

# GRADED BUNDLES

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# Plan of the talk

- Multiplication by reals is enough
- Smooth actions of  $(\mathbb{R}, \cdot)$  (homogeneity structures)
- Graded spaces (not graded vector spaces)
- Vector bundles and graded bundles (not graded vector bundles)
- Graded bundle=homogeneity structure
- Digressions on mechanics and field theory
- Transition functions and the tower
- Splitting graded bundles
- Lifts of graded structures

# What is a vector space?

- A (real) **vector space** is a set  $E$  with a distinguished element  $0^E$ , equipped with two operations:
  1. an addition

$$+ : E \times E \rightarrow E, \quad (u, v) \mapsto u + v,$$

2. and a multiplication by scalars

$$h : \mathbb{R} \times E \rightarrow E, \quad h(t, v) = h_t(v) = t \cdot v = tv,$$

satisfying a list of axioms.

- For instance,  $(E, +)$  is a commutative group with  $0^E$  being the neutral element, the **homotheties**  $h_t$  satisfy

$$h_t \circ h_s = h_{ts},$$

and  $h_0(v) = 0^E$  for all  $v \in E$ .

# One operation is enough

- To distinguish finite-dimensional real vector spaces among differentiable manifolds, a single operation of the above two is enough.
- If we know the addition, we get the multiplication by natural numbers in the obvious way:

$$nv = v + \cdots + v,$$

and we easily extend it to integers by  $(-n)v = n(-v)$ . The multiplication by rational numbers,  $(m/n)v$  we obtain as the solution of the equation  $nx = mv$ .

Assuming differentiability (in fact, continuity) of  $h$ , we extend this multiplication to all reals uniquely.

- If we know the multiplication by reals  $h$  instead, we use a version of Euler's Homogeneous Function Theorem: any differentiable  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is **homogeneous** of degree 1, i.e.

$$f(t \cdot x) = t \cdot f(x),$$

if and only if  $f$  is linear.

# Homogeneous Function Theorem

- Indeed,  $t \cdot f(x) = f(t \cdot x)$  and differentiability gives

$$f(x) = \frac{\partial f}{\partial x^i}(t \cdot x) \cdot x^i.$$

Putting  $t = 0$  we obtain further

$$f(x) = \frac{\partial f}{\partial x^i}(0) \cdot x^i$$

which means that  $f$  is linear. The other implication is obvious.

- Thus, from the multiplication by reals on  $E$  we get the dual space  $E^*$ , where the addition is well defined, and consequently the addition on  $E = (E^*)^*$ .
- All this can be reformulated for a vector bundle  $\tau : E \rightarrow M$ : the multiplication by reals  $h$  in  $E$  (**homotheties**) uniquely determines  $E$  with the projection  $\tau = h_0$ .

# Homogeneity structures

- We can consider now a general (smooth) action  $h : \mathbb{R} \times F \rightarrow F$  of the multiplicative monoid  $(\mathbb{R}, \cdot)$  on a manifold  $F$ ,  $h_t \circ h_s = h_{ts}$ . Such an action we will call a **homogeneity structure**.

A smooth function  $f : F \rightarrow \mathbb{R}$  will be called **homogeneous of degree  $w$**  if

$$f(h_t(x)) = t^w f(x) \quad \text{for } t \geq 0.$$

- It is a nontrivial observation (we will come to it later) that homogeneity degrees can only be non-negative integers and that we can choose local coordinates which are homogeneous (and have non-negative integers as degrees).
- Note that it is crucial that  $h_t$  is defined for  $t = 0$ , since, for instance, the action  $h : \mathbb{R}^\times \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , with  $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ , of the multiplicative group  $\mathbb{R}^\times$  on  $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$  given by  $h_t(x) = |t|x$  admits 'homogeneous' functions of arbitrary degree  $w$ , namely  $f(x) = x^w$ . Here  $(tx)^w = t^w x^w$  for  $t > 0$ . However this is not homogeneity in the sense we consider, as the projection  $h_0$  is not defined.

# Graded spaces

Assume now that for a homogeneity structure  $h$  on a manifold  $F$  there is a (necessary unique) point  $0^F \in F$  such that  $h_0(F) = \{0^F\}$ . Such a structure we will call a **graded space (they are not graded vector spaces)** by the following reasons.

## Theorem (Grabowski-Rotkiewicz)

Any graded space  $(F, h)$  is diffeomorphically equivalent (isomorphic) to a certain  $(\mathbb{R}^d, h^d)$ , where  $d = (d_1, \dots, d_k)$ , with positive integers  $d_i$ , and  $\mathbb{R}^d = \mathbb{R}^{d_1}[1] \times \dots \times \mathbb{R}^{d_k}[k]$  is equipped with the action  $h^d$  of multiplicative reals given by

$$h_t^d(y_1, \dots, y_k) = (t \cdot y_1, \dots, t^k \cdot y_k), \quad y_i \in \mathbb{R}^{d_i}.$$

In other words,  $F$  can be equipped with a system of (global) coordinates  $(y_i^j)$ ,  $i = 1, \dots, k$ ,  $j = 1, \dots, d_i$ , such that linear coordinates  $y_i^j$  in  $\mathbb{R}^{d_i}[i]$  are **homogeneous of degree  $i$**  with respect to the homogeneity structure  $h$ , i.e.

$$y_i^j \circ h_t = t^i \cdot y_i^j.$$

Of course, in these coordinates  $0^F = (0, \dots, 0)$ .

# How to recognize vector spaces?

- Note that the isomorphism in the above theorem is generally non-canonical. The number  $k$ , however, is uniquely determined and called the **minimal degree** of the graded space. By convention, a **degree** of  $h$  is any natural  $k' \geq k$ .
- How to recognize a vector space among graded spaces?
- **Answer:** Vector spaces are graded spaces of degree 1.
- **Regularity condition:** For any  $y \in F$ ,

$$\frac{d}{dt}\Big|_{t=0} (h_t(y)) = 0 \Leftrightarrow y = 0^F.$$

## Theorem

*The homogeneity structure in a graded space comes from a vector space structure if and only if it is regular. In this case, the vector space structure is uniquely determined by the homogeneity structure.*



# Weight vector field

- It is natural to call a **morphism** between homogeneity structures  $(F_a, h^a)$ ,  $a = 1, 2$ , a smooth map  $\Phi : F_1 \rightarrow F_2$  which intertwines the homogeneity structures:  $\Phi \circ h_t^1 = h_t^2 \circ \Phi$ .
- The  $(\mathbb{R}, \cdot)$ -action restricted to positive reals gives a one-parameter group of diffeomorphism of  $F$ , thus is generated by a vector field  $\nabla_F$ . It is called the **weight vector field** as it completely determines the homogeneity structure. For a graded space with homogeneous global coordinates  $(y_w^j)$

$$\nabla_F = \sum_w w y_w^j \partial_{y_w^j}.$$

- A function  $f$  is homogeneous of degree  $w$  if and only if  $\nabla_F(f) = w \cdot f$ , and a smooth map  $\Phi : F_1 \rightarrow F_2$  is a morphism of homogeneity structures iff it relates the corresponding weight vector fields.
- Note that morphisms need not to be linear, so the category of graded spaces is different from that of vector spaces. For instance, if  $(y, z) \in \mathbb{R}^2$  are coordinates of degrees  $1, 2$ , respectively, then the map  $(y, z) \mapsto (y, z + y^2)$  is an automorphism of the structure, but it is nonlinear.

# Vector bundles classically

- A **vector bundle** is a locally trivial fibration  $\tau : E \rightarrow M$  which, locally over  $U \subset M$ , reads  $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$  and admits an atlas in which local trivializations transform linearly in fibers

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n,$$

$$A(x) \in GL(n, \mathbb{R}).$$

- The latter property can also be expressed in the terms of the gradation in which base coordinates  $x$  have degrees 0, and 'linear coordinates'  $y$  have degree 1. Linearity of changes of coordinates is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow \tau_1 & & \downarrow \tau_2 \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

being linear (homogeneous) in fibres.

# Graded bundles

- A straightforward generalization is the concept of a **graded bundle**  $\tau : F \rightarrow M$  of rank  $d$ , with a local trivialization by  $U \times \mathbb{R}^d$ , and with the difference that the transition functions of local trivializations:

$$U \cap V \times \mathbb{R}^d \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^d,$$

respect the weights of coordinates  $(y^1, \dots, y^{|d|})$  in the fibres. In other words, a graded bundle of rank  $d$  is a locally trivial fibration with fibers modelled on the graded space  $\mathbb{R}^d$ .

## Theorem

$A(x, y)$  must be polynomial in homogeneous fiber coordinates  $y$ 's, i.e. any graded bundle is a **polynomial bundle**.

- As these polynomials need not to be linear, **graded bundles do not have, in general, vector space structure in fibers.**
- If all  $w_i \leq r$ , we say that the graded bundle is **of degree  $r$ .**
- In the above terminology, **vector bundles are just graded bundles of degree 1.**

# Graded bundles - examples

- Note that, according to our convention, any differential manifold  $M$  can be viewed as a graded bundle of degree 0.
- A trivial example is of course

$$F = M \times \mathbb{R}^d = M \times (\mathbb{R}^{d_1}[1] \oplus \cdots \oplus \mathbb{R}^{d_k}[k]).$$

- Another trivial example, is a **split graded bundle**, i.e. a **graded vector bundle**

$$F = E^1[1] \oplus_M \cdots \oplus_M E^k[k],$$

where  $E^i$  are vector bundles over  $M$ .

- For vector bundles  $E^0, E^1$  over  $M$ , we can consider the vector bundle  $E = E^0[0] \oplus E^1[1]$  as a vector bundle over  $E^0$ . The wedge product  $\wedge^2 E = \wedge^2 E^0 \oplus (E^0 \otimes E^1) \oplus \wedge^2 E^1$  can be then viewed as a graded vector bundle over  $\wedge^2 E^0$  of degree 2, with  $(E^0 \otimes E^1)$  being its part of degree 1 and  $\wedge^2 E^1$  being of degree 2.
- Note that objects similar to graded bundles have been used in supergeometry by Ševera, Voronov, Roytenberg et al. under the name **N-manifolds**. However, during this course we will work exclusively with classical purely even manifolds.

# Homogeneity structure of a graded bundle

- Note that the homogeneity structure in the typical fiber of a graded bundle  $F$ , i.e. the action  $h : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ , is preserved under the transition functions, that defines a globally defined homogeneity structure  $h : \mathbb{R} \times F \rightarrow F$ .
- In local homogeneous coordinates,

$$h_t(x^A, y_w^a) = (x^A, t^w y_w^a).$$

- We call a function  $f : F \rightarrow \mathbb{R}$  **homogeneous of degree (weight)  $w$**  if

$$f \circ h_t = t^w f.$$

- The whole information about the degree of homogeneity is contained in the **weight vector field** (called for vector bundles the **Euler vector field**)

$$\nabla_F = \sum_a w y_w^a \partial_{y_w^a}.$$

- A function  $f : F \rightarrow \mathbb{R}$  is homogeneous of degree  $w$  if and only if

$$\nabla_F(f) = w f.$$

# The category of graded bundles

Mimicking the definition of a vector bundle morphism, we get the following.

## Definition

**Morphisms** in the **category of graded bundles** are represented by commutative diagram of smooth maps

$$\begin{array}{ccc} F^1 & \xrightarrow{\Phi} & F^2 \\ \downarrow \tau_1 & & \downarrow \tau_2 \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

which are morphisms of graded spaces in fibers, i.e. which locally preserve the weight of homogeneous coordinates.

One can equivalently say that the fiber bundle morphism  $\Phi$  is a smooth map which relates the weight vector fields  $\nabla_{F^1}$  and  $\nabla_{F^2}$ .

**Example.** Morphisms  $\Phi : F \rightarrow F$ , for  $F = \mathbb{R} \times \mathbb{R}^{(1,1)}$  with local coordinates  $(x, y, z)$  of degrees  $(0, 1, 2)$ , respectively, are of the form  $\Phi(x, y, z) = (\phi(x), a(x)y, b(x)z + d(x)y^2)$ .

# Graded bundle = homogeneity structure

The fundamental fact (cf. [Grabowski-Rotkiewicz]) says that **graded bundles and homogeneity structures are in fact equivalent concepts.**

## Theorem

*Associating the homogeneity structure with a graded bundle is an isomorphism of categories. In particular, for any homogeneity structure  $h$  on a manifold  $F$ , there is a smooth submanifold  $M = h_0(F) \subset F$  and a non-negative integer  $k \in \mathbb{N}$  such that  $h_0 : F \rightarrow M$  is canonically a graded bundle of degree  $k$  whose homogeneity structure coincides with  $h$ . In other words, there is an atlas on  $F$  consisting of local homogeneous functions.*

Since **morphisms** of two homogeneity structures are defined as smooth maps  $\Phi : F_1 \rightarrow F_2$  intertwining the  $\mathbb{R}$ -actions:  $\Phi \circ h_t^1 = h_t^2 \circ \Phi$ , this describes also morphism of graded bundles.

Consequently, a **graded subbundle** of a graded bundle  $F$  is a smooth submanifold  $S$  of  $F$  which is invariant with respect to homotheties,  $h_t(S) \subset S$  for all  $t \in \mathbb{R}$ .

# Consequences for vector bundles

Vector bundles can be recognized as graded bundles  $\tau : F \rightarrow M$  of degree 1, i.e. satisfying the following **regularity condition**:

$$\frac{d}{dt}\Big|_{t=0} h_t(p) = 0 \Leftrightarrow p \in M.$$

The principle **multiplication by reals is enough** has now the following consequences for vector bundles.

## Corollary

*A smooth map  $\Phi : E_1 \rightarrow E_2$  between the total spaces of two vector bundles  $\pi_i : E_i \rightarrow M_i$ ,  $i = 1, 2$  is a morphism of vector bundles if and only if it intertwines the multiplications by reals:*

$$\Phi(t \cdot v) = t \cdot \Phi(v).$$

*In this case the map  $\phi = \Phi|_{M_1}$  is a smooth map between the base manifolds covered by  $\Phi$ .*



# Graded bundles - further examples

- **Example.** Consider the second-order tangent bundle  $T^2M$ , i.e. the bundle of second jets of smooth maps  $(\mathbb{R}, 0) \rightarrow M$ .

Writing paths in local coordinates  $(x^A)$  on  $M$ :

$$x^A(t) = x^A(0) + \dot{x}^A(0)t + \ddot{x}^A(0)\frac{t^2}{2} + o(t^2),$$

we get local coordinates  $(x^A, \dot{x}^B, \ddot{x}^C)$  on  $T^2M$ , which transform

$$x'^A = x'^A(x),$$

$$\dot{x}'^A = \frac{\partial x'^A}{\partial x^B}(x) \dot{x}^B,$$

$$\ddot{x}'^A = \frac{\partial x'^A}{\partial x^B}(x) \ddot{x}^B + \frac{\partial^2 x'^A}{\partial x^B \partial x^C}(x) \dot{x}^B \dot{x}^C.$$

- This shows that associating with  $(x^A, \dot{x}^B, \ddot{x}^C)$  the weights  $0, 1, 2$ , respectively, will give us a graded bundle structure of degree  $2$  on  $T^2M$ . **Due to the quadratic terms above, this is not a vector bundle!**

# Digression about higher order mechanics

- All this can be generalised to higher tangent bundles  $T^k M$  (which are not vector bundles). Any smooth map  $\phi : M_1 \rightarrow M_2$  induces a canonical morphism of graded bundles  $T^k \phi : T^k M_1 \rightarrow T^k M_2$ .
- Note that the bundle  $T^k M$  is actually the bundle of  $k$ -th jets of curves in  $M$  and therefore it is the starting point for Lagrangian mechanics with higher-order Lagrangians.
- A  $k$ -th order Lagrangian  $L$  is a function  $L : T^k M \rightarrow \mathbb{R}$ .
- The motion of a system is given by a parameterized curve  $\gamma : (a, b) \rightarrow M$  in the manifold  $M$ .
- A  $k$ -th order infinitesimal piece of the motion is the  $k$ -th jet of the curve,  $j^k(\gamma)(t)$ .
- The dynamics should be an equation (in general, implicit) for curves in the phase space, i.e.  $\mathcal{D} \subset TT^*M$ .
- A curve  $\sigma : (a, b) \rightarrow T^*M$  in the phase space is a solution of  $\mathcal{D}$  if and only if its tangent extension  $t\sigma : (a, b) \rightarrow TT^*M$  takes values in  $\mathcal{D}$ .
- How to construct  $\mathcal{D}$  from  $L$  we will discuss later using the geometry of (double) graded bundles.

# Graded bundles - further example

- **$n$ -vectors on a vector bundle** If  $\tau : E \rightarrow M$  is a vector bundle, then  $\wedge^2 TE$  is canonically a graded bundle of degree 2 with respect to the projection

$$\wedge^2 T\tau : \wedge^2 TE \rightarrow \wedge^2 TM.$$

- The adapted coordinates  $(x^\rho, y^a, \dot{x}^{\mu\nu}, y^{\sigma b}, z^{cd})$  on  $\wedge^2 E$ , with  $\dot{x}^{\mu\nu} = -\dot{x}^{\nu\mu}$ ,  $z^{cd} = -z^{dc}$ , coming from the decomposition of a bivector

$$\wedge^2 TE \ni u = \frac{1}{2} \dot{x}^{\mu\nu} \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} + y^{\sigma b} \frac{\partial}{\partial x^\sigma} \wedge \frac{\partial}{\partial y^b} + \frac{1}{2} z^{cd} \frac{\partial}{\partial y^c} \wedge \frac{\partial}{\partial y^d},$$

are of degrees 0, 1, 0, 1, 2, respectively.

- All this can be generalized to a graded bundle structure of degree  $r$  on  $\wedge^r TE$ :

$$\wedge^r T\tau : \wedge^r TE \rightarrow \wedge^r TM.$$

- We want to build a framework for generating (first-order) dynamics for higher dimensional objects, being motivated by the study of dynamics of one-dimensional ones (curves).

# Digression on field theories

- The **motion** of a system is given by an  $n$ -dimensional submanifold in the manifold  $M$  ("space-time").
- An infinitesimal piece of the motion is the first jet of the submanifold.
- However, this model leads to essential complications even in one-dimensional case (relativistic particle).
- For instance, the infinitesimal action (Lagrangian) is not a function on first jets, but a section of certain line bundle over the first-jet manifold, a 'dual' of the bundle of "first jets with volumes".
- **Compromise:** take for the space of infinitesimal pieces of motions the space of simple  $n$ -vectors, which represent first jets of  $n$ -dimensional submanifolds together with an infinitesimal volume.
- It is technically convenient to extend this space to all  $n$ -vectors, i.e. to the vector bundle  $\wedge^n TM$  of  $n$ -vectors on  $M$ .

# Dynamics with the configuration space $\wedge^n TM$

- A **Lagrangian**  $L$  is a function

$$L : \wedge^n TM \rightarrow \mathbb{R}.$$

- The phase space of the system  $\wedge^n T^*M$ .
- The **dynamics** should be an equation (in general, implicit) for  $n$ -dimensional submanifolds in the phase space, i.e.

$$\mathcal{D} \subset \wedge^n T \wedge^n T^*M.$$

- A submanifold  $S$  in the phase space  $\wedge^n T^*M$  is a **solution** of  $\mathcal{D}$  if and only if its tangent space  $T_\alpha S$  at  $\alpha \in \wedge^n T^*M$  is represented by a  $n$ -vector from  $\mathcal{D}_\alpha$ .
- How to construct  $\mathcal{D}$  from  $L$  we will discuss later using the geometry of (double) graded bundles.

# Transition functions for graded bundles

- Let us go back to graded bundles. For a graded bundle  $F$  one can pick an atlas of  $F$  consisting of charts for which we have homogeneous local coordinates  $(x^A, y_w^a)$  with weight  $\text{deg}$ , where  $\text{deg}(x^A) = 0$  and  $\text{deg}(y_w^a) = w$  with  $1 \leq w \leq k$ , where  $k$  is the degree of the graded bundle. Here,  $a$  should be considered as a 'generalised index' running over all the possible weights. The label  $w$  in this respect is somewhat redundant, but it will come in very useful.
- The local changes of coordinates are of the form

$$\begin{aligned}x'^A &= x'^A(x), \\y_w'^a &= y_w^b T_b^a(x) + \sum_{\substack{1 \leq n \\ w_1 + \dots + w_n = w}} \frac{1}{n!} y_{w_1}^{b_1} \cdots y_{w_n}^{b_n} T_{b_n \dots b_1}^a(x),\end{aligned}\tag{1}$$

where  $T_b^a$  are invertible and  $T_{b_n \dots b_1}^a$  are symmetric in indices  $b$ .

- In particular, the transition functions of coordinates of degree  $r$  involve only coordinates of degree  $\leq r$ , defining a reduced graded bundle  $F_r$  of degree  $r$  (we simply 'forget' coordinates of degrees  $> r$ ).

# The tower of affine fibrations

- Transformations for the canonical projection  $F_r \rightarrow F_{r-1}$  are linear modulo a shift by a polynomial in variables of degrees  $< r$ ,

$$y_r^a = y_r^b T_b^a(x) + \sum_{\substack{1 \leq n \\ w_1 + \dots + w_n = r}} \frac{1}{n!} y_{w_1}^{b_1} \cdots y_{w_n}^{b_n} T_{b_n \dots b_1}^a(x),$$

so the fibrations  $F_r \rightarrow F_{r-1}$  are **affine**. The linear part of  $F_r$  corresponds to a vector subbundle  $\bar{F}_r$  over  $M$  (we put  $y_w^a$ , with  $0 < w < r$ , equal to 0).

- In this way we get for any graded bundle  $F$  of degree  $k$ , like for jet bundles, a tower of affine fibrations

$$F = F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M.$$

- **Example.** In the case of the canonical graded bundle  $F = T^k M$ , we get exactly the tower of projections of jet bundles

$$T^k M \xrightarrow{\tau^k} T^{k-1} M \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} T^2 M \xrightarrow{\tau^2} TM \xrightarrow{\tau^1} F_0 = M.$$

# Further constructions

- The reduced manifold  $F_r$  will also be denoted  $F[\nabla \leq r]$  if we want to stress which weight vector field  $\nabla$  we have in mind (sometimes we will work with many).
- There is also a “dual” sequence of submanifolds and their inclusions

$$M := F_0 = F^{[k]} \hookrightarrow F^{[k-1]} \hookrightarrow \dots \hookrightarrow F^{[0]} = F_k, \quad (2)$$

where we define, locally but correctly,

$$F^{[i]} := \{p \in F_k \mid y_w^a = 0 \text{ if } w \leq i\}.$$

- In words, “you project higher to lower, but set to 0 lower to higher”.
- Note that the  $C^\infty(M)$ -module  $\mathcal{A}^r(F)$  of homogeneous functions of degree  $r$  on  $F$  is finitely generated and projective, so it corresponds to sections of a vector bundle  $\mathcal{A}^r(F)$  over  $M$ . The graded algebra

$$\mathcal{A}(F) = \bigoplus_{i=0}^{\infty} \mathcal{A}^i(F)$$

generated by homogeneous functions is called the **polynomial algebra of  $F$** .



# Splitting of graded bundles

- Homogeneous local coordinates  $(y_i^a)$  of degree  $i > 0$  represent locally a basis of the quotient vector bundle

$$\mathcal{A}^i(F)/\bar{\mathcal{I}}^i(F) \simeq \bar{F}_i^*,$$

where  $\bar{\mathcal{I}}^i(F) = (\mathcal{I}(F) \cdot \mathcal{I}(F))^i$  is the degree  $i$  part of the algebraic square  $\mathcal{I}(F) \cdot \mathcal{I}(F)$  of the ideal  $\mathcal{I}(F)$  in the polynomial algebra  $\mathcal{A}(F)$  generated by homogeneous functions of degrees  $> 0$ ,

$$\mathcal{I}(F) = \bigoplus_{i>0} \mathcal{A}^i(F).$$

- Choosing vector subbundles  $\bar{\mathcal{A}}^i(F) \subset \mathcal{A}^i(F)$  complementary to  $\bar{\mathcal{I}}^i(F)$ , we can pick up local coordinates of degree  $i$  from  $\bar{\mathcal{A}}^i(F) \subset \mathcal{A}^i(F)$ , killing therefore the higher order polynomial parts in transition rules. This gives the following.

## Theorem

Any graded bundle  $F$  of degree  $k$  is isomorphic with the split graded bundle  $\bar{F} = \bar{F}^1 \oplus \dots \oplus \bar{F}^k$ .

# Splitting of graded bundles – comments

- The point is that this isomorphism is **not canonical**. Also the morphism of graded vector bundles in the category of graded bundles differ from graded vector bundle morphism which makes these categories different.
- The situation is similar to the celebrated **Batchelor Theorem** in supergeometry stating that any supermanifold is (non-canonically) diffeomorphic with the 'superization'  $\Pi E$  of a vector bundle  $E$ . Of course, morphisms of such supermanifolds are different from that of vector bundles, so these categories are completely different.
- The Betchelor Theorem was actually proved first by Polish physicist Gawędzki, that provides therefore another example of the **Arnold's law** saying that *"Discoveries are rarely attributed to the correct person"*.
- Of course Arnold's law is self-referential, as Whitehead claimed earlier that *"Everything of importance has been said before by someone who did not discover it"*.

# Tangent lifts of graded structures

- Consider an arbitrary graded bundle  $F_k$  over  $M$  of minimal degree  $k$  with homogeneous coordinates  $(x^A, y_w^a)$ ,  $1 \leq w \leq k$ . The corresponding homogeneity structure is then

$$h_t(x^A, y_w^a) = (x^A, t^w y_w^a)$$

and the weight vector field:  $\nabla_F := \sum_w w y_w^a \frac{\partial}{\partial y_w^a}$ .

- Applying the tangent functor to all  $h_t$ , we get a homogeneity structure  $(d_T h)_t = T(h_t)$  on  $TF$ :

$$(d_T h)_t(x^A, y_w^a, \dot{x}^B, \dot{y}_w^b) = (x^A, t^w y_w^a, \dot{x}^B, t^w \dot{y}_w^b).$$

- The corresponding weight vector field is the **tangent lift** of  $\nabla_F$ :

$$\nabla_{TF} = d_T \nabla_F = \sum_w w y_w^a \frac{\partial}{\partial y_w^a} + \sum_w w \dot{y}_w^a \frac{\partial}{\partial \dot{y}_w^a}.$$

# Phase lifts of graded structures

- Similarly we can try to lift  $h_t$  to the cotangent bundle  $T^*F$  with the adapted coordinates  $(x^A, y_w^a, p_B, p_b^w)$ ; for  $t \neq 0$ :

$$(Th_t)^*(x^A, y_w^a, p_B, p_b^w) = (x^A, t^{-w} y_w^a, p_B, t^w p_b^w).$$

- As this cannot be directly extended to an action of  $\mathbb{R}$ , we define the **phase lift** as  $(d_{T^*}^k h)_t = t^k (T(h_{t-1})^*)$ :

$$(d_{T^*}^k h)_t(x^A, y_w^a, p_B, p_b^w) = (x^A, t^w y_w^a, t^k p_B, t^{k-w} p_b^w).$$

- The associated weight vector field reads

$$\nabla_{T^*F} = d_{T^*}^k \nabla_F = \sum_w w y_w^a \frac{\partial}{\partial y_w^a} + k p_B \frac{\partial}{\partial p_B} + \sum_w (k - w) p_a^w \frac{\partial}{\partial p_a^w}.$$

- In this way, the tangent and cotangent bundles are canonically graded bundles of degree  $k$  over  $F$  and  $\bar{F}_k^*$ , respectively.

# Higher lifts and canonical isomorphisms

- Using higher tangent functors  $T^k$ , we can lift homogeneity structures on  $F$  to homogeneity structures on  $T^k F$  simply putting

$$(d_{T^k h})_t = T^k(h_t) : T^k F \rightarrow T^k F.$$







- We have fundamental isomorphisms between iterated higher tangent and cotangent bundles.

## Theorem (Cantrijn-Crampin-Sarlet-Saunders-Tulczyjew)

For any manifold  $M$  and any  $k \in \mathbb{N}$ , there is a canonical isomorphism

$$T^*T^k M \simeq T^k T^* M.$$

- The corresponding graded bundle structure  $T^k T^* M \rightarrow T^* M$  and the vector bundle structure  $T^* T^k M \rightarrow T^k M$  are compatible in a natural sense, so that  $T^* T^k M \simeq T^k T^* M$  is a canonical example of a **double graded bundle**, which will be discussed in the next talk.

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**THANK YOU FOR YOUR ATTENTION!**

# Homework

- **Problem 1.** Prove that any real vector space structure on  $\mathbb{R}^n$ , with zero at  $0 \in \mathbb{R}^n$ , coincides with the standard one.
- **Problem 2.** Prove directly that any smooth function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , which satisfies  $f(t \cdot x) = t^k \cdot f(x)$  for some  $k \in \mathbb{N}$  and all  $t \in \mathbb{R}$ , is a polynomial.
- **Problem 3.** Show that a submanifold  $E_0$  of a vector bundle  $E$  over  $M$  is a vector subbundle (possibly covering a submanifold  $M_0 \subset M$ ) if and only if it is invariant with respect to all homotheties, i.e.  $h_t(E_0) \subset E_0$  for all  $t \in \mathbb{R}$ .
- **Problem 4.** Find a split graded bundle isomorphic to the graded bundle  $T^2M$ .
- **Problem 5.** Let  $\tau : E \rightarrow M$  be a vector bundle. What is the base of the vector bundle structure on  $T^*E$  being the 1-phase lift of the vector bundle (graded bundle of degree 1) structure on  $E$ ?