Lines on quartic surfaces

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joint work with:
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V. González-Alonso (Leibniz University Hannover) - singular case
arXiv:1212.3511 + 1303.1304 + 1409.7485 + work in progress

Outline

Classical results.

Segre's argument in modern language.

A counterexample.

Main results.

Sketch of the proof.

Cubics

Basic notions:

- A line in $\mathbb{P}_3$:= set of zeroes of two linearly independent linear forms.
- A smooth degree-d surface := a smooth degree-d algebraic hypersurface $\mathrm{Z}(f) \subset \mathbb{P}_3(\mathbb{K})$.

Fix $d \in \mathbb{N}$.
Question. What is the maximal number of lines on a smooth projective algebraic degree-d surface?

Cubics:

d=3: 1847 - Cayley/Salmon + Clebsch (later):
Answer: Exactly 27 lines on every smooth $S_3 \subset \mathbb{P}_3$.
If S_3 is not a cone, but $\mathrm{sing}(S_3) \neq \emptyset$, then S_3 contains strictly less than 27 lines.

Proof: Computation of the degree and ramification locus of a cover.

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any d ≥ 3**:** 1860 - Salmon/Clebsch:

Thm. There exists a degree-$(11d-24)$ polynomial F_d such that

$$Z(F_d) \cap S_d = \{P \in S_d;\ \text{there exists a line } L \text{ with } i_P(S_d, L) \geq 4\},$$

where $i_P(S_d, L)$ is the multiplicity of vanishing of $f|_L$ in the point P.

Flecnodal divisor $\mathcal{F}_d :=$ the cycle of zeroes of F_d on the surface S_d.

Corollary.

$$(\text{Number of lines on degree-d surfaces}) \leq \deg(\mathcal{F}_d) = d \cdot (11d - 24)$$

Example. The Fermat surface $Z(x_1^d + x_2^d + x_3^d + x_4^d)$ contains $3d^2$ lines.

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$$\rho(S_d) = 1 \text{ and } \mathrm{NS}(S_d) = \mathbb{Z}\,\mathcal{O}_{S_d}(1)$$

Consequently: no lines on S_d.

Example. [Shioda 81] For certain $d \geq 4$ the surface $\mathrm{Z}(x_4^d + x_1 x_2^{d-1} + x_2 x_3^{d-1} + x_3 x_1^{d-1})$ has Picard number one. In particular, it contains no lines.

d=4: 1882 - Schur:
The quartic surface $\mathrm{Z}(x_1^4 - x_1 x_2^3 - x_3^4 + x_3 x_4^3) \subset \mathbb{P}_3$ contains exactly 64 lines.

1943 - Segre claims to show:
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Segre's argument in modern language I

Step 1. S a smooth quartic surface, $\ell \subset S$ a line. Then ℓ is **met by at most** 18 **other lines on** S.

Step 2. If there exists a plane Π such that $\Pi \cap S$ consists of four lines, then S contains at most 64 lines.

Step 3. If there exists a line $\ell \subset S$ met by **at least 13** other lines, then there exists a plane Π such that $\Pi \cap S$ consists of four lines.

Assumption A: Each line on S met by at most 12 other lines, no four of them coplanar.

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Assumption B: Each pair of coplanar lines on S defines a conic in $\mathrm{supp}(\mathcal{F}_S)$.

Step 5. If S contains at least 9 pairs of coplanar lines, then it contains at most 62 lines.

Assumption C: S contains $N \leq 8$ pairs of coplanar lines.

Step 6. The pairs of lines span a sublattice of $\mathrm{NS}(S)$ of rank $\geq (N+1)$. The number of lines is bounded by

$$2N + 22 - (N+1) = 21 + N \leq 29\,.$$

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Example [R-S 2012]
The quartic S given by vanishing of

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with $r = -16/27$,
- is smooth outside characteristics $2, 3, 5$,
- contains 60 lines,
- contains the line $\{x_3 = x_4 = 0\}$,
- contains **20 other lines that meet the line** $\{x_3 = x_4 = 0\}$.

Over $\mathbb{C}$, we obtain a K3 surface with Picard number $\rho = 20$.

The claim of Step 1 is false.

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The quartic S given by vanishing of

$$x_1^3x_3 + x_1x_2x_3^2 + x_2^3x_4 + rx_3^3x_4 - x_1x_2x_4^2 - rx_3x_4^3$$

with $r = -16/27$,
- is smooth outside characteristics $2, 3, 5$,
- contains 60 lines,
- contains the line $\{x_3 = x_4 = 0\}$,
- contains **20 other lines that meet the line** $\{x_3 = x_4 = 0\}$.
Over $\mathbb{C}$, we obtain a K3 surface with Picard number $\rho = 20$.

The claim of Step 1 is false.

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Main results I

Assumption: $\mathbb{K}$ is alg closed.

Thm 1 [R-S, 2012].

1. A line ℓ on a smooth quartic surface S in $\mathbb{P}^3(\mathbb{K})$ intersects at most 20 other lines provided that $\mathrm{char}(\mathbb{K}) \neq 2, 3$.
2. If ℓ meets more than 18 lines on S, then S can be given by a quartic polynomial

$$x_3 x_1^3 + x_4 x_2^3 + x_1 x_2 q(x_3, x_4) + x_3 g(x_3, x_4)$$

where $q, g \in \mathbb{K}[x_3, x_4]$ are homogeneous of degree 2 resp. 3.

3. The line $\ell = \{x_3 = x_4 = 0\}$ meets 20 lines on S if and only if $x_4 \mid g$.

Thm 2 [R-S, 2012]. Let $\mathbb{K}$ be an alg. closed field with $\mathrm{char}(\mathbb{K}) \neq 2, 3$. A smooth quartic $S \subset \mathbb{P}^3(\mathbb{K})$ contains at most 64 lines.

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Main results II

Different approach is needed to deal with $\mathrm{char}(\mathbb{K}) \in \{2,3\}$, because flecnodal divisor can degenerate.

Thm 3 [R-S, 2014]. An irreducible conic on a smooth quartic $S \subset \mathbb{P}^3(\mathbb{K})$ may meet at most 48 lines on S.

Thm 4 [R-S, 2014]. Let $\mathrm{char}(\mathbb{K}) = 3$.
(a) A smooth quartic $S \subset \mathbb{P}^3(\mathbb{K})$ contains at most 112 lines.
(b) Up to the action of $\mathrm{PGL}(4)$ there is a unique smooth quartic surface in $\mathbb{P}^3(\mathbb{K})$ containing 112 lines.

Example The quartic

$$S_3 = \{x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\} \subset \mathbb{P}^3(\mathbb{K}).$$

contains exactly 112 lines (each defined over $\mathbb{F}_9$).
It was known already to B. Segre.

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Remark: On the quartic from the example a line on S_3 is met by 30 other lines.

Question: What for characteristic 2? (work in progress)

Thm 4 [R-S, 2014]. Let $\mathrm{char}(\mathbb{K}) = 2$. A smooth quartic $S \subset \mathbb{P}^3(\mathbb{K})$ contains at most 84 lines.

Best known example (inspired by Barth quintic): 60 lines.

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Question: What for **complex** quartics with singularities?

Thm 5 [Davide Veniani, 2014]. Assume that $S \subset \mathbb{P}_3(\mathbb{C})$ is a $K3$ quartic surface, then it contains at most 64 lines.

Thm 6 [GA-R, 2015]. Assume that $S \subset \mathbb{P}_3(\mathbb{C})$ is a non-$K3$ quartic surface. If S is not ruled by lines, then it contains at most 48 lines.

Thm 7 [GA-R, 2015]. (a) Let $S \subset \mathbb{P}_3(\mathbb{C})$ be a quartic surface with at most isolated singularities, none of which is a fourfold point. Then S contains at most 64 lines.

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Question: What for **complex** quartics with singularities?

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Sketch of the proof I

Set-up. $\ell \subset S$ a line,

- We consider the family of planes $H_\lambda \supset \ell$.
- $H_\lambda \cap S =: \ell + F_\lambda$
- for general $\lambda \in \mathbb{P}_1$ the cubic F_λ is smooth.

We constructed an genus-1 fibration

$$\pi : S \supset C_\lambda \ni P \mapsto \lambda \in \mathbb{P}_1 ,$$

Definition. The line ℓ is of the first kind iff it intersects at least one smooth fibre $Z(f_\lambda)$ only in the points where the determinant of Hessian of f_λ does not vanish.

Lemma 0. If ℓ of the first kind, then ℓ is met by at most 18 other lines.

Proof: Computation of the resultant of the equation of $F_\lambda|_\ell$ and of its Hessian$|_\ell$. □

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We consider the restriction $\pi|_\ell : \ell \to \mathbb{P}^1$ and get a degree-3 morphism.

Definition. The line ℓ is of ramification type 1^4 (resp. $2, 1^2$), (resp. 2^2) iff $\pi|_\ell$ has 4, (resp. 3) or (resp. 2) ramification points.

Definition. The line ℓ is of the second kind iff it intersects all smooth fibres $Z(f_\lambda)$ in the points where Hessian of f_λ vanishes.

Support of the closure of the inflection points of smooth fibers:

fibre type	configuration
I_1	3 smooth points, the node
I_2	3 smooth points of one component, both nodes
I_3	3 smooth points on each component
II	1 smooth point, the cusp
III	1 smooth point of one component, the node
IV	1 smooth point on each component, the node

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Sketch of the proof III

Assumption: ℓ of the second kind.

Definition A fiber F of π is (un)ramified iff $\pi|_\ell$ (un)ramified at F.

Lemma 1. Let F a singular fibre of π. If F is unramified, then F has type I_1, I_3 or IV.

Proof: ℓ meets F is 3 smooth points, so F contains 3 smooth flex points.
Table $\Rightarrow$ F of type I_1, I_2, I_3 or IV.
ℓ meets each component of F, so I_2 excluded. □

Lemma 2. Let F a ramified fibre of π. Then F has type I_1, I_2, II or IV, according to the ramification type as follows:

fibre type	II	I_1, I_2, IV
ramification type	1	2

Proof: Tate's algorithm + base changes. □

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Sketch of the proof IV

Lemma 3. Semi-stable fibres on S occur in pairs (I_1, I_3) and triples (I_2, I_3, I_3).

Proposition 4. Let R be the ramification type of ℓ. Let G_R be defined as follows:

R	1^4	$2, 1^2$	2^2
G_R	$\{12\}$	$\{15, 16\}$	$\{18, 19, 20\}$

Then ℓ meets exactly N other lines contained in S, where $N \in G_R$.

Proof: Case-by-case analysis of ramification types,
e.g. for $\boldsymbol{R = 1^4}$ we have 4 type-II fibers by Lemma 2. This gives Euler number 8.
Lemma 1 implies remaining fibers of type I_1, I_3 or IV.
By Lemma 3 we get:

$$(24 - 8)/4 \cdot 3 \text{ lines}$$

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Sketch of the proof V

Lemma 5. Let ℓ be of the ramification type $R = 2^2$. Then S is projectively equivalent to a quartic in the family $\mathcal{Z}$

$$\{x_3x_1^3 + x_4x_2^3 + x_1x_2q(x_3, x_4) + g(x_3, x_4) = 0\},$$

where $q \in k[x_3, x_4]$ (resp. $g \in k[x_3, x_4]$) is a polynomial of degree 2 (resp. 4).

Proof: After a linear transformation,
- ℓ given by $x_3 = x_4 = 0$,
- the ramification occurs at $x_3 = 0$, $x_4 = 0$.

After further normalisation the equation:

$$x_3x_1^3 + x_4x_2^3 + x_3^2q_1 + x_3x_4q_2 + x_4^2q_3 = 0$$

where the q_j are homogeneous quadratic forms in $x_1, \ldots, x_4$.
Solve for ℓ to be a line of the second kind, i.e. for the Hessian to vanish identically on ℓ . □

Sketch of the proof V

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Solve for ℓ to be a line of the second kind, i.e. for the Hessian to vanish identically on ℓ . □

Sketch of the proof V

Lemma 5. Let ℓ be of the ramification type $R = 2^2$. Then S is projectively equivalent to a quartic in the family $\mathcal{Z}$

$$\{x_3 x_1^3 + x_4 x_2^3 + x_1 x_2 q(x_3, x_4) + g(x_3, x_4) = 0\},$$

where $q \in k[x_3, x_4]$ (resp. $g \in k[x_3, x_4]$) is a polynomial of degree 2 (resp. 4).

Proof: After a linear transformation,
- ℓ given by $x_3 = x_4 = 0$,
- the ramification occurs at $x_3 = 0$, $x_4 = 0$.

After further normalisation the equation:

$$x_3 x_1^3 + x_4 x_2^3 + x_3^2 q_1 + x_3 x_4 q_2 + x_4^2 q_3 = 0$$

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Sketch of the proof VI

We study quartics given by

$$\{x_3 x_1^3 + x_4 x_2^3 + x_1 x_2 q(x_3, x_4) + g(x_3, x_4) = 0\},$$

Lemma 6. A surface $S \in \mathcal{Z}$ is a smooth quartic such that the fibration $\pi : S \to \mathbb{P}_1$ attains a fibre of Kodaira type I_2 (necessarily at 0 or ∞) iff x_3 or x_4 divides g. The ramified fibres degenerate to Kodaira type IV iff x_3 or x_4 divides q.

Proof: Generically, there are six singular fibres of Kodaira type I_1 located at $0, \infty$ and at the zeroes of g.
Formulas for the Jacobian of the fibration π give 6 fibres of Kodaira type I_3 at the zeroes of $q^3 + 27 x_3 x_4 g$. □

That is the way we found our counterexample:

$$x_3 x_1^3 + x_4 x_2^3 + x_1 x_2 x_3^2 - x_1 x_2 x_4^2 + r x_3^3 x_4 - r x_3 x_4^3$$

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Sketch of the proof VII - 66 lines

We fix $S \in \mathcal{Z}$ and the line ℓ_0 of the second kind with induced elliptic fibration

$$\pi_0 : S \to \mathbb{P}^1$$

By direct, computer-aided computation we get

Lemma 7. A line in a fibre of π_0 is of the second kind iff S is the Schur quartic.

Proposition 8. A smooth quartic contains at most 66 lines.

Proof: If S lies away from $\mathcal{Z}$ we are done.

We can assume we deal with the line $\ell_0 \subset S$.

By Proposition 4 we can assume ℓ_0 of ramification type 2^2.

By Lemma 6 π_0 has an I_3-fiber or a type-IV fiber.

By Lemma 7 either S is Schur quartic or number of lines bounded by

$$17 + 3 \cdot 15 + 4 = 66.$$

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Sketch of the proof VIII

Assumption: S contains 65 or 66 lines.

- We fix $S \in \mathcal{Z}$ and the line ℓ_0 of the second kind with induced elliptic fibration $\pi_0 : S \to \mathbb{P}_1$.
- By Lemma 6 π_0 admits a (ramified) fibre of Kodaira type I_2 (i.e. line + conic). The fibre consists of:
  - ℓ_1 a line of the first kind
  - Q a conic, that does not come up in the flecnodal divisor $\text{supp}(\mathcal{F}_S)$.
- The line ℓ_1 induces a second elliptic fibration $\pi_1 : S \to \mathbb{P}_1$.
- The quartic S admits the automorphism of order 3

$$\sigma : \begin{array}{ccc} S & \to & S \\ [x_1, x_2, x_3, x_4] & \mapsto & [\varrho x_1, \varrho^2 x_2, x_3, x_4] \end{array}$$

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Sketch of the proof IX

- the lines ℓ_0, ℓ_1 are fixed by σ,
- the resolution of S/σ is a K3 surface S',
- π_0, π_1 induce elliptic fibrations on S'.

We exploit the above properties to get:

Thm 2 [R-S, 2012]. Let $\mathbb{K}$ be an alg. closed field with $\mathrm{char}(\mathbb{K}) \neq 2, 3$. A smooth quartic $S \subset \mathbb{P}_3(\mathbb{K})$ contains at most 64 lines.

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