DOUBLE STRUCTURES AND ALGEBROIDS

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Plan of the talk

- Double vector bundles
- n-fold graded bundles
- Canonical examples
- Canonical isomorphism
- Graded-linear bundles
- Lie algebroids
- General algebroids
- Non-holonomic reduction
- Groupoids
- Lie groupoids and their Lie algebroids
- Examples: Pair and Ehresmann groupoids

Double vector bundles

In geometry and applications one often encounters double vector bundles, i.e. manifolds equipped with two vector bundle structures which are compatible in a categorical sense. They were defined by Pradines and studied by Mackenzie, Konieczna (Grabowska), and Urbański as vector bundles in the category of vector bundles. More precisely:

Definition

A double vector bundle $(D; A, B; M)$ is a system of four vector bundle structures

$$\begin{array}{ccc} D & \xrightarrow{q^D_B} & B \\ {\scriptstyle q^D_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow{q_A} & M \end{array}$$

in which D has two vector bundles structures, on bases A and B. The latter are themselves vector bundles on M, such that each of the four structure maps of each vector bundle structure on D (namely the bundle projection, zero section, addition and scalar multiplication) is a morphism of vector bundles with respect to the other structures.

The structure of double vector bundles

- In the above figure, we refer to A and B as the side bundles of D, and to M as the double base.
- In the two side bundles, the addition and scalar multiplication are denoted by the usual symbols $+$ and juxtaposition, respectively.
- We distinguish the two zero-sections, writing $0^A : M \to A$, $m \mapsto 0^A_m$, and $0^B : M \to B$, $m \mapsto 0^B_m$.
- In the vertical bundle structure on D with base A, the vector bundle operations are denoted by $+_A$ and $\cdot_A$, with $\tilde{0}^A : A \to D$, $a \mapsto \tilde{0}^A_a$, for the zero-section.
- Similarly, in the horizontal bundle structure on D with base B we write $+_B$ and $\cdot_B$, with $\tilde{0}^B : B \to D$, $b \mapsto \tilde{0}^B_b$, for the zero-section.
- The two structures on D, namely (D, q^D_B, B) and (D, q^D_A, A) will also be denoted, respectively, by $\tilde{D}_B$ and $\tilde{D}_A$, and called the horizontal bundle structure and the vertical bundle structure.

Double vector bundles - compatibility conditions

The condition that each vector bundle operation in D is a morphism with respect to the other is equivalent to the following conditions, known as the interchange laws:

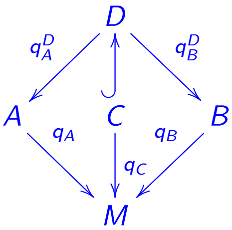
$$
\begin{aligned}
(d_1 +_B d_2) +_A (d_3 +_B d_4) &= (d_1 +_A d_3) +_B (d_2 +_A d_4),\\
t \cdot_A (d_1 +_B d_2) &= t \cdot_A d_1 +_B t \cdot_A d_2,\\
t \cdot_B (d_1 +_A d_2) &= t \cdot_B d_1 +_A t \cdot_B d_2,\\
t \cdot_A (s \cdot_B d) &= s \cdot_B (t \cdot_A d),\\
\tilde{0}^A_{a_1+a_2} &= \tilde{0}^A_{a_1} +_B \tilde{0}^A_{a_2},\\
\tilde{0}^A_{ta} &= t \cdot_B \tilde{0}^A_{a},\\
\tilde{0}^B_{b_1+b_2} &= \tilde{0}^B_{b_1} +_A \tilde{0}^A_{b_2},\\
\tilde{0}^B_{tb} &= t \cdot_A \tilde{0}^B_{b}.
\end{aligned}
$$

The core

We denote by C the intersection of the two kernels:

$$C = \{c \in D \mid \exists\ m \in M \text{ such that } q^D_B(c) = 0^B_m, \quad q^D_A(c) = 0^A_m\},$$

which is called the core, and together with the map $q_C : c \mapsto m$, (C, q_C, M) is also a vector bundle over M.
Eventually we can write the diagram below to emphasis the core of the relevant double vector bundle.



Double vector bundles - reference example

- Let $q_A : A \to M$, $q_B : B \to M$, $q_C : C \to M$ be vector bundles.
- Consider the manifold
$$D = A \times_M B \times_M C\,.$$
- D is a double vector bundle (with side bundles A and B, and the core C) with respect to the obvious projections
$$q^D_A : D \ni (a_m, b_m, c_m) \mapsto a_m \in A\,, \quad q^D_B : D \ni (a_m, b_m, c_m) \mapsto b_m \in B\,,$$
obvious embeddings
$$\tilde{0}^A : A \ni a_m \mapsto (a_m, 0^B_m, 0^C_m) \in D\,, \quad \tilde{0}^B : B \ni b_m \mapsto (0^A_m, b_m, 0^C_m) \in D\,,$$
and obvious vector space structures in fibers:
$$(a_m, b_m, c_m) +_A (a_m, b'_m, c'_m) = (a_m, b_m + b'_m, c_m + c'_m)\,, \text{ etc.}$$
- Actually, every double vector bundle is locally of this form.
- In particular, any Whitney direct sum $A \oplus_M B$, identified with $\simeq A \times_M B$, can be given a double vector bundle structure.

Double Graded Bundles

- We can extend the concept of a double vector bundle of Pradines to double graded bundles.
- However, thanks to our simple description in terms of a homogeneity structure, the 'diagrammatic' definition of Pradines can be substantially simplified.
- As two graded bundle structure on the same manifold are just two homogeneity structures, the obvious concept of compatibility leads to the following:

Definition (Grabowski-Rotkiewicz)

A double graded bundle is a manifold equipped with two homogeneity structures h^1, h^2 which are compatible in the sense that

$$h^1_t \circ h^2_s = h^2_s \circ h^1_t \quad \text{for all } s, t \in \mathbb{R}\,.$$

n-fold Graded Bundles

- The above condition can also be formulated as commutation of the corresponding weight vector fields, $[\nabla^1, \nabla^2] = 0$.
- For vector bundles this is equivalent to the concept of a double vector bundle in the sense of Pradines and Mackenzie.

Theorem (Grabowski-Rotkiewicz)

The concept of a double vector bundle, understood as a particular double graded bundle in the above sense, coincides with that of Pradines.

- All this can be extended to n-fold graded bundles in the obvious way:

Definition

A n-fold graded bundle is a manifold equipped with n homogeneity structures $h^1, \dots, h^n$ which are compatible in the sense that

$$h^i_t \circ h^j_s = h^j_s \circ h^i_t \quad \text{for all } s, t \in \mathbb{R} \quad \text{and} \quad i, j = 1, \dots, n\,.$$

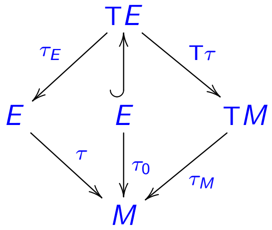
Double graded bundles - examples

Proposition

The tangent and phase lifts of graded bundles are compatible with the vector bundle structures of the tangent (resp., cotangent) bundle.

First example: $\mathsf{T}E$.

$$\begin{aligned} \tau : E &\longrightarrow M \\ (x^a, y^i) &\longmapsto (x^a) \end{aligned} \qquad \begin{aligned} \tau_M : \mathsf{T}M &\longrightarrow M \\ (x^a, \dot{x}^b) &\longmapsto (x^a) \end{aligned}$$



$$\nabla^1 = \dot{x}^a \partial_{\dot{x}^a} + \dot{y}^i \partial_{\dot{y}^i}$$

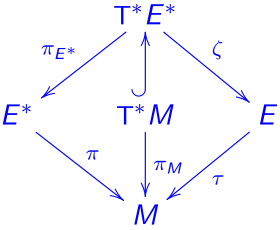
$$\nabla^2 = \mathrm{d}_{\mathsf{T}}(y^i \partial_{y^i}) = y^i \partial_{y^i} + \dot{y}^j \partial_{\dot{y}^j}$$

Double graded bundles - examples

Second example: T^*E^*.

$$\pi_{E^*} : \mathsf{T}^*E^* \longrightarrow E^*$$
$$(x^a, \xi_i, p_b, y^j) \longmapsto (x^a, \xi_i)$$

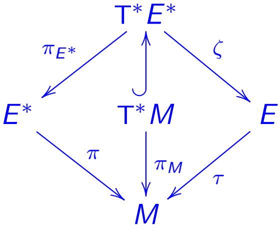
$$\zeta : \mathsf{T}^*E^* \longrightarrow E$$
$$(x^a, \xi_i, p_b, y^j) \longmapsto (x^a, y^j)$$



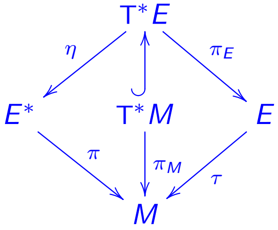
$$\nabla^1 = p_a \partial_{p_a} + y^i \partial_{y^i}\,, \qquad \nabla^2 = p_a \partial_{p_a} + \xi_i \partial_{\xi_i}\,.$$

Canonical isomorphism

Canonical isomorphism: $\mathsf{T}^*E^* \simeq \mathsf{T}^*E$.



(x^a, ξ_i, p_b, y^j)



(x^a, y^i, p_b, ξ_j)

T^*E^* is (symplectically) isomorphic to T^*E. The graph of the canonical d.v.b. anti-symplectic isomorphism $\mathcal{R}$ is the lagrangian submanifold generated in

$$\mathsf{T}^*(E^* \times E) \simeq \mathsf{T}^*E^* \times \mathsf{T}^*E \quad \text{by} \quad E^* \times_M E \ni (\xi, y) \longmapsto \xi(y) \in \mathbb{R}.$$
$$\mathcal{R} : (x^a, y^i, p_b, \xi_j) \longmapsto (x^a, \xi_i, -p_b, y^j)\,.$$

Graded linear bundles

- A double graded bundle whose one structure is linear we will call a graded linear bundle (GrL-bundle). Canonical examples are $\mathsf{T}F$ and T^*F with the lifted and the vector bundle structures.
  Iterated lifts, $\mathsf{T}\mathsf{T}^*F \simeq \mathsf{T}^*\mathsf{T}F$ lead to triple structures of this kind.
- Example. The weight vector field of the lifted graded structure on $\mathsf{T}\mathsf{T}^2M$ with coordinates $(x^a, \dot{x}^b, \ddot{x}^c, \delta x^d, \delta\dot{x}^e, \delta\ddot{x}^f)$ is

$$\nabla^2 = \dot{x}^b\partial_{\dot{x}^b} + 2\ddot{x}^c\partial_{\ddot{x}^c} + \delta\dot{x}^e\partial_{\delta\dot{x}^e} + 2\delta\ddot{x}^f\partial_{\delta\ddot{x}^f}\,.$$

  It yields a GrL-bundle with the standard Euler vector field of the tangent bundle structure $\nabla^1 = \delta x^d\partial_{\delta x^d} + \delta\dot{x}^e\partial_{\delta\dot{x}^e} + \delta\ddot{x}^f\partial_{\delta\ddot{x}^f}$.
- Another example: if $\tau : E \to M$ is a vector bundle, then $\wedge^k\mathsf{T}E$ is canonically a GrL-bundle:

$$\begin{array}{ccc} & \wedge^k\mathsf{T}E & \\ \swarrow & & \searrow \\ E & & \wedge^k\mathsf{T}M \\ \searrow & & \swarrow \\ & M & \end{array}\,.$$

Linearity

Linearity of different geometrical structures is usually related to some double vector bundle structures.

- A bivector field Π on a vector bundle E is linear if the corresponding map
$$\widetilde{\Pi}:\mathsf{T}^*E\longrightarrow\mathsf{T}E$$
is a morphism of double vector bundles.
- A two-form ω on a vector bundle E is linear if the corresponding map
$$\widetilde{\omega}:\mathsf{T}E\longrightarrow\mathsf{T}^*E$$
is a morphism of double vector bundles.
- A (linear) connection on a vector bundle E is a morphisms of double vector bundles $\Gamma:E\times_M\mathsf{T}M\to\mathsf{T}E$, that acts as the identity on the vector bundles E and $\mathsf{T}M$:
$$(\nabla_X\sigma)^v=\mathsf{T}\sigma(X)-\Gamma(\sigma,X)\,,$$
where $\sigma^v=y^i(x)\partial_{\dot{y}^i}$ is the vertical lift of the section σ:
$$M\ni x\mapsto\sigma(x)=(y^i(x))\in E\,.$$

Lie algebroids

- $\tau : E \to M$ is a rank-n vector bundle over an m-dimensional manifold M, and $\pi : E^* \to M$ its dual;
- $\mathcal{A}^i(E) = \mathrm{Sec}(\wedge^i E)$, for $i = 0, 1, 2, \ldots$, the module of sections of the bundle $\wedge^i E$.
- $\mathcal{A}(E) = \bigoplus_{i\in\mathbb{N}} \mathcal{A}^i(E)$ the Grassmann algebra of multi-sections of E.

We use affine coordinates (x^a, ξ_i) on E^* and the dual coordinates (x^a, y^i) on E.

Definition

A Lie algebroid structure on E is given by a linear Poisson tensor Π on E^*, $[\Pi, \Pi]_{Schouten} = 0$. In local coordinates,

$$\Pi = \frac{1}{2} c_{ij}^k(x) \xi_k \partial_{\xi_i} \wedge \partial_{\xi_j} + \rho_i^b(x) \partial_{\xi_i} \wedge \partial_{x^b} ,$$

where $c_{ij}^k(x) = -c_{ji}^k(x)$.

Lie algebroids - equivalent definitions

The bivector field Π defines a Poisson bracket $\{\cdot,\cdot\}_\Pi$ on the algebra $C^\infty(E^*)$ of smooth functions on E^* by $\{\phi,\psi\}_\Pi = \langle \Pi, \mathrm{d}\phi \wedge \mathrm{d}\psi\rangle$.

Theorem

A Lie algebroid structure (E,Π) can be equivalently defined as

- *a Lie bracket $[\cdot,\cdot]_\Pi$ on the space $\mathrm{Sec}(E)$, together with a vector bundle morphisms $\rho\colon E \to TM$ (the anchor), such that*

$$[X, fY]_\Pi = \rho(X)(f)Y + f[X,Y]_\Pi, \tag{1}$$

for all $f \in C^\infty(M)$, $X, Y \in \mathrm{Sec}(E)$,

- *or as a homological derivation d^Π of degree 1 in the Grassmann algebra $\mathcal{A}(E^*)$ (de Rham derivative). The latter is a map $\mathrm{d}^\Pi : \mathcal{A}(E^*) \to \mathcal{A}(E^*)$ such that $\mathrm{d}^\Pi : \mathcal{A}^i(E^*) \to \mathcal{A}^{i+1}(E^*)$, $(\mathrm{d}^\Pi)^2 = 0$, and that, for $\alpha \in \mathcal{A}^a(E^*)$, $\beta \in \mathcal{A}^b(E^*)$ we have*

$$\mathrm{d}^\Pi(\alpha\wedge\beta) = \mathrm{d}^\Pi\alpha\wedge\beta + (-1)^a\alpha\wedge\mathrm{d}^\Pi\beta\,. \tag{2}$$

Lie algebroids - equivalent definitions

These objects are related to Π according to the formulae

$$\begin{aligned} \iota([X,Y]_\Pi) &= \{\iota(X),\iota(Y)\}_\Pi, \\ \pi^*(\rho(X)(f)) &= \{\iota(X),\pi^*f\}_\Pi\,, \\ (\mathrm{d}^\Pi\mu)^v &= [\Pi,\mu^v]_S\,. \end{aligned}$$

where $\iota(X)(e^*_p)=\langle X(p),e^*_p\rangle$, μ^v is the natural vertical lift of a k-form $\mu\in\mathcal{A}^k(E^*)$ to a vertical k-vector field on E^*, and $[\cdot,\cdot]_S$ is the Schouten bracket of multivector fields. In a local basis of sections $\{e_1,\dots,e_n\}$ of E and the corresponding local coordinates,

$$\begin{aligned} [e_i,e_j]_\Pi(x) &= c^k_{ij}(x)e_k, \\ \rho(e_i)(x) &= \rho^a_i(x)\partial_{x^a}, \\ \mathrm{d}^\Pi f(x) &= \rho^a_i(x)\frac{\partial f}{\partial x^a}(x)e^i, \\ \mathrm{d}^\Pi e^i(x) &= c^i_{lk}(x)e^k\wedge e^l\,. \end{aligned}$$

Lie algebroids - examples

- A Lie algebroid over a single point, with the zero anchor, is a Lie algebra.
- The tangent bundle, $\mathsf{T}M$, of a manifold M, with bracket the Lie bracket of vector fields and with anchor the identity of $\mathsf{T}M$, is a Lie algebroid over M. Any integrable sub-bundle of $\mathsf{T}M$, in particular the tangent bundle along the leaves of a foliation, is also a Lie algebroid.
- If (M, Λ) is a Poisson manifold, then the cotangent bundle T^*M is a Lie algebroid over M. The anchor is the map $\Lambda^\# : \mathsf{T}^*M \to \mathsf{T}M$ The Lie bracket $[,]_\Lambda$ of differential 1-forms satisfies $[\mathrm{d}f, \mathrm{d}g]_\Lambda = \mathrm{d}\{f, g\}_\Lambda$.
- If P is a principal bundle with structure group G, base M and projection p, the G-invariant vector fields on P are the sections of a vector bundle with base M, denoted $E = \mathsf{T}P/G$, and called the Atiyah algebroid of the principal bundle P. This vector bundle is a Lie algebroid, with bracket induced by the Lie bracket of G-invariant vector fields on P, and with surjective anchor induced by $\mathsf{T}p : \mathsf{T}P \to \mathsf{T}M$.

Lie algebroids - related objects

- For any section $X \in \mathrm{Sec}(E)$, the *Lie derivative* $\mathcal{L}_X$, acting in $\mathcal{A}(E)$ and $\mathcal{A}(E^*)$, is defined in the standard way:

$$\begin{aligned} \mathcal{L}_X(f) &= \rho(X)(f)\,, \quad \text{for} \quad f \in C^\infty(M)\,, \\ \mathcal{L}_X(Y_1 \wedge \cdots \wedge Y_l) &= \sum_{i=1}^{l} Y_1 \wedge \cdots \wedge [X, Y_i]_\Pi \wedge \cdots \wedge Y_a\,, \\ \mathcal{L}_X(\alpha) &= i_X \mathrm{d}^\Pi + \mathrm{d}^\Pi i_X\,. \end{aligned}$$

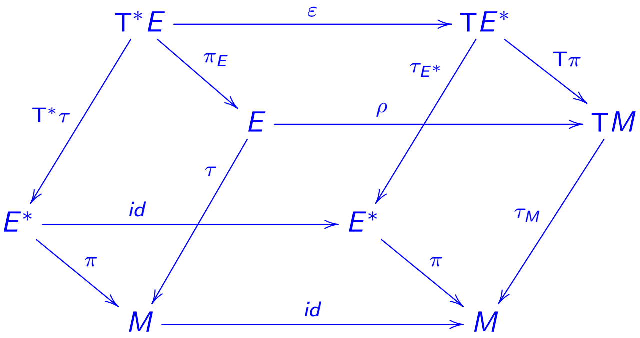
- We know that the linear bivector field Π on E^* induces a morphism of double vector bundles $\Pi^\# : \mathsf{T}^*E^* \to \mathsf{T}E^*$, covering the identity on E^*. Composing it with the canonical isomorphism $\mathcal{R} : \mathsf{T}^*E \to \mathsf{T}^*E^*$, we get a morphism of double vector bundles

$$\varepsilon_\Pi : \mathsf{T}^*E \to \mathsf{T}E^*\,,$$

covering the identity on E^*.

General algebroids

- A general algebroid is a double vector bundle morphism covering the identity on E^*:



In local coordinates,

$$\varepsilon(x^a, y^i, p_b, \xi_j) = (x^a, \xi_i, \rho_k^b(x) y^k, c_{ij}^k(x) y^i \xi_k + \sigma_j^a(x) p_a) \, .$$

Algebroids

Any such morphism is associated with a linear tensor field on E^*,

$$\Pi_\varepsilon = c^k_{ij}(x)\xi_k\partial_{\xi_i}\otimes\partial_{\xi_j} + \rho^b_i(x)\partial_{\xi_i}\otimes\partial_{x^b} - \sigma^a_j(x)\partial_{x^a}\otimes\partial_{\xi_j}.$$

We speak about a skew algebroid (resp., Lie algebroid) if the tensor Π_ε is skew-symmetric (resp., Poisson tensor).

Theorem

An algebroid structure (E,ε) can be equivalently defined as a bilinear bracket $[\cdot,\cdot]_\varepsilon$ on sections of $\tau\colon E\to M$, together with vector bundle morphisms $a^\varepsilon_l,\ a^\varepsilon_r\colon E\to \mathsf{T}M$ (left and right anchors), such that

$$[fX,gY]_\varepsilon = f(a^\varepsilon_l\circ X)(g)Y - g(a^\varepsilon_r\circ Y)(f)X + fg[X,Y]_\varepsilon$$

for $f,g\in\mathcal{C}^\infty(M)$, $X,Y\in\mathrm{Sec}(E)$.
For skew-algebroids the bracket is skew-symmetric, thus $a^\varepsilon_l = a^\varepsilon_r = \rho^\varepsilon$, and for Lie algebroids it satisfies the Jacobi identity,

$$[[X,Y]_\varepsilon,Z]_\varepsilon = [X,[Y,Z]_\varepsilon]_\varepsilon - [Y,[X,Z]_\varepsilon]_\varepsilon\,.$$

Non-holonomic reduction

Let ε be a Lie algebroid structure on a vector bundle E over M associated with the tensor Π_ε.
For a linear subbundle D in E, supported on the whole M, consider a decomposition

$$E = D \oplus_M D^\perp \tag{3}$$

and the associated projection $p : E \to D$. With such a decomposition we can associate a skew-algebroid structure on D.
The projection P induces a map on sections: $p : \mathrm{Sec}(E) \to \mathrm{Sec}(D)$ and thus a bracket

$$[X, Y]_{\varepsilon_p} = p[X, Y]_\varepsilon \tag{4}$$

on sections of D – the nonholonomic restriction of $[\cdot, \cdot]$ along p.
This is a skew algebroid bracket with the original anchor.
A particular case of this construction can be applied to a vector subbundle D of $\mathsf{T}M$, for M equipped with a Riemannian structure, e.g. nonholonomic systems with mechanical Lagrangians.

Groupoids

Definition

A groupoid over a set Γ_0 is a set Γ equipped with source and target mappings $\alpha, \beta : \Gamma \to \Gamma_0$, a multiplication map m from $\Gamma_2 \stackrel{\text{def}}{=} \{(g,h) \in \Gamma \times \Gamma \mid \beta(g) = \alpha(h)\}$ to Γ, an injective map $\epsilon : \Gamma_0 \to \Gamma$, and an involution $\iota : \Gamma \to \Gamma$, satisfying the following properties (where we write gh for $m(g,h)$ and g^{-1} for $\iota(g)$):

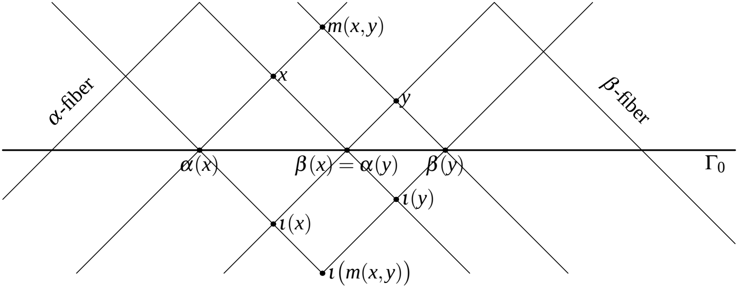
- (anchor) $\alpha(gh) = \alpha(g)$ and $\beta(gh) = \beta(h)$;
- (associativity) $g(hk) = (gh)k$ in the sense that, if one side of the equation is defined, so is the other, and then they are equal;
- (identities) $\epsilon(\alpha(g))g = g = g\epsilon(\beta(g))$;
- (inverses) $gg^{-1} = \epsilon(\alpha(g))$ and $g^{-1}g = \epsilon(\beta(g))$.

The elements of Γ_2 are sometimes referred to as composable (or admissible) pairs.

A groupoid Γ over a set Γ_0 will be denoted $\Gamma \rightrightarrows \Gamma_0$.

Groupoids: α- and β-fibers

- We can regard Γ_0 as a subset in Γ, and thus ϵ as the identity, that simplifies the picture, since α, β become just projections in Γ.
- The inverse images of points under the source and target maps we call α- and β-fibres. The fibres through a point g, will be denoted by $\mathcal{F}^\alpha(g)$ and $\mathcal{F}^\beta(g)$, respectively.



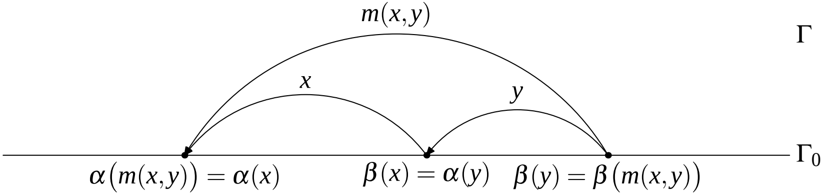
- Another approach to groupoids is that of Zakrzewski: in the definition of a group just replace maps with relations.

Groupoid as a small category

- The full information about the groupoid is contained in the multiplication relation:

$$\Gamma_3 = \{(x, y, z) \in \Gamma \times \Gamma \times \Gamma \mid (x, y) \in \Gamma_2 \text{ and } z = xy\} \ .$$

- Alternatively, a groupoid $\Gamma \rightrightarrows \Gamma_0$ is defined as a small category, i.e. a category whose objects form a set Γ_0, in which every morphism is an isomorphism. Elements of Γ represent morphisms in this category.



- Any group G is a groupoid over its neutral element, $G \rightrightarrows \{e\}$. Here, any morphism is an automorphism.

Lie groupoids

- In differential geometry we consider differentiable (Lie) groupoids (introduced by Ehresmann), i.e. groupoids $G \rightrightarrows M$, where G, G_2, G_3, M are smooth manifolds, α, β are smooth submersions, ϵ is an immersion and ι is a diffeomorphism.
- The anchor property implies that each element g of G determines the left and right translation maps

$$l_g : \mathcal{F}^\alpha(\beta(g)) \to \mathcal{F}^\alpha(\alpha(g))\,, \quad r_g : \mathcal{F}^\beta(\alpha(g)) \to \mathcal{F}^\beta(\beta(g))\,,$$

- Let us consider the vector bundle $\tau : A(G) \to M$, whose fiber at a point $x \in M$ is $A_x G = V_{\epsilon(x)}\alpha = Ker(T_{\epsilon(x)}\alpha)$.
- With any sections X of τ, $X \in \mathrm{Sec}(\tau)$, there is canonically associated a left-invariant vector field $\overleftarrow{X}$ on G, the left prolongation of X, namely,

$$\overleftarrow{X}(g) = (T_{\epsilon(\beta(g))} l_g)(X(\beta(g)))\,.$$

for $g \in G$. It is, by definition, tangent to α-fibers.

Lie algebroid of a Lie groupoid

- We can now introduce a Lie algebroid structure $([\cdot,\cdot],\rho)$ on $A(G)$, which is defined by

$$\overleftarrow{[X,Y]} = [\overleftarrow{X},\overleftarrow{Y}], \quad \rho(X)(x) = (T_{\epsilon(x)}\beta)(X(x)), \tag{5}$$

for $X, Y \in \Gamma(\tau)$ and $x \in M$.

- We recall that a Lie algebroid A over a manifold M is a real vector bundle $\tau : A \to M$ together with a skew-symmetric bracket $[\cdot,\cdot]$ on the space $\Gamma(\tau)$ of sections of $\tau : A \to M$ and a bundle map, called the anchor map, such that

$$[X, fY] = f[X,Y] + \rho(X)(f)Y,$$

for $X, Y \in \Gamma(\tau)$ and $f \in C^\infty(M)$. Here we denoted by ρ also the map induced by ρ on sections.

Theorem

For any groupoid $G \rightrightarrows M$, the formulae (5) define on $\tau : A(G) \to M$ the structure of a Lie algebroid.

Pair groupoid

Example

- Let M be a set and $\Gamma = M \times M$. define the source and target maps as
$$\alpha(u,v) = u\,, \quad \beta(u,v) = v\,.$$
- Then, $M \times M$ is a groupoid over M with the units mapping $\epsilon(u) = (u,u)$, and the partial composition by $(u,v)(v,z) = (u,z)$. In other words,
$$\Gamma_3 = \{(u,v,v,z,u,z) \in \Gamma \times \Gamma \times \Gamma \mid u,v,z \in M\}\,.$$
- Note that M can be viewed as embedded into $M \times M$ as the diagonal.
- If M is a manifold, we deal with a Lie groupoid. We can identify α-fibers as
$$\mathcal{F}^\alpha(u,u) = \{(u,v) \mid v \in M\} \simeq M\,,$$
so $A(\Gamma)$ with $\mathsf{T}M$. The left invariant vector field $\overleftarrow{X}$ tangent to α-fibers in Γ and corresponding to $X \in \mathcal{X}(M)$ is, under this identification, $\overleftarrow{X}(u,v) \simeq X(v)$. In consequence, the Lie algebroid of Γ is $\mathsf{T}M$ with the bracket of vector fields.

Ehresmann gauge groupoid

Example

- For $p : P \to M$ being a principal bundle with the structure group G, consider the set $\Gamma = (P \times P)/G$ of G-orbits, where G acts on $P \times P$ diagonally, $(v, u)g = (vg, ug)$.
- For the coset $\langle v|u\rangle$ of (v, u), define the source and target maps
$$\alpha\langle v|u\rangle = p(u) \quad \beta\langle v|u\rangle = p(v)\,,$$
and the (partial) multiplication $\langle w|v\rangle\langle v|u\rangle = \langle w|u\rangle$.
- It is well defined, as
$$\alpha\langle w|v\rangle = \beta\langle v'|u'\rangle \Leftrightarrow v' = vg$$
and
$$\langle w|v\rangle\langle vg|ug\rangle = \langle wg|vg\rangle\langle vg|ug\rangle = \langle wg|ug\rangle = \langle w|u\rangle = \langle w|v\rangle\langle v|u\rangle\,.$$
In this way we obtained a Lie groupoid $\Gamma = (P \times P)/G \rightrightarrows M = M$, the Ehresmann gauge groupoid of P.
- The Lie algebroid of Γ is the Atyiah algebroid $\mathsf{T}P/G$.

THANK YOU FOR YOUR ATTENTION!

Homework

- Problem 1. Prove that the tangent and cotangent bundles of a double graded bundle are canonically triple graded bundles.
- Problem 2. On the space of curves $\gamma : \mathbb{R} \to M$ in a manifold M, consider the $(\mathbb{R}, \cdot)$-action $\hat{h}_t(\gamma)(s) = \gamma(ts)$.
  Prove that this action induces the canonical homogeneity structure on the space $\mathsf{T}^2 M$ of second jets of these curves.
- Problem 3. Show that the second tangent lift of a homogeneity structure h on F, defined by $(\mathsf{T}^2 h)_t = \mathsf{T}^2(h_t)$, is a homogeneity structure on $\mathsf{T}^2 F$. Here $\mathsf{T}^2\phi : \mathsf{T}^2 M \to \mathsf{T}^2 N$ denotes the obvious second-jet prolongation of $\phi : M \to N$ to the second tangent bundles.
- Problem 4. Prove that the lifted homogeneity structure $\mathsf{T}^2 h$ from the previous problem is compatible with the canonical homogeneity structure on the second tangent bundle $\mathsf{T}^2 F$.
- Problem 5. Show that the anchor map induces, for any Lie algebroid E, a homomorphism of the Lie algebroid bracket into the Lie bracket of vector fields:

$$\rho([X, Y]_\varepsilon) = [\rho(X), \rho(Y)]_{vf}\,.$$