Birational invariance of the bounded negativity

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Joint work with

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Definition

We say that a surface X has the *Bounded Negativity Property* if there exists a number $b(X)$ such that

$$C^2 \geq -b(X)$$

holds for all reduced (and irreducible) curves $C \subset X$.

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Example

- For $\mathbb{P}^2$ it suffices to take $b(\mathbb{P}^2) = 0$.
- For the Hirzebruch surface $\mathbb{F}_n$, $b(\mathbb{F}_n) = n$ suffices.

Conjecture

Every **complex** *surface has the Bounded Negativity Property.*

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Remark

This conjecture **fails** in the finite characteristic!

Theorem (Th. Bauer, B. Harbourne, A. L. Knutsen, A. Küronya, S. Müller-Stach, X. Roulleau, TS)

Let X be a complex surface which admits an endomorphism which is not an isomorphism. Then X has the Bounded Negativity Property.

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Remark

There is a full classification of such surfaces available, due to Fujimoto and Nakayama. The statement follows from this classification and the next result (plus additional arguments for elliptic surfaces).

Theorem (Th. Bauer, C. Bocci, S. Cooper, S. Di Rocco, M. Dumnicki, B. Harbourne, K. Jabbusch, A. L. Knutsen, A. Küronya, R. Miranda, J. Roé, H. Schenck, TS, and Z. Teitler)

If X is a complex surface with $\mathbb{Q}$-effective anticanonical divisor, then X has the Bounded Negativity Property.

Problem

Let X and Y be birationally equivalent complex projective surfaces. Has X the Bounded Negativity Property if and only if Y does?

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Of course, if BNC is true, then the above Problem has positive answer.

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Even if BNC is true, it is interesting to compare the numbers

$$b(X) \quad \textit{and} \quad b(Y)$$

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Example

Let $f : X \to \mathbb{P}^2$ be the blow up of s **general** points. Then the Segre-Harbourne-Gimigliano-Hirschowitz Conjecture predicts that $b(X) = 1$ is independent of the number of points blown up.

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Remark

The above statement fails completely for **arbitrary** points.

Definition

Let X be a smooth projective surface and let $\mathcal{P} = \{P_1, \dots, P_s\}$ be a set of $s \geq 1$ mutually distinct points of X. Then the *H–constant of X at $\mathcal{P}$* is defined as

$$H(X; \mathcal{P}) := \inf_C \frac{\widetilde{C}^2}{s},$$

where $\widetilde{C}$ is the proper transform of C with respect to the blow-up $f : Y \to X$ of X at the set $\mathcal{P}$ and the infimum is taken over all *reduced* curves $C \subset X$. Note that

$$\widetilde{C}^2 = (f^*C - \sum_{i=1}^s (\mathrm{m}_{P_i}(C))E_i)^2 = C^2 - \sum_{i=1}^s (\mathrm{m}_{P_i}(C))^2$$

where $E_1, \dots, E_s$ are the exceptional divisors of the blown up points. Thus $\frac{\widetilde{C}^2}{s}$ can be thought of as the average of the numbers $-((\mathrm{m}_{P_i}(C))^2 - \frac{C^2}{s})$.

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Finally, we define the *global H–constant of* X as

$$H(X) := \inf_{s \geq 1} H(X;s).$$

Remark

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For an arbitrary surface X one has always $H(X) \leq -2$.

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If $H(X)$ is finite, then BNC holds on a blow up of X at arbitrary number of points at arbitrary position.

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Remark

Even if $H(X) = -\infty$, the Bounded Negativity Property might still hold for X and its blow ups.

Conjecture

$$H(\mathbb{P}^2) = -4$$

Definition

Let $\mathcal{P} = \{P_1, \ldots, P_s\}$ be a set of mutually distinct $s \geq 1$ points in the projective $\mathbb{P}^2$. Then the *linear H–constant at* $\mathcal{P}$ is defined as

$$H_L(\mathcal{P}) := \inf_{\mathcal{L}} H_L(\mathcal{P}, \mathcal{L}),$$

where for $\mathcal{L}$, a line arrangement $H_L(\mathcal{P}, \mathcal{L}) = \frac{d^2 - \sum_{i=1}^{s} \mathrm{m}_{P_i}(\mathcal{L})^2}{s}$.

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As before, the *s–tuple linear H–constant* is

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The *global linear H–constant of* $\mathbb{P}^2$ is

$$H_L := \inf_{s \geq 1} H_L(s).$$

Theorem (Bounded linear negativity on $\mathbb{P}^2$)

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Theorem (Hirzebruch)

Let $\mathcal{L}$ be an arrangement of d lines in the complex projective plane $\mathbb{P}^2$. Then

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{k \geq 5}(k-4)t_k, \tag{1}$$

provided $t_d = t_{d-1} = 0$.

Here $t_k = t_k(\mathcal{L})$ denotes the number of points where exactly k lines from $\mathcal{L}$ meet, for $k \geq 2$.

Remark

The proof of the Hirzebruch inequality is based on the logarithmic Miyaoka–Yau–Sakai inequality, which assumes the complex numbers.

Example (Wiman configuration)

There exists a configuration of **45** lines with

- $t_3 = 120$
- $t_4 = 45$
- $t_5 = 36$

With $\mathcal{P}$ the set of all singular points of the configuration, this configuration gives

$$H_L(\mathcal{P}, \mathcal{L}) = -\frac{225}{67} \approx -3.36.$$

Proposition (P. Pokora, X. Roulleau, H. Tutaj-Gasińska)

The conical H-constant of $\mathbb{P}^2$ is bounded below by

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Proposition (X. Roulleau)

There exist a configuration of cubic curves with the H-constant equal -4. Moreover

$$-4$$

is the lower bound for the cubical H-constant of $\mathbb{P}^2$.

Proposition

Over the reals we have

$$H_L(\mathbb{P}^2(\mathbb{R})) = -3.$$

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Remark

Over an arbitrary field **Marcin Dumnicki** and **Justyna Szpond** give explicit formulas for *absolute* values of linear Harbourne constants depending on the number of involved lines (work in progress).

The story goes on and has (unexpected) connections to other interesting problems, see the poster session!

Last but not least