$\mathcal{O}$: a maximal order of B

$\bar{a}$: the canonical involution of $a \in B$

For a divisor L on A

$$\begin{array}{rccc} \varphi_L : & A & \longrightarrow & \mathrm{Pic}^0(A) \\ & x & \mapsto & T_x^* L - L, \end{array}$$

T_x : the translation by $x \in A$

$$H = \{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha, \delta \in \mathbf{Z},\ \beta, \gamma \in \mathcal{O},\ \gamma = \bar{\beta} \}.$$

$$H \subset M_2(\mathcal{O}) \cong \mathrm{End}(A)$$

$$\begin{array}{rccc} j : & \mathrm{NS}(A) & \longrightarrow & H \\ & L & \mapsto & \varphi_Y^{-1} \circ \varphi_L \end{array}$$

is a bijective homomorphism.

$$j(E\times\{P_\infty\}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; j(\{P_\infty\}\times E) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

For $L_1, L_2 \in NS(A)$, set

$$j(L_1) = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \; j(L_2) = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}.$$

Then, the intersection number

$$(L_1, L_2) = \alpha_1\delta_2 + \alpha_2\delta_1 - \gamma_1\beta_2 - \gamma_2\beta_1.$$

$$L_1^2 = 2\det\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$$

addition of E : $m : E \times E \to E$

Set $\Delta = \text{Ker}\, m$.

$$\Delta = \{(P, -P) \mid P \in E\}$$

For $a_1, a_2 \in \mathcal{O} = \mathrm{End}(E)$, set

$$\Delta_{a_1,a_2} = (a_1 \times a_2)^* \Delta$$

(In particular $\Delta = \Delta_{1,1}$)

Then,

$$j(\Delta_{a_1,a_2}) = \begin{pmatrix} \bar{a}_1 a_1 & \bar{a}_1 a_2 \\ \bar{a}_2 a_1 & \bar{a}_2 a_2 \end{pmatrix}$$

For $g \in \mathrm{End}(E)$, Φ_g is the graph of g

$$\Phi_g \subset E \times E$$

$$\Phi_g = (-g \times \mathrm{id})^* \Delta$$

The intersection number

C: a nonsingular curve of $g \geq 1$

$\eta_i : C \longrightarrow E \quad (i = 1, 2)$: finite morphisms

$$\eta = (\eta_1, \eta_2) : C \longrightarrow E \times E = A.$$

$$\Gamma[\eta] : \text{ the image of } \eta$$

If η is an immersion, then

$$(\Gamma[\eta], E \times \{P_\infty\}) = \deg \eta_2, \; (\Gamma[\eta], \{P_\infty\} \times E) = \deg \eta_1.$$

The graph $\Phi_g = ((-g) \times \mathrm{id})^*\Delta$

$$\theta : \; C \xrightarrow{\eta} E \times E \xrightarrow{(-g)\times\mathrm{id}} E \times E \xrightarrow{m} E$$

The intersection number of $\Gamma[\eta]$ and Φ_g:

Proposition 1.5 *Assume that η is a birational mapping from C to $\Gamma[\eta]$.*

Then, $(\Gamma[\eta], \Phi_g) = \deg\theta$.

Proof

$$\begin{aligned}(\Gamma[\eta], \Phi_g) &= \deg \eta^*\Phi_g = \deg(\eta^* \circ ((-g) \times \mathrm{id})^*\Delta) \\ &= \deg(\eta^* \circ ((-g) \times \mathrm{id})^* \circ m^{-1}(P_\infty)) \\ &= \deg((m \circ ((-g) \times \mathrm{id}) \circ \eta)^*(\mathrm{P}_\infty)) \\ &= \deg\theta.\end{aligned}$$

E : supersingular elliptic curve

$[n]_E$: the multiplication by an integer n.

F : Frobenius morphism

Lemma.

E : supersingular elliptic curve defined over $\mathbf{F}_p$.

Then, $\mathrm{Ker}[p+1]_E = E(\mathbf{F}_{p^2})$.

In particular, $| E(\mathbf{F}_{p^2}) | = (p+1)^2$.

Proof A point $P \in \bar{E}$ is contained in $E(\mathbf{F}_{p^2})$ if and only if $F^2(P) = P$. Since $F^2 = -p$, we have $F^2(P) = P$ if and only if $[p+1]_E(P) = 0$. ■

Assume the characteristic $p = 3$.

The supersingular elliptic curve (unique)

$$E:\ y^2 = x^3 - x$$

$$E(\mathbf{F}_3) = \{P_\infty = (0,1,0), P_0 = (0,0,1), P_1 = (1,0,1), P_{-1} = (-1,0,1)\}$$

2-torsion points of E; P_∞ : the zero point of E

$$E(\mathbf{F}_9) = \{(0,1,0),(0,0,1),(1,0,1),(-1,0,1),(\zeta,\pm\zeta^3,1),(\zeta^2,\pm\zeta,1),$$
$$(\zeta^3,\pm\zeta,1),(\zeta^5,\pm\zeta,1),(\zeta^6,\pm\zeta^3,1),(\zeta^7,\pm\zeta^3,1)\}.$$

4-torsion points of E

ζ : a primitive eighth root of unity which satisfies $\zeta^2 + \zeta = 1$, $\zeta^2 = \sqrt{-1}$.

$E: \; y^2 = x^3 - x$

E has the following automorphisms σ and τ:

$$\sigma : x \mapsto x+1, \; y \mapsto y$$
$$\tau : x \mapsto -x, \; y \mapsto \sqrt{-1}y,$$

which satisfiy

$$\sigma^3 = \mathrm{id}, \; \tau^2 = -\mathrm{id}, \; \tau \circ \sigma = \sigma^2 \circ \tau.$$

Structure of $B = \mathrm{End}(E) \otimes \mathbf{Q}$

$$B = \mathbf{Q} + \mathbf{Q}F + \mathbf{Q}\tau + \mathbf{Q}F\tau$$
$$F^2 = -3, \; \tau^2 = -1, \; F\tau = -\tau F$$

A maximal order $\mathcal{O}$ of B:

$$\mathcal{O} = \mathbf{Z} + \mathbf{Z}\tau + \mathbf{Z}(-\sigma) + \mathbf{Z}(-\tau\sigma).$$

$A = E_1 \times E_2$ with $E_1 = E_2 = E$

A basis of $\mathrm{NS}(A)$

$$E_1,\ E_2,\ \Delta = \Delta_{1,1},\ \Delta_{1,\tau},\ \Delta_{1,-\sigma},\ \Delta_{1,-\tau\sigma}$$

Lemma

$$j(E_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ j(E_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ j(\Delta) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$j(\Delta_{1,\tau}) = \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix},\ j(\Delta_{1,-\sigma}) = \begin{pmatrix} 1 & -\sigma \\ -\sigma^2 & 1 \end{pmatrix},$$

$$j(\Delta_{1,-\tau\sigma}) = \begin{pmatrix} 1 & -\tau\sigma \\ \sigma^2\tau & 1 \end{pmatrix}.$$

The intersection numbers.

	E_1	E_2	Δ	$\Delta_{1,\tau}$	$\Delta_{1,-\sigma}$	$\Delta_{1,-\tau\sigma}$
E_1	0	1	1	1	1	1
E_2	1	0	1	1	1	1
Δ	1	1	0	2	1	2
$\Delta_{1,\tau}$	1	1	2	0	2	1
$\Delta_{1,-\sigma}$	1	1	1	2	0	2
$\Delta_{1,-\tau\sigma}$	1	1	2	1	2	0

One more morphism

The translation T_{P_0} by a 2-torsin point P_0:

$$T_{P_0}^* x_1 = -1/x_1,\ T_{P_0}^* y_1 = y_1/x_1^2.$$

Set $G = \langle T_{P_0} \rangle$ and consider the quotient of E by G:

$$\pi : E \longrightarrow E/G \cong E$$
$$x = \zeta^2(1/x_1 - x_1),\ y = -\zeta(y_1/x_1^2 + y_1)).$$

Relations:

$$\pi = \mathrm{id} - \tau$$
$$\tau \circ \pi = \pi \circ \tau = \mathrm{id} + \tau$$
$$\sigma \circ \pi = \pi \circ \sigma$$
$$\pi \circ \pi \circ \tau = \tau \circ \pi \circ \pi = [2]_E$$

$E: \ y^2 = x^3 - x$

C : the non-singular complete curve of genus 4 defined by

$$Y^2 = X^9 - X.$$

The morphism $\varphi : C \longrightarrow E$ defined by

$$x = X^3 + X, \ y = Y.$$

Take an automorphism η of C:

$$X \mapsto (X - \zeta^2)/X, \ Y \mapsto \zeta Y/X^5$$

Take an automorphism η' of the elliptic curve E:

$$x \mapsto -x - 1, \ y \mapsto \zeta^2 y.$$

The morphism $\varphi' : C \longrightarrow E$ defined by

$$\varphi' = \eta' \circ T_{P_{-1}} \circ \varphi \circ \eta : C \longrightarrow E$$

$$x = \zeta^2 X^3/(X^2 - 1), \ x = -\zeta^3 XY/(X^2 - 1)^2$$

We set

$$\psi = (\varphi, \varphi') : C \longrightarrow E_1 \times E_2 \quad \text{with } E_1 = E_2 = E$$

This morphism is an immersion.

We set

$$C_\infty = C = \mathrm{Im}\ \psi.$$

Lemma

$$C^2 = 6$$
$$(C, E_1) = 3,\ (C, E_2) = 3$$
$$(C, \Delta) = 6,\ (C, \Delta_{1,\tau}) = 6$$
$$(C, \Delta_{1,-\sigma}) = 3,\ (C, \Delta_{1,-\tau\sigma}) = 3$$

$C_\infty = C = \text{Im}\ \psi$

C_∞ is a curve of genus 4 and contains the following ten 2-torsion points:

$$C_\infty \ni \quad (P_\infty, P_\infty), (P_1, P_\infty), (P_{-1}, P_\infty), (P_1, P_1), (P_{-1}, P_1), (P_0, P_1),$$
$$(P_1, P_{-1}), (P_{-1}, P_{-1}), (P_0, P_{-1}), (P_0, P_0),$$

Translate C by 16 two torsion points of A :

We get 16 cureves of genus 4 on A.

Theorem

In $NS(A)$, we have a decomposition

$$C = E_1 + E_2 - \Delta - \Delta_{1,\tau} + 2\Delta_{1,-\sigma} + 2\Delta_{1,-\tau\sigma}.$$

In particular,

$$j(C) = \begin{pmatrix} 3 & -(1+\tau)(1+2\sigma) \\ -(1+2\sigma^2)(1-\tau) & 3 \end{pmatrix}$$

Proof Using the basis $\{E_1,\ E_2,\ \Delta,\ \Delta_{1,\tau},\ \Delta_{1,-\sigma},\ \Delta_{1,-\tau\sigma}\}$ of $NS(A)$, we suppose that C is expressed as

$$C = aE_1 + bE_2 + c\Delta + d\Delta_{1,\tau} + e\Delta_{1,-\sigma} + f\Delta_{1,-\tau\sigma}$$

with integers a, b, c, d, e, f. Considering the intersection of C with the elements of the basis, we have equations:

$$\begin{aligned} 3 &= b + c + d + e + f \\ 3 &= a + c + d + e + f \\ 6 &= a + b + 2d + e + 2f \\ 6 &= a + b + 2c + 2e + f \\ 3 &= a + b + c + 2d + 2f \\ 3 &= a + b + 2c + d + 2e. \end{aligned}$$

Solving these equations, we get the result. ■

$$j(C) = \begin{pmatrix} 3 & -(1+\tau)(1+2\sigma) \\ -(1+2\sigma^2)(1-\tau) & 3 \end{pmatrix}$$

Find $a_1, a_2, b_1, b_2 \in \mathcal{O}$ such that

$$j(C) = \begin{pmatrix} \bar{a}_1 a_1 & \bar{a}_1 a_2 \\ \bar{a}_2 a_1 & \bar{a}_2 a_2 \end{pmatrix} + \begin{pmatrix} \bar{b}_1 b_1 & \bar{b}_1 b_2 \\ \bar{b}_2 b_1 & \bar{b}_2 b_2 \end{pmatrix}$$

$$\Delta_{a_1,a_2} = (a_1 \times a_2)^* \Delta, \quad \Delta_{b_1,b_2} = (b_1 \times b_2)^* \Delta$$

$$j(\Delta_{a_1,a_2}) = \begin{pmatrix} \bar{a}_1 a_1 & \bar{a}_1 a_2 \\ \bar{a}_2 a_1 & \bar{a}_2 a_2 \end{pmatrix}, \quad j(\Delta_{b_1,b_2}) = \begin{pmatrix} \bar{b}_1 b_1 & \bar{b}_1 b_2 \\ \bar{b}_2 b_1 & \bar{b}_2 b_2 \end{pmatrix}$$

$$C \equiv \Delta_{a_1,a_2} + \Delta_{b_1,b_2}$$

Consider the equation in $\mathcal{O}$:

$$-(1+\tau)(1+2\sigma) = \bar{a}_1 a_2 + \bar{b}_1 b_2$$

$$j(C) = \begin{pmatrix} 3 & -(1+\tau)(1+2\sigma) \\ -(1+2\sigma^2)(1-\tau) & 3 \end{pmatrix}$$

In $\mathcal{O} = \mathrm{End}(\mathrm{E})$, we get 10 decompositions of $-(1+\tau)(1+2\sigma)$:

$$\begin{aligned} -(1+\tau)(1+2\sigma) &= 0 + \bar{\pi}F \\ &= 0 + \bar{V}\pi \\ &= \overline{(\sigma+\tau)}(-\sigma^2-\tau) + \bar{1}\cdot(-\sigma) \\ &= \overline{(1-\tau)}(-\sigma) + \bar{1}\cdot\bar{\pi}\sigma^2 \\ &= \overline{(1+\sigma^2\tau)}(\tau-\sigma) + \bar{1}\cdot\tau\sigma^2 \\ &= \bar{1}\cdot\sigma^2\pi + \overline{-\bar{\pi}\sigma^2}\cdot 1 \\ &= \overline{(\sigma^2+\tau)}(\sigma+\tau) + \bar{1}\cdot\sigma^2 \\ &= \bar{1}\cdot(-\bar{\pi}\sigma) + \overline{\sigma\pi}\cdot 1 \\ &= \overline{(-\sigma^2+\tau)}(1+\tau\sigma^2) + \bar{1}\cdot(-\tau\sigma) \\ &= \overline{(\sigma+\sigma^2\tau)}\cdot 1 + \bar{1}\cdot(-\sigma+\tau\sigma^2) \end{aligned}$$

Corresponding to these decompositions, we get 20 elliptic curves

class(∞)	$E_2 + \Delta_{\pi,F}$
class(0)	$E_1 + \Delta_{V,\pi}$
class(1)	$\Delta_{\sigma+\tau,-\sigma^2-\tau} + \Delta_{1,-\sigma}$
class(ζ)	$\Delta_{1-\tau,-\sigma} + \Delta_{1,\bar{\pi}\sigma^2}$
class(ζ^2)	$\Delta_{1+\sigma^2\tau,\tau-\sigma} + \Delta_{1,\tau\sigma^2}$
class(ζ^3)	$\Delta_{1,\sigma^2\pi} + \Delta_{-\bar{\pi}\sigma^2,1}$
class(-1)	$\Delta_{\sigma^2+\tau,\sigma+\tau} + \Delta_{1,\sigma^2}$
class($-\zeta$)	$\Delta_{1,-\bar{\pi}\sigma} + \Delta_{\sigma\pi,1}$
class($-\zeta^2$)	$\Delta_{-\sigma^2+\tau,1+\tau\sigma^2} + \Delta_{1,-\tau\sigma}$
class($-\zeta^3$)	$\Delta_{\sigma+\sigma^2\tau,1} + \Delta_{1,-\sigma+\tau\sigma^2}$

Translate these 20 elliptic curves by 16 two torsion points of A :

We get 80 elliptic cureves on A.

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{f} & A \\ \downarrow g & & \downarrow \\ \mathrm{Km}(A) & \xrightarrow{f'} & A/\langle \iota \times \iota \rangle. \end{array}$$

Here, f' is the minimal resolution of singularities and f is the blowings-up at sixteen 2-torsion points of A. The morphism g is the quotient map to the quotient surface by the group of order 2.

$\mathcal{A}$: the set of exceptional curves for the resolution f'

$\mathcal{B}$: the set of rational curves on Km(A) which come from the 16 curves of genus 4

$\mathcal{E}$: the set of rational curves on Km(A) which come from the 80 elliptic curves

We set

$$\mathcal{R} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{E}.$$

Then, $\mathcal{R}$ contains in total 112 nonsigular rational curves defined over $\mathbf{F}_9$ whose self-intersection numbers are all equal to -2.

On one rational curve ℓ in $\mathcal{R}$ there exists ten $\mathbf{F}_9$-rational points.

At each $\mathbf{F}_9$-rational point 4 rational curves in $\mathcal{R}$ intersect transversely two by two.

Therefore we have $10 \times 112 \div 4 = 280$ $\mathbf{F}_9$-rational points.

We denote by $\mathcal{P}$ the set of these 280 points.

Theorem

$\mathcal{P}$ and $\mathcal{R}$ make $(280_4, 112_{10})$-configuration.

Remark

The 112 smooth rational curves coinside with the lines defined over $\mathbf{F}_9$ on the Fermat K3 surface

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \quad \text{in } \mathbf{P}^3.$$

cf. Next Rams' talk

Theorem

The sets $\mathcal{A}$ and $\mathcal{B}$ make a 16_{10}-configuration on a supersingular K3 surface $\mathrm{Km}(A)$.

k: an algebraically closed field of characteristic 5

$E : y^2 = x^3 - 1$, the supersingular elliptic curve

$$A = E \times E$$

$$\begin{array}{ccc} A & & \mathrm{Km}(A) \\ \pi \downarrow & \swarrow & \\ A/\langle \iota \rangle & & \end{array}$$

$\mathcal{S}_{00}$ exceptional curves, $\quad$ $\mathcal{S}_{10}$ curves of genus 5

$\mathcal{S}_{01}$ curves of genus 2, $\quad$ $\mathcal{S}_{11}$ elliptic curves

$\mathcal{S}_{02}$ curves of genus 2, $\quad$ $\mathcal{S}_{12}$ elliptic curves

k: an algebraically closed field of characteristic 5

$E : y^2 = x^3 - 1$, the supersingular elliptic curve

$P_0 = (1, 0)$: a 2-torsion point

$$\begin{array}{rccc} \phi_{E,2} : & E & \longrightarrow & E/\langle P_0 \rangle \simeq E \\ & (x, y) & \mapsto & (u, v). \end{array}$$

$$u = \frac{2x^2 + 3x + 1}{(x-1)}, \quad v = \frac{2\sqrt{2}y(x^2 + 3x + 3)}{(x-1)^2}$$

$$\phi_{E,2}^2 = -[2]_E.$$

$\omega = 2 + 3\sqrt{2}$: a primitive cube root of unity

$\gamma \in \mathrm{End}(E)$:

$$\gamma : x \mapsto \omega x,\ y \mapsto -y$$

$$\gamma^6 = \mathrm{id}$$

(id : the identity of E.)

$B = \mathrm{End}^0(E) = \mathrm{End}(E) \otimes_{\mathbf{Z}} \mathbf{Q}$: a division algebra with discriminant 5

$\mathcal{O} = \mathrm{End}(E)$: a maximal order

a basis :

$$\omega_1 = 1,\ \omega_2 = \gamma,\ \omega_3 = \phi_{E,2},\ \omega_4 = \gamma\phi_{E,2}$$

A basis of Néron-Severi group NS(A)

$$B_1 = E \times \{P_\infty\},\ B_2 = \{P_\infty\} \times E,$$
$$B_3 = \Phi_{\mathrm{id}} = (-\mathrm{id} \times \mathrm{id})^*\Delta,\ \mathrm{B}_4 = \Phi_\gamma = (-\gamma \times \mathrm{id})^*\Delta,$$
$$B_5 = \Phi_{\phi_{E,2}} = (-\phi_{E,2} \times \mathrm{id})^*\Delta,\ \mathrm{B}_6 = \Phi_{\gamma\phi_{\mathrm{E},2}} = (-\gamma\phi_{\mathrm{E},2} \times \mathrm{id})^*\Delta$$

$$j(B_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\quad j(B_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad j(B_3) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$
$$j(B_4) = \begin{pmatrix} 1 & -\gamma^5 \\ -\gamma & 1 \end{pmatrix},\quad j(B_5) = \begin{pmatrix} 2 & \phi_{E,2} \\ -\phi_{E,2} & 1 \end{pmatrix},$$
$$j(B_6) = \begin{pmatrix} 2 & -\phi_{E,2}\gamma^2 \\ -\gamma\phi_{E,2} & 1 \end{pmatrix}.$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 0 & 2 & 3 \\ 2 & 1 & 3 & 2 & 0 & 2 \\ 2 & 1 & 4 & 3 & 2 & 0 \end{bmatrix}$$

F : the curve of genus 2 defined by the equation

$$y^2 = x^6 - 1.$$

$$\phi_{F,2} : F \longrightarrow E \qquad (x, y) \mapsto (x^2, \ \ y) \,,$$

$$h_F : F \longrightarrow F \qquad (u, v) \mapsto \left(\frac{2\sqrt{2}\, u + 4}{u + 2\sqrt{2}}, \ \ \frac{v}{(u + 2\sqrt{2})^3} \right),$$

$$(\phi_{F,2}, \phi_{F,2} \circ h_F) : F \longrightarrow E \times E$$

This morphism is an immersion.

$$(F, B_1) = 2,\ (F, B_2) = 2,\ (F, B_3) = 4,$$
$$(F, B_4) = 2,\ (F, B_5) = 8,\ (F, B_6) = 7$$

In NS(A), $F \subset A$ is expressed as

$$F = aB_1 + bB_2 + cB_3 + dB_4 + eB_5 + fB_6$$

with 6 variables a, b, c, d, e, f.

Linear equations

$$
\begin{aligned}
2 &= (F, B_1) = a(B_1, B_1) + b(B_2, B_1) + c(B_3, B_1) + d(B_4, B_1) + e(B_5, B_1) + f(B_6, B_1) \\
2 &= (F, B_2) = a(B_1, B_2) + b(B_2, B_2) + c(B_3, B_2) + d(B_4, B_2) + e(B_5, B_1) + f(B_6, B_2) \\
4 &= (F, B_3) = a(B_1, B_3) + b(B_2, B_3) + c(B_3, B_3) + d(B_4, B_3) + e(B_5, B_3) + f(B_6, B_3) \\
2 &= (F, B_4) = a(B_1, B_4) + b(B_2, B_4) + c(B_3, B_4) + d(B_4, B_4) + e(B_5, B_4) + f(B_6, B_4) \\
8 &= (F, B_5) = a(B_1, B_5) + b(B_2, B_5) + c(B_3, B_5) + d(B_4, B_5) + e(B_5, B_5) + f(B_6, B_5) \\
7 &= (F, B_6) = a(B_1, B_6) + b(B_2, B_6) + c(B_3, B_6) + d(B_4, B_6) + e(B_5, B_6) + f(B_6, B_6)
\end{aligned}
$$

Solving these equations, we have

$$F = 2B_1 + 3B_2 - B_3 + 2B_4 - B_5$$

Therefore, we have

$$j(F) = \begin{pmatrix} 2 & 1 + 2\gamma^2 - \phi_{E,2} \\ 1 - 2\gamma^2 + \phi_{E,2} & 1 \end{pmatrix}$$

To construct smooth rational curves on $\mathrm{Km}(A)$, we use the following curves:

E : the supersingular elliptic curve

$$y^2 = x^3 - 1$$

F : the curve of genus 2 defined by

$$y^2 = x^6 - 1$$

G : the curve of genus 5 defined by

$$y^2 = \sqrt{2}\,(x^{12} + 2\,x^8 + 2\,x^4 + 1)$$

$$\phi_{E,2} : E \to E \qquad (u,v) \mapsto \left(\frac{2u^2+3u+1}{u-1}, \quad \frac{2\sqrt{2}\,v\,(u^2+3u+3)}{(u-1)^2} \right),$$

$$\phi_{G,3} : G \to E \qquad (u,v) \mapsto \left(\frac{4\sqrt{2}\,(u+3\sqrt{2}+4)^2\,(u+2\sqrt{2}+4)}{f}, \quad \frac{(4+4\sqrt{2})\,v}{f^2} \right),$$

ただし, $\quad f := (u+\sqrt{2})(u+4\sqrt{2}+1)\,(u+3\sqrt{2}+2),$

$$\phi_{G,4} : G \to E \qquad (u,v) \mapsto \left(\frac{u^4+(1+4\sqrt{2})u^2+2}{g}, \quad \frac{v\,u}{g^2} \right),$$

ただし, $\quad g := u^4 + (1+2\sqrt{2})u^2 + (4+\sqrt{2})$

Automorphisms

$$\gamma : E \to E \qquad (x,y) \mapsto (\omega x, -y),$$

$$h'_F : F \to F \qquad (u,v) \mapsto \left(\frac{2\sqrt{2}\,u+1}{u+3\sqrt{2}}, \quad \frac{v}{(u+3\sqrt{2})^3} \right),$$

$$h_G : G \to G \qquad (u,v) \mapsto \left(\frac{2u+3}{u+1}, \quad \frac{4v}{(u+1)^6} \right)$$

$$\begin{array}{rccc} \tau : & A & \longrightarrow & A \\ & (P,Q) & \mapsto & (Q, \iota_E(P)) \end{array}$$

$\mathcal{T}(\Gamma)$: curves translated by two torsion points A_2 for a curve Γ on A.

$$
\begin{aligned}
\mathcal{L}_{01} &:= \mathcal{T}(\Gamma[(\phi_{F,2}, \phi_{F,2} \circ h_F)]\,), \\
\mathcal{L}_{02} &:= \mathcal{T}(\Gamma[(\phi_{F,3}, \phi_{F,3} \circ h'_F]\,), \\
\mathcal{L}_{10,(4,3)} &:= \mathcal{T}(\Gamma[(\phi_{G,4}, \phi_{G,3})]\,), \\
\mathcal{L}_{10,(4,4)} &:= \mathcal{T}(\Gamma[(\gamma^2 \circ \phi_{G,4}, \gamma \circ \phi_{G,4} \circ h_G)]\,), \\
\mathcal{L}_{10} &:= \mathcal{L}_{10,(4,3)} \cup \tau(\mathcal{L}_{10,(4,3)}) \cup \mathcal{L}_{10,(4,4)} \cup \tau(\mathcal{L}_{10,(4,4)}), \\
\mathcal{L}_{11,(1,2)} &:= \mathcal{T}(\Gamma[(\gamma^2, \gamma^2 \circ \phi_{E,2})]\,), \\
\mathcal{L}_{11,(2,2)} &:= \mathcal{T}(\Gamma[(\phi_{E,2} \circ \gamma, \gamma \circ \phi_{E,2})]\,), \\
\mathcal{L}_{11} &:= \mathcal{L}_{11,(1,2)} \cup \tau(\mathcal{L}_{11,(1,2)}) \cup \mathcal{L}_{11,(2,2)} \cup \tau(\mathcal{L}_{11,(2,2)}), \\
\mathcal{L}_{12} &:= \mathcal{T}(B_1) \cup \mathcal{T}(B_2) \cup \mathcal{T}(B_4) \cup \mathcal{T}(\Gamma[(\mathrm{id}, \gamma^2)]\,).
\end{aligned}
$$

$\mathcal{S}_{\nu i}$: Curves on $Km(A)$ induced by $\mathcal{L}_{\nu i}$

$\mathcal{S}_{00}$: Exceptional curves on $Km(A)$

Our 96 smooth rational curves on $Km(A)$

$$\mathcal{S}_{00}, \mathcal{S}_{01}, \mathcal{S}_{02}, \mathcal{S}_{10}, \mathcal{S}_{11}, \mathcal{S}_{12}$$

Theorem 1.3 (**T.K.-S.Kondo-I.Shimada**)

k : an algebraically closed field of characteristic 5

X : the superspecial K3 surface over k.

Then, there exists a set $\mathcal{S}$ of 96 smooth rational curves on X which are divided into six sets

$$\mathcal{S}_{00}, \mathcal{S}_{01}, \mathcal{S}_{02}, \mathcal{S}_{10}, \mathcal{S}_{11}, \mathcal{S}_{12}$$

of disjoint 16 smooth rational curves with the following properties:

(i) *If $i \neq j$, then $\mathcal{S}_{\nu i}$ and $\mathcal{S}_{\nu j}$ form a $(16)_6$-configuration for $\nu = 1$ and 1.*

(ii) *For $i = 0, 1, 2$, the sets $\mathcal{S}_{0i}$ and $\mathcal{S}_{1i}$ form a $(16)_{12}$-confguration*

(iii) *If $i \neq j$, then $\mathcal{S}_{0i}$ and $\mathcal{S}_{1j}$ form a $(16)_4$-confguration.*

Lattice Theory

Λ : Leech lattice,

H: hyperbolic lattice of rank 2

Set

$$L \cong H \oplus \Lambda$$

L : even unimodular lattice with index $(1, 25)$. (unique)

$$\mathcal{P}_L \subset \{x \in L \otimes \mathbf{R} \mid x^2 > 0\}.$$

a connected component (positive cone)

Set

$$\mathcal{R}_L = \{v \in L \mid v^2 = -2\}$$

Reflection of L: For $r \in \mathcal{R}_L$,

$$s_r : x \mapsto x + \langle x, r \rangle r$$

Set

$$W(L) = \langle \{s_r \mid r \in \mathcal{R}_L\} \rangle \quad \text{(Weyl group)}$$

hyperplanes: $\mathcal{R}_L^* = \{(v)^\perp \mid v \in \mathcal{R}_L\}$

$$\mathcal{P}_L \setminus \cup_{v\in\mathcal{R}_L}(v)^\perp$$

Definition:

The closure of a connected component of $\mathcal{P}_L \setminus \cup_{v\in\mathcal{R}_L}(v)^\perp$ in $\mathcal{P}_L$ is called an $\mathcal{R}_L^*$-chamber.

In particular, an $\mathcal{R}_L^*$-chamber of even unimodular lattice L with index $(1,25)$ is said to be a Conway chamber.

Facts:

Weyl group $W(L)$ acts on $\mathcal{P}_L$.

Each $\mathcal{R}_L^*$-chamber is a fundamental domain.

$w \in L$: a non-zero primitive vector with $w^2 = 0$.
w is called a Weyl vector if w satisfied the following two conditions.

(i) w is contained in the closure of $\mathcal{P}_L$ in $L \otimes \mathbf{R}$.

(ii) $\langle w \rangle^{\perp} / \langle w \rangle \cong \Lambda$.

For a Weyl vector w, set

$$\Delta(w) := \{r \in \mathcal{R}_L \ | \ (r, w) = 1\}$$

Theorem 1.6 (Conway) *For a Weyl vector w,*

$$\mathcal{D}(w) := \{x \in \mathcal{P}_L \ | \ (r, x) \geq 0 \ \ r \in \Delta(w)\}$$

is a Conway chamber. Conversely, for a Conway chamber $\mathcal{D}$, there exists a unique Weyl vector w such that

$$\mathcal{D} = \mathcal{D}(w)$$

Set $X = \mathrm{Km}(A)$ and $S = \mathrm{NS}(X)$

Ample cone $\subset \mathcal{P}_S \subset \mathrm{NS}(X)$: a positive cone.

A primitive embedding: $S = \mathrm{NS}(X) \hookrightarrow L$

$$\begin{array}{ccc} S \otimes \mathbf{R} & \hookrightarrow & L \otimes \mathbf{R} \\ \uparrow & & \uparrow \\ \mathcal{P}_S & \hookrightarrow & \mathcal{P} \end{array}$$

Set

$$\mathrm{NC}(X) = \{C \in \mathrm{NS}(X) \mid C^2 > 0,\ (C, C') \geq 0 \text{ for any curve } C' > 0\}$$

$\mathrm{NC}(X)$ is a $\mathcal{R}_S^*$-chamber of $S \otimes \mathbf{R}$ (Rudakov-Shafarevich) .

For $x \in L \otimes \mathbf{R}$,

$$x \mapsto x_S$$

the projection to $S \otimes \mathbf{R}$.

Set

$$\mathcal{R}_{L|S} = \{r_S \mid r \in \mathcal{R}_L,\ (r_S, r_S) < 0\},$$
$$\mathcal{R}_{L|S}^* = \{(r_S)^\perp \mid r_S \in \mathcal{R}_{L|S}\}$$

Definition:

Conway chamber $\mathcal{D}$ is said to be S-nondegenerate if $\mathcal{D} \cap \mathcal{P}_S$ contains a non-empty open set of $\mathcal{P}_S$.

If $\mathcal{D}$ is a S-nondegenerate Conway chamber,

$D := \mathcal{D} \cap \mathcal{P}_S$ is a $\mathcal{R}^*_{L|S}$-chamber of $\mathcal{P}_S$.

Definition:

D is called an induced chamber.

Since $\mathcal{P}_L$ is coverd by Conway chambers, $\mathcal{P}_S$ is covered by induced chambers.

Since $\mathcal{R}_S$ is a subset of $\mathcal{R}_{L|S}$, $\mathcal{R}_S^*$-chamber is a union of induced chambers.

Therefore, NC(X) is a union of induced chambers.

Among these induced chambers, we have 3 induced chambers:

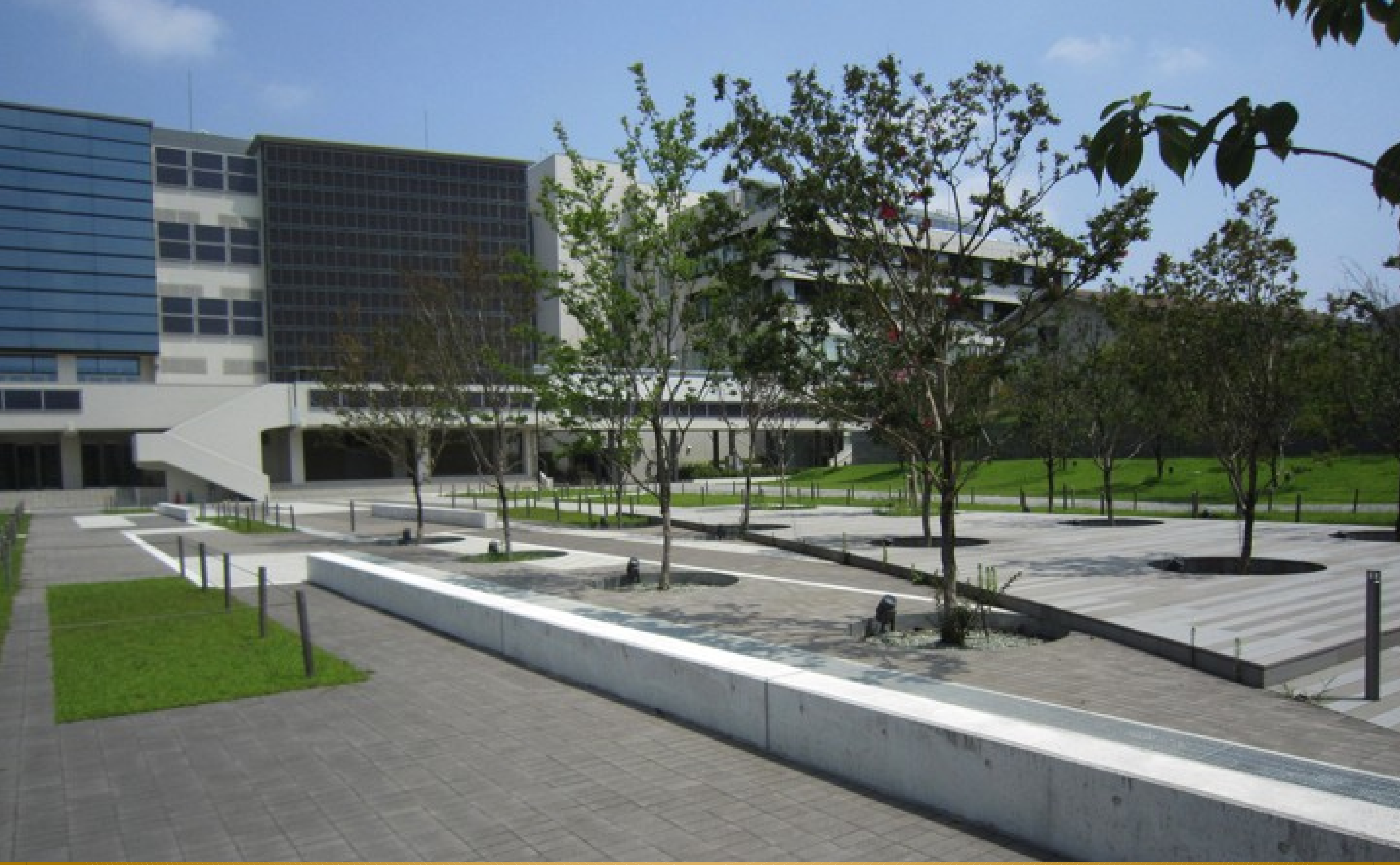
D_0 : the induced chamber which contains 252 (-2)-vectors as walls.

D_1 : the induced chamber which contains 168 (-2)-vectors as walls.

D_2 : the induced chamber which contains 96 (-2)-vectors as walls.

D_0 is adjacent to D_1.
The number of common (-2)-vectors is 126.

D_0 is adjacent to D_2.
The number of common (-2)-vectors is 48.



Thank you for your attention