Lines on quartic surfaces

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joint work with:
M. Schütt (Leibniz University Hannover) - smooth case
V. González-Alonso (Leibniz University Hannover) - singular case
arXiv:1212.3511 + 1303.1304 + 1409.7485 + work in progress
Outline

Classical results.

Segre’s argument in modern language.

A counterexample.

Main results.

Sketch of the proof.
Cubics

Basic notions:

- A line in $\mathbb{P}^3 :=$ set of zeroes of two linearly independent linear forms.
- A smooth degree-$d$ surface := a smooth degree-$d$ algebraic hypersurface $Z(f) \subset \mathbb{P}^3(K)$.

Fix $d \in \mathbb{N}$.

**Question.** What is the maximal number of lines on a smooth projective algebraic degree-$d$ surface?

**Cubics:**

**d=3:** 1847 - Cayley/Salmon + Clebsch (later):

**Answer:** Exactly 27 lines on every smooth $S_3 \subset \mathbb{P}_3$.

If $S_3$ is not a cone, but $\text{sing}(S_3) \neq \emptyset$, then $S_3$ contains strictly less than 27 lines.

Proof: Computation of the degree and ramification locus of a cover.
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Higher-degree surfaces

any $d \geq 3$: 1860 - Salmon/Clebsch:

**Thm.** There exists a degree-$(11d - 24)$ polynomial $F_d$ such that

$$Z(F_d) \cap S_d = \{ P \in S_d; \text{ there exists a line } L \text{ with } i_P(S_d, L) \geq 4 \},$$

where $i_P(S_d, L)$ is the multiplicity of vanishing of $f|_L$ in the point $P$.

**Flecnodal divisor** $\mathcal{F}_d :=$ the cycle of zeroes of $F_d$ on the surface $S_d$.

**Corollary.**

$$(\text{Number of lines on degree-$d$ surfaces}) \leq \deg(\mathcal{F}_d) = d \cdot (11d - 24)$$

**Example.** The Fermat surface $Z(x_1^d + x_2^d + x_3^d + x_4^d)$ contains $3d^2$ lines.
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Lefschetz Thm. For very general smooth $S_d \subset \mathbb{P}^3(\mathbb{C})$ with $d \geq 4$

$$\rho(S_d) = 1 \text{ and } NS(S_d) = \mathbb{Z} \mathcal{O}_{S_d}(1)$$

Consequently: no lines on $S_d$.

Example. [Shioda 81] For certain $d \geq 4$ the surface $Z(x_4^d + x_1x_2^{d-1} + x_2x_3^{d-1} + x_3x_1^{d-1})$ has Picard number one. In particular, it contains no lines.

$d=4$: 1882 - Schur:
The quartic surface $Z(x_1^4 - x_1x_2^3 - x_3^4 + x_3x_4^3) \subset \mathbb{P}_3$ contains exactly 64 lines.

1943 - Segre claims to show:
• a line on a smooth quartic is never met by more than 18 other lines.
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**Segre’s argument in modern language I**

**Step 1.** \( S \) a smooth quartic surface, \( l \subset S \) a line. Then \( l \) is **met by at most** 18 other lines on \( S \).

**Step 2.** If there exists a plane \( \Pi \) such that \( \Pi \cap S \) consists of four lines, then \( S \) contains at most 64 lines.

**Step 3.** If there exists a line \( l \subset S \) met by **at least** 13 other lines, then there exists a plane \( \Pi \) such that \( \Pi \cap S \) consists of four lines.

**Assumption A:** Each line on \( S \) met by at most 12 other lines, no four of them coplanar.
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Step 4. If there exist coplanar lines $l_1, l_1 \subset S$ such that the conic $C_2 \in |\mathcal{O}_S(1) - l_1 - l_2|$ is no component of the flecnodal divisor $\mathcal{F}_S$, then $S$ contains at most 60 lines.

Assumption B: Each pair of coplanar lines on $S$ defines a conic in $\text{supp}(\mathcal{F}_S)$.

Step 5. If $S$ contains at least 9 pairs of coplanar lines, then it contains at most 62 lines.

Assumption C: $S$ contains $N \leq 8$ pairs of coplanar lines.

Step 6. The pairs of lines span a sublattice of $\text{NS}(S)$ of rank $\geq (N + 1)$. The number of lines is bounded by

$$2N + 22 - (N + 1) = 21 + N \leq 29.$$
**Segre’s argument in modern language II**

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A counterexample

Example [R-S 2012]
The quartic $S$ given by vanishing of

$$x_1^3 x_3 + x_1 x_2 x_3^2 + x_2^3 x_4 + r x_3^3 x_4 - x_1 x_2 x_4^2 - r x_3 x_4^3$$

with $r = -16/27$,
- is smooth outside characteristics 2, 3, 5,
- contains 60 lines,
- contains the line $\{x_3 = x_4 = 0\}$,
- contains 20 other lines that meet the line $\{x_3 = x_4 = 0\}$.

Over $\mathbb{C}$, we obtain a K3 surface with Picard number $\rho = 20$.

The claim of Step 1 is false.
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x_1^3 x_3 + x_1 x_2 x_3^2 + x_2^3 x_4 + r x_3^3 x_4 - x_1 x_2 x_4^2 - r x_3 x_4^3
\]

with \( r = -16/27 \),
- is smooth outside characteristics 2, 3, 5,
- contains 60 lines,
- contains the line \( \{x_3 = x_4 = 0\} \),
- contains 20 other lines that meet the line \( \{x_3 = x_4 = 0\} \).

Over \( \mathbb{C} \), we obtain a K3 surface with Picard number \( \rho = 20 \).

The claim of Step 1 is false.
Example [R-S 2012]
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A counterexample

**Example** [R-S 2012]
The quartic $S$ given by vanishing of

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Main results I

Assumption: $\mathbb{K}$ is alg closed.

**Thm 1** [R-S, 2012].

1. A line $\ell$ on a smooth quartic surface $S$ in $\mathbb{P}^3(\mathbb{K})$ intersects at most 20 other lines provided that $\text{char}(\mathbb{K}) \neq 2, 3$.

2. If $\ell$ meets more than 18 lines on $S$, then $S$ can be given by a quartic polynomial

   $$x_3x_1^3 + x_4x_2^3 + x_1x_2q(x_3, x_4) + x_3g(x_3, x_4)$$

   where $q, g \in \mathbb{K}[x_3, x_4]$ are homogeneous of degree 2 resp. 3.

3. The line $\ell = \{x_3 = x_4 = 0\}$ meets 20 lines on $S$ if and only if $x_4 \mid g$.

**Thm 2** [R-S, 2012]. Let $\mathbb{K}$ be an alg. closed field with $\text{char}(\mathbb{K}) \neq 2, 3$. A smooth quartic $S \subset \mathbb{P}^3(\mathbb{K})$ contains at most 64 lines.
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Different approach is needed to deal with $\text{char}(\mathbb{K}) \in \{2, 3\}$, because flecnodal divisor can degenerate.

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**Thm 4** [R-S, 2014]. Let $\text{char}(\mathbb{K}) = 3$.

(a) A smooth quartic $S \subset \mathbb{P}^3(\mathbb{K})$ contains at most 112 lines.

(b) Up to the action of $\text{PGL}(4)$ there is a unique smooth quartic surface in $\mathbb{P}^3(\mathbb{K})$ containing 112 lines.

**Example** The quartic

$$S_3 = \{x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\} \subset \mathbb{P}^3(\mathbb{K}).$$

contains exactly 112 lines (each defined over $\mathbb{F}_9$).

It was known already to B. Segre.
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Remark: On the quartic from the example a line on $S_3$ is met by 30 other lines.

Question: What for characteristic 2? (work in progress)

Thm 4 [R-S, 2014]. Let $\text{char}(\mathbb{K}) = 2$. A smooth quartic $S \subset \mathbb{P}^3(\mathbb{K})$ contains at most 84 lines.

Best known example (inspired by Barth quintic): 60 lines.
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**Thm 5** [Davide Veniani, 2014]. Assume that $S \subset \mathbb{P}_3(\mathbb{C})$ is a $K3$ quartic surface, then it contains at most 64 lines.

**Thm 6** [GA-R, 2015]. Assume that $S \subset \mathbb{P}_3(\mathbb{C})$ is a non-$K3$ quartic surface. If $S$ is not ruled by lines, then it contains at most 48 lines.

**Thm 7** [GA-R, 2015]. (a) Let $S \subset \mathbb{P}_3(\mathbb{C})$ be a quartic surface with at most isolated singularities, none of which is a fourfold point. Then $S$ contains at most 64 lines.

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Sketch of the proof I

Set-up. \( \ell \subset S \) a line,

- We consider the family of planes \( H_\lambda \supset \ell \).
- \( H_\lambda \cap S =: \ell + F_\lambda \)
- for general \( \lambda \in \mathbb{P}_1 \) the cubic \( F_\lambda \) is smooth.

We constructed an genus-1 fibration

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\pi : S \supset C_\lambda \ni P \mapsto \lambda \in \mathbb{P}_1,
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Definition. The line \( \ell \) is of the first kind iff it intersects at least one smooth fibre \( Z(f_\lambda) \) only in the points where the determinant of Hessian of \( f_\lambda \) does not vanish.

Lemma 0. If \( \ell \) of the first kind, then \( \ell \) is met by at most 18 other lines.

Proof: Computation of the resultant of the equation of \( F_\lambda|_\ell \) and of its Hessian\( |_\ell \). \qed


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We consider the restriction $\pi|_\ell : \ell \to \mathbb{P}^1$ and get a degree-3 morphism.

**Definition.** The line $\ell$ is of **ramification type** $1^4$ (resp. $2, 1^2$), (resp. $2^2$) iff $\pi|_\ell$ has 4, (resp. 3) or (resp. 2) ramification points.

**Definition.** The line $\ell$ is of the **second kind** iff it intersects all smooth fibres $Z(f_\lambda)$ in the points where Hessian of $f_\lambda$ vanishes.

Support of the closure of the inflection points of smooth fibers:

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We consider the restriction $\pi|_{\ell} : \ell \to \mathbb{P}^1$ and get a degree-3 morphism.

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**Sketch of the proof III**

**Assumption:** $\ell$ of the second kind.

**Definition** A fiber $F$ of $\pi$ is (un)ramified iff $\pi|_{\ell}$ (un)ramified at $F$.

**Lemma 1.** Let $F$ a singular fibre of $\pi$. If $F$ is unramified, then $F$ has type $I_1, I_3$ or $IV$.

**Proof:** $\ell$ meets $F$ is 3 smooth points, so $F$ contains 3 smooth flex points. Table $\Rightarrow F$ of type $I_1, I_2, I_3$ or $IV$. $\ell$ meets each component of $F$, so $I_2$ excluded. $\square$

**Lemma 2.** Let $F$ a ramified fibre of $\pi$. Then $F$ has type $I_1, I_2, II$ or $IV$, according to the ramification type as follows:

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<td>ramification type</td>
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**Proof:** Tate’s algorithm + base changes. $\square$
Assumption: \( \ell \) of the second kind.

Definition A fiber \( F \) of \( \pi \) is \((un)\)ramified iff \( \pi|_{\ell} \) \((un)\)ramified at \( F \).

Lemma 1. Let \( F \) a singular fibre of \( \pi \). If \( F \) is unramified, then \( F \) has type \( I_1, I_3 \) or \( IV \).

Proof: \( \ell \) meets \( F \) is 3 smooth points, so \( F \) contains 3 smooth flex points. Table \( \Rightarrow \) \( F \) of type \( I_1, I_2, I_3 \) or \( IV \).
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**Sketch of the proof III**

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**Definition** A fiber \( F \) of \( \pi \) is (un)ramified iff \( \pi |_\ell \) (un)ramified at \( F \).

**Lemma 1.** Let \( F \) a singular fibre of \( \pi \). If \( F \) is unramified, then \( F \) has type \( I_1, I_3 \) or \( IV \).

**Proof:** \( \ell \) meets \( F \) is 3 smooth points, so \( F \) contains 3 smooth flex points. Table \( \Rightarrow \) \( F \) of type \( I_1, I_2, I_3 \) or \( IV \). \( \ell \) meets each component of \( F \), so \( I_2 \) excluded. \( \square \)

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**Proof:** Tate’s algorithm + base changes. \( \square \)
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Sketch of the proof III

Assumption: \( \ell \) of the second kind.

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Proof: \( \ell \) meets \( F \) is 3 smooth points, so \( F \) contains 3 smooth flex points. Table \( \Rightarrow \) \( F \) of type \( l_1, l_2, l_3 \) or \( IV \).
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Lemma 2. Let \( F \) a ramified fibre of \( \pi \). Then \( F \) has type \( l_1, l_2, ll \) or \( IV \), according to the ramification type as follows:

<table>
<thead>
<tr>
<th>fibre type</th>
<th>ll</th>
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<tbody>
<tr>
<td>ramification type</td>
<td>1</td>
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Proof: Tate’s algorithm + base changes. \( \square \)
Assumption: \( \ell \) of the second kind.

Definition A fiber \( F \) of \( \pi \) is (un)ramified iff \( \pi|_\ell \) (un)ramified at \( F \).

Lemma 1. Let \( F \) a singular fibre of \( \pi \). If \( F \) is unramified, then \( F \) has type \( I_1, I_3 \) or \( IV \).

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| fibre type | \( II \) | \( I_1, I_2, IV \) |
| ramification type | 1 | 2 |

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Proof: Tate’s algorithm + base changes. □
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**Proposition 4.** Let $R$ be the ramification type of $\ell$. Let $G_R$ be defined as follows:

<table>
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<tr>
<td>$G_R$</td>
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Then $\ell$ meets exactly $N$ other lines contained in $S$, where $N \in G_R$.

**Proof:** Case-by-case analysis of ramification types, e.g. for $R = 1^4$ we have 4 type-II fibers by Lemma 2. This gives Euler number 8. Lemma 1 implies remaining fibers of type $I_1, I_3$ or $IV$. By Lemma 3 we get:

$$\frac{(24 - 8)}{4 \cdot 3}$$ lines
Sketch of the proof IV

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\[
\begin{array}{c|c|c|c}
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G_R & \{12\} & \{15, 16\} & \{18, 19, 20\}
\end{array}
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Sketch of the proof V

**Lemma 5.** Let $\ell$ be of the ramification type $R = 2^2$. Then $S$ is projectively equivalent to a quartic in the family $\mathcal{Z}$

$$\{x_3x_1^3 + x_4x_2^3 + x_1x_2q(x_3, x_4) + g(x_3, x_4) = 0\},$$

where $q \in k[x_3, x_4]$ (resp. $g \in k[x_3, x_4]$) is a polynomial of degree 2 (resp. 4).

**Proof:** After a linear transformation,
- $\ell$ given by $x_3 = x_4 = 0$,
- the ramification occurs at $x_3 = 0, x_4 = 0$.

After further normalisation the equation:

$$x_3x_1^3 + x_4x_2^3 + x_3^2q_1 + x_3x_4q_2 + x_4^2q_3 = 0$$

where the $q_j$ are homogeneous quadratic forms in $x_1, \ldots, x_4$.

Solve for $\ell$ to be a line of the second kind, i.e. for the Hessian to vanish identically on $\ell$. □
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Solve for $\ell$ to be a line of the second kind, i.e. for the Hessian to vanish identically on $\ell$. □
Sketch of the proof VI

We study quartics given by

\[ \{x_3x_1^3 + x_4x_2^3 + x_1x_2q(x_3, x_4) + g(x_3, x_4) = 0\}, \]

Lemma 6. A surface \( S \in \mathcal{Z} \) is a smooth quartic such that the fibration \( \pi : S \to \mathbb{P}_1 \) attains a fibre of Kodaira type \( I_2 \) (necessarily at 0 or \( \infty \)) iff \( x_3 \) or \( x_4 \) divides \( g \). The ramified fibres degenerate to Kodaira type \( IV \) iff \( x_3 \) or \( x_4 \) divides \( q \).

Proof: Generically, there are six singular fibres of Kodaira type \( I_1 \) located at 0, \( \infty \) and at the zeroes of \( g \).
Formulas for the Jacobian of the fibration \( \pi \) give 6 fibres of Kodaira type \( I_3 \) at the zeroes of \( q^3 + 27x_3x_4g \).

That is the way we found our counterexample:

\[ x_3x_1^3 + x_4x_2^3 + x_1x_2x_3^2 - x_1x_2x_4^2 + rx_3^3x_4 - rx_3x_4^3 \]
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**Proof:** Generically, there are six singular fibres of Kodaira type $I_1$ located at $0, \infty$ and at the zeroes of $g$. Formulas for the Jacobian of the fibration $\pi$ give 6 fibres of Kodaira type $I_3$ at the zeroes of $q^3 + 27x_3 x_4 g$. \(\square\)

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Sketch of the proof VII - 66 lines

We fix $S \in \mathcal{Z}$ and the line $\ell_0$ of the second kind with induced elliptic fibration

$$\pi_0 : S \rightarrow \mathbb{P}^1$$

By direct, computer-aided computation we get

**Lemma 7.** A line in a fibre of $\pi_0$ is of the second kind iff $S$ is the Schur quartic.

**Proposition 8.** A smooth quartic contains at most 66 lines.

**Proof:** If $S$ lies away from $\mathcal{Z}$ we are done.

We can assume we deal with the line $\ell_0 \subset S$.

By Proposition 4 we can assume $\ell_0$ of ramification type $2^2$.

By Lemma 6 $\pi_0$ has an I$_3$-fiber or a type-IV fiber.

By Lemma 7 either $S$ is Schur quartic or number of lines bounded by

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Sketch of the proof VII - 66 lines

We fix $S \in Z$ and the line $\ell_0$ of the second kind with induced elliptic fibration

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By direct, computer-aided computation we get

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Sketch of the proof VII - 66 lines

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**Proof:** If $S$ lies away from $\mathcal{Z}$ we are done.

We can assume we deal with the line $\ell_0 \subset S$.

By Proposition 4 we can assume $\ell_0$ of ramification type $2^2$.

By Lemma 6 $\pi_0$ has an $I_3$-fiber or a type-$IV$ fiber.

By Lemma 7 either $S$ is Schur quartic or number of lines bounded by

$$17 + 3 \cdot 15 + 4 = 66.$$
Sketch of the proof VII - 66 lines

We fix $S \in \mathcal{Z}$ and the line $\ell_0$ of the second kind with induced elliptic fibration

$$\pi_0 : S \to \mathbb{P}^1$$

By direct, computer-aided computation we get

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Sketch of the proof VIII

Assumption: \( S \) contains 65 or 66 lines.

- We fix \( S \in \mathbb{Z} \) and the line \( \ell_0 \) of the second kind with induced elliptic fibration \( \pi_0 : S \to \mathbb{P}_1 \).
- By Lemma 6 \( \pi_0 \) admits a (ramified) fibre of Kodaira type \( I_2 \) (i.e. line + conic). The fibre consists of:
  - \( \ell_1 \) a line of the first kind
  - \( Q \) a conic, that does not come up in the flecnodal divisor \( \text{supp}(\mathcal{F}_S) \).
- The line \( \ell_1 \) induces a second elliptic fibration \( \pi_1 : S \to \mathbb{P}_1 \).
- The quartic \( S \) admits the automorphism of order 3

\[
\sigma : S \quad \rightarrow \quad S \\
[x_1, x_2, x_3, x_4] \quad \mapsto \quad [\varrho x_1, \varrho^2 x_2, x_3, x_4]
\]

where \( \varrho \) is a primitive third root of unity,
**Assumption:** \( S \) contains 65 or 66 lines.

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Sketch of the proof IX

- the lines $\ell_0, \ell_1$ are fixed by $\sigma$,
- the resolution of $S/\sigma$ is a K3 surface $S'$,
- $\pi_0, \pi_1$ induce elliptic fibrations on $S'$.

We exploit the above properties to get:

**Thm 2** [R-S, 2012]. Let $K$ be an alg. closed field with $\text{char}(K) \neq 2, 3$. A smooth quartic $S \subset \mathbb{P}_3(K)$ contains at most 64 lines.
Sketch of the proof IX

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