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Vienna, Preprint ESI 452 (1997) June 9, 1997

Supported by Federal Ministry of Science and Research, Austria
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Completely integrable systems: a generalization

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Abstract

We present a slight generalization of the notion of completely integrable systems to get them being integrable by quadratures. We use this generalization to integrate dynamical systems on double Lie groups.

1 Introduction

A Hamiltonian system on a $2n$-dimensional symplectic manifold M is said to be *completely integrable* if it has n first integrals in involution, which are functionally independent on some open dense submanifold of M. This definition of a completely integrable system is usually found, with some minor variants, in any modern text on symplectic mechanics [AM, Ar, LM, MSS, Th].

Starting with this definition, one uses the so called Liouville-Arnold theorem to introduce action-angle variables and write the Hamiltonian system

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in the form

$$
\begin{aligned}
\dot{I}^k &= 0, && (1.1)\\
\dot{\phi}_k &= \frac{\partial H}{\partial I_k} = \nu_k(I), && (1.2)
\end{aligned}
$$

where $k \in \{1, \ldots, n\}$. The corresponding flow is given by

$$
\begin{aligned}
I^k(t) &= I^k(0), && (1.3)\\
\phi_k(t) &= \phi_k(0) + \nu_k t. && (1.4)
\end{aligned}
$$

The main interest in completely integrable systems relies on the fact that they can be integrated by quadratures.

It is clear, however, that even if $\nu_k \mathrm{d}I^k$ is not an exact (or even a closed) 1-form, as long as $\dot{\nu}_k = 0$, the system can always be integrated by quadratures.

In this letter, we would like to take up this remark from the view point of Lie groups and their cotangent bundles, as well as double Lie groups, plying the role of deformed cotangent bundles, to show how the notion of a completely integrable system can be generalized by relaxing the property of being Hamiltonian and retaining only that it has enough constants of the motion (first integrals) to warrant it to be integrable by quadratures. More precisely, we define some class of dynamical systems on a Poisson manifold (M, Λ), defined by a vector field $\Gamma = \Lambda(\eta)$, where η is a 1-form, which are integrable by quadratures. Here M is the cotangent bundle of a Lie group or its appropriate deformation.

2 A universal model for completely integrable system

If we consider the abelian Lie group $\mathbb{R}^n$, we can construct a Hamiltonian action of $\mathbb{R}^n$ on $T^*\mathbb{R}^n$ induced by the group addition:

$$\mathbb{R}^n \times T^*\mathbb{R}^n \longrightarrow T^*\mathbb{R}^n. \qquad (2.1)$$

This can be generalized to the Hamiltonian action

$$\mathbb{R}^n \times T^*(\mathbb{R}^k \times T^{n-k}) \longrightarrow T^*(\mathbb{R}^k \times T^{n-k}), \qquad (2.2)$$

of $\mathbb{R}^n$, where T^m stands for the m-dimensional torus, and reduces to $\mathbb{R}^n \times T^*T^n$ or $T^n \times T^*T^n$, when $k = 0$.

By using the standard symplectic structure on $T^*\mathbb{R}^n$, we find the momentum map $\mu : T^*\mathbb{R}^n \longrightarrow (\mathbb{R}^n)^*,\ (q,p) \mapsto p$, induced by the natural action of $\mathbb{R}^n$ on itself via translations, which is a Poisson map if $(\mathbb{R}^n)^*$ is endowed with the (trivial) natural Poisson structure of the dual of a Lie algebra. It is now clear that any function on $(\mathbb{R}^n)^*$, when pulled back to $T^*\mathbb{R}^n$ or T^*T^n, gives rise to a Hamiltonian system which is completely integrable (in the Liouville sense). Because the level sets of this function carry on the action of $\mathbb{R}^n$, the completely integrable system gives rise to a one-dimensional subgroup of the action of $\mathbb{R}^n$ on the given level set. The specific subgroup will, however, depend on the particular level set, i.e. the 'frequencies' are first integrals. The property of being integrable by quadratures is captured by the fact that it is a subgroup of the $\mathbb{R}^n$-action on each level set.

It is now clear, how we can preserve this property, while giving up the requirement that our system is Hamiltonian. We can indeed consider any 1-form η on $(\mathbb{R}^n)^*$ and pull it back to $T^*\mathbb{R}^n$ or T^*T^n, then associated vector field $\Gamma_\eta = \Lambda_0(\mu^*(\eta))$, where Λ_0 is the canonical Poisson structure in the cotangent bundle, is no more Hamiltonian, but it is still integrable by quadratures. In action-angle variables, if $\eta = \nu_k \mathrm{d}I^k$ is the 1-form on $(\mathbb{R}^n)^*$, the associated equations of motion on T^*T^n will be

$$\dot{I}^k = 0, \tag{2.3}$$
$$\dot{\phi}_k = \nu_k, \tag{2.4}$$

with $\dot{\nu}_k = 0$, therefore the flow will be as in (1.3), (1.4), even though

$$\frac{\partial \nu_k}{\partial I^j} \neq \frac{\partial \nu_j}{\partial I^k}. \tag{2.5}$$

We can now generalize this construction to any Lie group G. We consider the Hamiltonian action

$$G \times T^*G \longrightarrow T^*G, \tag{2.6}$$

of G on the cotangent bundle, induced by the right action of G on itself. The associated momentum map

$$\mu : T^*G \simeq \mathcal{G}^* \times G \longrightarrow \mathcal{G}^* \tag{2.7}$$

It is a Poisson map with respect to the natural Poisson structure on $\mathcal{G}^*$ (see e.g. [AG1, LM]).

Now, we consider any differential 1-form η on $\mathcal{G}^*$ which is annihilated by the natural Poisson structure $\Lambda_{\mathcal{G}^*}$ on $\mathcal{G}^*$ associated with the Lie bracket. Such form will be called a *Casimir form.* We define the vector field $\Gamma_\eta = \Lambda_0(\mu^*(\eta))$. Then, the corresponding dynamical system can be written as (for the proof we refer to the general case described in Theorem 1)

$$
\begin{aligned}
g^{-1}\dot{g} &= \eta(g,p) = \eta(p), && (2.8)\\
\dot{p} &= 0, && (2.9)
\end{aligned}
$$

since $\omega_0 = \mathrm{d}(< p, g^{-1}\mathrm{d}g >)$ (cf. [AG1]). Here we interpret the covector $\eta(p)$ on $\mathcal{G}^*$ as a vector of $\mathcal{G}$. Again, our system can be integrated by quadratures, because on each level set, obtained by fixing p's in $\mathcal{G}^*$, our dynamical system coincides with a one-parameter group of the action of G on that particular level set.

We give a familiar example: the rigid rotator and its generalizations.

Example 1 *In the case of* $G = \mathrm{SO}(3)$ *the (right) momentum map*

$$\mu : T^*\mathrm{SO}(3) \longrightarrow \mathrm{so}(3)^* \qquad (2.10)$$

is a Poisson map onto $\mathrm{so}(3)^*$ *with the linear Poisson structure*

$$\Lambda_{\mathrm{so}(3)^*} = \varepsilon_{ijk} p_i \partial_{p_j} \otimes \partial_{p_k}. \qquad (2.11)$$

Casimir 1-forms for $\Lambda_{so(3)^*}$ *read* $\eta = F\mathrm{d}H_0$*, where* $H_0 = \sum p_i^2/2$ *is the 'free Hamiltonian' and* $F = F(p)$ *is an arbitrary function. Clearly,* $F\mathrm{d}H_0$ *is not a closed form in general, but* (p_i) *are first integrals for the dynamical system* $\Gamma_\eta = \Lambda_0(\mu^*(\eta))$*. It is easy to see that*

$$\Gamma_\eta = F(p)\Gamma_0 = F(p)p_i\widehat{X_i}, \qquad (2.12)$$

where $\widehat{X_i}$ *are left-invariant vector fields on* $\mathrm{SO}(3)$*, corresponding to the basis* (X_i) *of* $\mathrm{so}(3)$ *identified with* $(\mathrm{d}p_i)$*. Here we used the identification* $T^*\mathrm{SO}(3) \simeq \mathrm{SO}(3)\times\mathrm{so}(3)^*$ *given by the momentum map* μ*. In other words, the dynamics is given by*

$$
\begin{aligned}
\dot{p}_i &= 0 && (2.13)\\
g^{-1}\dot{g} &= F(p)p_iX_i \in \mathrm{so}(3) && (2.14)
\end{aligned}
$$

and it is completely integrable, since it reduces to left-invariant dynamics on SO(3) *for every value of* p. *We recognize the usual isotropic rigid rotator, when* $F(p)=1$.

We can generalize our construction once more, replacing the cotangent bundle T^*G by its deformation, namely a group double $D(G,\Lambda_G)$ associated with a Lie-Poisson structure Λ_G on G (see e.g. [AG2, Lu1]). This double, denoted simply by D, carry on a natural Poisson tensor Λ_D^+ which is non-degenerate on the open-dense subset $D^+ = G\cdot G^* \cap G^*\cdot G$ of D (here $G^* \subset D$ is the dual group of G with respect to Λ_G). We refer to D as being *complete* if $D^+ = D$. Identifying D with $G\times G^*$ if D is complete (or D^+ with an open submanifold of $G\times G^*$ in general case; we assume completeness for simplicity) via the group product, we can write Λ_D^+ in 'coordinates' $(g,u)\in G\times G^*$ in the form

$$\Lambda_D^+(g,u) = \Lambda_G(g) + \Lambda_{G^*}(u) - X_i^l(g)\wedge Y_i^r(u), \tag{2.15}$$

where X_i^l and Y_i^r are, respectively, the left- and right-invariant vector fields on G and G^* relative to dual bases X_i and Y_i in the Lie algebras $\mathcal{G}$ and $\mathcal{G}^*$, and where Λ_G and Λ_{G^*} are the corresponding Lie-Poisson tensors on G and G^* (see [Lu1, AG2]). It is clear now that the projections μ_{G^*} and μ_G of (D,Λ_D^+) onto (G,Λ_G) and (G^*,Λ_{G^*}), respectively, are Poisson maps. Note that we get the cotangent bundle $(D,\Lambda_D^+) = (T^*G,\Lambda_0)$ if we put $\Lambda_G = 0$.

The group G acts on (D,Λ_D^+) by left translations which, in general are not canonical transformations. This is, however, a Poisson action with respect to the inner Poisson structure Λ_G on G, which is sufficient to develop the momentum map reduction theory (see [Lu2]). For our purposes, let us take a Casimir 1-form η for Λ_{G^*}, i.e. $\Lambda_{G^*}(\eta)=0$. By means of the momentum map $\mu_{G^*}: D \longrightarrow G^*$, we define the vector field on D:

$$\Gamma_\eta = \Lambda_D^+(\mu_{G^*}^*(\eta)). \tag{2.16}$$

In 'coordinates' (g,u), due to the fact that η is a Casimir, we get

$$\Gamma_\eta(g,u) =< Y_i^r, \eta > (u) X_i^l(g), \tag{2.17}$$

so that Γ_η is associated with the 'Legendre map'

$$L_\eta : D \simeq G\times G^* \longrightarrow TG \simeq G\times\mathcal{G}, \quad L_\eta(g,u) =< Y_i^r,\eta > (u)X_i, \tag{2.18}$$

which can be viewed also as a map $L_\eta : G^* \longrightarrow \mathcal{G}$. Thus we get the following.

Theorem 1 *The dynamics Γ_η on the group double $D(G,\Lambda_G)$, associated with a 1-form η which is a Casimir for the Lie-Poisson structure Λ_{G^*} on the dual group, is given by the system of equations*

$$\dot{u} = 0, \tag{2.19}$$
$$g^{-1}\dot{g} = < Y_i^r, \eta > (u) X_i \in \mathcal{G}, \tag{2.20}$$

and is therefore completely integrable by quadratures.

Example 2 *We consider now the double Lie group $D = SL(2,\mathbb{C})$ with $G = SU(2)$ and $G^* = SB(2,\mathbb{C})$ (see e.g. [AG2]. We will write the elements as follows:*

$$D = SL(2,\mathbb{C}) \ni a = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}, \quad \text{where} \quad z_i \in \mathbb{C}, \quad z_1 z_4 - z_2 z_3 = 1,$$

$$G = SU(2) \ni g = \begin{pmatrix} \alpha & -\bar{\nu} \\ \nu & \bar{\alpha} \end{pmatrix}, \quad \text{where} \quad \alpha,\nu \in \mathbb{C}, \quad |\alpha|^2 + |\nu|^2 = 1,$$

$$G^* = SB(2,\mathbb{C}) \ni u = \begin{pmatrix} r & \gamma \\ 0 & r^{-1} \end{pmatrix}, \quad \text{where} \quad r > 0, \gamma \in \mathbb{C}.$$

The Poisson structure $\Lambda_{SL(2,\mathbb{C})}$ is the restriction of the following quadratic (real) Poisson brackets on $\mathbb{C}^4$:

$$\begin{array}{lcl@{\qquad}lcl}
\{z_1,z_2\} & = & -\frac{i}{2}z_1z_2 & \{z_2,z_3\} & = & iz_1z_4 \\
\{z_1,z_3\} & = & \frac{i}{2}z_1z_3 & \{z_2,z_4\} & = & \frac{i}{2}z_2z_4 \\
\{z_1,z_4\} & = & 0 & \{z_3,z_4\} & = & -\frac{i}{2}z_3z_4 \\
\{z_1,\bar{z}_1\} & = & -\frac{i}{2}|z_1|^2 - i|z_3|^2 & \{z_2,\bar{z}_2\} & = & -\frac{i}{2}|z_2|^2 - i|z_1|^2 - i|z_4|^2 \\
\{z_3,\bar{z}_3\} & = & -\frac{i}{2}|z_3|^2 & \{z_4,\bar{z}_4\} & = & -\frac{i}{2}|z_4|^2 - i|z_3|^2 \\
\{z_1,\bar{z}_2\} & = & -iz_3\bar{z}_4 & \{z_2,\bar{z}_3\} & = & \frac{i}{2}z_2\bar{z}_3 \\
\{z_1,\bar{z}_3\} & = & 0 & \{z_2,\bar{z}_4\} & = & -iz_1\bar{z}_3 \\
\{z_1,\bar{z}_4\} & = & \frac{i}{2}z_1\bar{z}_4 & \{z_3,\bar{z}_4\} & = & 0.
\end{array} \tag{2.21}$$

The lacking commutators may be obtained from this list if we remember that the Poisson bracket is real, e.g., $\{\bar{z}_i,\bar{z}_j\} = \overline{\{z_i,z_j\}}$. One can then check that, indeed, det *and* $\overline{\det}$ *are Casimir functions, and that $z_1 \leftrightarrow z_4$, $z_2 \mapsto -z_2$, and $z_3 \mapsto -z_3$ defines a symmetry of the bracket associated to the inverse $a \mapsto a^{-1}$ in $SL(2,\mathbb{C})$.*

Our double group is complete, since we have the following unique (Iwasawa) decompositions:

$$SL(2,\mathbb{C}) \simeq SU(2).SB(2,\mathbb{C}), \text{ where } \quad s = \frac{1}{\sqrt{|z_1|^2+|z_3|^2}},$$

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} sz_1 & -s\bar{z}_3 \\ sz_3 & s\bar{z}_1 \end{pmatrix} \begin{pmatrix} 1/s & s(\bar{z}_1 z_2 + \bar{z}_3 z_4) \\ 0 & s \end{pmatrix} \tag{2.22}$$

$$SL(2,\mathbb{C}) \simeq SB(2,\mathbb{C}).SU(2), \text{ where } \quad t = \frac{1}{\sqrt{|z_3|^2+|z_4|^2}},$$

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} t & t(z_1\bar{z}_3 + z_2\bar{z}_4) \\ 0 & 1/t \end{pmatrix} \begin{pmatrix} t\bar{z}_4 & -t\bar{z}_3 \\ tz_3 & tz_4 \end{pmatrix}. \tag{2.23}$$

Hence, the bracket $\{\ \ ,\ \ \}$ *is globally symplectic on* $SL(2,\mathbb{C})$*. This bracket is projectable on the subgroups* $SU(2)$ *and* $SB(2,\mathbb{C})$*, and for the 'left trivialization'* $SL(2,\mathbb{C}) = SU(2).SB(2,\mathbb{C})$ *it gives us the Poisson Lie brackets on* $SU(2)$*:*

$$\begin{array}{rclrcl} \{\alpha,\bar{\alpha}\} &=& -i|\nu|^2 & \{\nu,\bar{\nu}\} &=& 0 \\ \{\alpha,\nu\} &=& \frac{i}{2}\alpha\nu & \{\bar{\alpha},\bar{\nu}\} &=& -\frac{i}{2}\bar{\alpha}\bar{\nu} \\ \{\alpha,\bar{\nu}\} &=& \frac{i}{2}\alpha\bar{\nu} & \{\bar{\alpha},\nu\} &=& -\frac{i}{2}\bar{\alpha}\nu, \end{array} \tag{2.24}$$

and on $SB(2,\mathbb{C})$*:*

$$\{\gamma, r\} = \frac{i}{2}\gamma r, \qquad \{\bar{\gamma},\gamma\} = i(r^2 - r^{-2}). \tag{2.25}$$

The 'interaction' between $SU(2)$ *and* $SB(2,\mathbb{C})$ *is described by*

$$\begin{array}{rclrcl} \{\nu,\gamma\} &=& -\frac{i}{4}\nu\gamma - i\bar{\alpha}r^{-1} & \{\alpha,\gamma\} &=& -\frac{i}{4}\alpha\gamma + i\bar{\nu}r^{-1} \\ \{\bar{\nu},\gamma\} &=& \frac{i}{4}\bar{\nu}\gamma & \{\bar{\alpha},\gamma\} &=& \frac{i}{4}\bar{\alpha}\gamma \\ \{\nu, r\} &=& -\frac{i}{4}\nu r & \{\alpha, r\} &=& -\frac{i}{4}\alpha r, \end{array} \tag{2.26}$$

where the lacking commutations can be derived, due to the fact that the bracket is real.

One can check that the Casimir 1-forms for $\Lambda_{SB(2,\mathbb{C})}$ *read* $\eta = F\mathrm{d}H_0$*, where* $F = F(\gamma, r)$ *is an arbitrary function on* $SB(2,\mathbb{C})$ *and*

$$H_0 = \frac{1}{2}Tr(aa^*) = \frac{1}{2}\sum |z_i|^2 = \frac{1}{2}(|\gamma|^2 + r^2 + r^{-2})$$

is the 'free' Hamiltonian. For the dynamics Γ_η on $SL(2,\mathbb{C})$ induced by η, we calculate (using 2.21) that

$$\begin{array}{rclrcl} \dot z_1 &=& -\frac{i}{2}F(H_0z_1-\bar z_4) & \dot z_2 &=& -\frac{i}{2}F(H_0z_2+\bar z_3) \\ \dot z_3 &=& -\frac{i}{2}F(H_0z_3+\bar z_2) & \dot z_4 &=& -\frac{i}{2}F(H_0z_4-\bar z_1). \end{array} \tag{2.27}$$

Since F and H_0 are constants of the motion, it is clear that the system is completely integrable. For $F=1$ this system was considered in [MSS, Za]. In variables $g\in SU(2)$ and $u\in SB(2,\mathbb{C})$, we have

$$\dot u = \begin{pmatrix} \dot r & \dot\gamma \\ 0 & -\dot r r^{-2} \end{pmatrix} \tag{2.28}$$

and

$$g^{-1}\dot g = -\frac{i}{2}F(H_0I + J\bar uJu^{-1}) = const \in su(2), \tag{2.29}$$

where $J=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In a more transparent form

$$g^{-1}\dot g = -\frac{i}{4}F\begin{pmatrix} r^2-r^{-2}+|\gamma|^2 & 2\gamma r^{-1} \\ 2\bar\gamma r^{-1} & r^{-2}-r^2-|\gamma|^2 \end{pmatrix} \in su(2). \tag{2.30}$$

It follows that we get a 'free motion' on $SU(2)$ along trajectories of left-invariant vector fields corresponding to

$$L_\eta = -\frac{i}{2}F(u)(H_0(u)I + J\bar uJu^{-1}) \in su(2) \tag{2.31}$$

(cf. [MSS, Za]). The mapping

$$SB(2,\mathbb{C}) \ni u \mapsto L_\eta(u) = -\frac{i}{2}F(u)(H_0(u)I + J\bar uJu^{-1}) \in su(2) \tag{2.32}$$

is a sort of a 'Legendre transform', transforming momenta from $SB(2,\mathbb{C})$ into velocities from $su(2)$. It is easy to see that, e.g. in the case $F=1$, L_η is invertible and the momenta corresponding to the velocity

$$-\frac{i}{2}\begin{pmatrix} s & w \\ \bar w & -s \end{pmatrix} \in su(2), \quad s\in\mathbb{R}, w\in\mathbb{C}, \tag{2.33}$$

are

$$r = s+\sqrt{s^2+|w|^2+1}, \quad \gamma = rw. \tag{2.34}$$

Example 3 *Since the dual subgroups in the group double play entirely symmetric role, let us consider now* $SU(2)$ *to be the set of momenta and* $SB(2,\mathbb{C})$ *to be the configuration space.*

The Lie-Poisson structure $\Lambda_{SU(2)}$ *admits, however no global Casimir function, so that we cannot produce a globally Hamiltonian system on* $SL(2,\mathbb{C})$ *by means of the momentum map* $\mu_{SU(2)} : SL(2,\mathbb{C}) \longrightarrow SU(2)$*, though we easily find out that*

$$\eta = iF(\alpha,\nu)(\nu \mathrm{d}\bar{\nu} - \bar{\nu}\mathrm{d}\nu) \tag{2.35}$$

is a general Casimir 1-form (which is real for real functions F*). The equations of the dynamical system* Γ_η *read*

$$\begin{aligned}
\dot{\alpha} &= 0 && (2.36)\\
\dot{\nu} &= 0 && (2.37)\\
\dot{r} &= -\frac{1}{2}F(\alpha,\nu)|\nu|^2\gamma && (2.38)\\
\dot{\gamma} &= -F(\alpha,\nu)\left(\frac{1}{2}|\nu|^2\gamma + \frac{i}{r}\alpha(\Re(\nu)-\Im(\nu))\right). && (2.39)
\end{aligned}$$

In other words, α *and* ν *are constant of the motion and*

$$\dot{u}u^{-1} = L_\eta(\alpha,\nu) = -\frac{1}{2}F(\alpha,\nu)\begin{pmatrix} |\nu|^2 & 2i\alpha(\Re(\nu)-\Im(\nu)) \\ 0 & -|\nu|^2 \end{pmatrix} \in sb(2,\mathbb{C}) \tag{2.40}$$

is time independent. The 'Legendre map' $L_\eta : SU(2) \longrightarrow sb(2,\mathbb{C})$ *is never bijective, since our set of momenta* $SU(2)$ *is a compact manifold, so that 'admissible' velocities form a compact subset of* $sb(2,\mathbb{C})$*.*

3 A further generalization

We have seen that if we concentrate on the possibility of integrating our system by quadratures, then we can do without the requirement that the system is Hamiltonian.

By considering again the equations of motion in action-angle variables, we have, classically,

$$\begin{aligned}
\dot{I}^k &= 0, && (3.1)\\
\dot{\varphi}_k &= \nu^k(I). && (3.2)
\end{aligned}$$

Clearly, if we have

$$\begin{aligned} \dot{I}^k &= F_k(I), \\ \dot{\varphi} &= A_k^j(I)\varphi_j, \end{aligned} \tag{3.3}$$

and we are able to integrate the first equation by quadratures, we again have the possibility to integrate by quadratures the system (3.3), if only the matrices $(A_k^j(I(t)))$ commute:

$$\varphi(t) = \exp\left(\int_0^t A(I(s))\mathrm{d}s\right)\varphi_0. \tag{3.4}$$

Of course, because φ_k are discontinues functions on the torus, we have to be more careful here. We show, however, how this idea works for double groups. In the case when the 1-form η on G^* is not a Casimir 1-form for the Lie-Poisson structure Λ_{G^*}, we get, in view of (2.15),

$$\Gamma_\eta(g,u) =< Y_i^r, \eta > (u)X_i^l(g) + \Lambda_{G^*}(\eta)(u). \tag{3.5}$$

Now, the momenta evolve according to the dynamics $\Lambda_{G^*}(\eta)$ on G^* (which can be interpreted, as we will see later, as being associated with an interaction of the system with an external field) and 'control' the evolution of the field of velocities on G (being left-invariant for a fixed time) by a 'variation of constants'. Let us summarize our observations in the following.

Theorem 2 *The vector field Γ_η on the double group $D(G,\Lambda_G)$, associated with a 1-form η on G^*, defines the following dynamics*

$$\dot{u} = \Lambda_{G^*}(\eta)(u), \tag{3.6}$$

$$g^{-1}\dot{g} = < Y_i^r, \alpha > (u)X_i \in \mathcal{G}, \tag{3.7}$$

and is therefore completely integrable, if only we are able to integrate the equation 3.6 and $< Y_i^r, \eta > (u(t))X_i$ lie in a commutative subalgebra of $\mathcal{G}$ for all t.

Example 4 *Let us take now $SU(2)$ for momenta and consider the Hamiltonian $H = \frac{1}{2}|\nu|^2$. The 1-form $\eta = \mathrm{d}H$ is exact, but not a Casimir 1-form for*

$\Lambda_{SU(2)}$. *The dynamical system* Γ_η *on* $SL(2,\mathbb{C})$ *induces the following dynamics of momenta*

$$\begin{aligned} \dot{\nu} &= 0, & (3.8)\\ \dot{\alpha} &= \frac{i}{2}|\nu|^2\alpha. & (3.9)\end{aligned}$$

This is the dynamics described in [LMS]. Additionally, we get from 2.26:

$$\begin{aligned} \dot{r} &= 0, & (3.10)\\ \dot{\gamma} &= -\frac{i}{2}\bar{\alpha}(\Re(\nu)+\Im(\nu))r^{-1}. & (3.11)\end{aligned}$$

Hence, $\nu(t)=\nu_0$, $\alpha(t)=\alpha_0\exp(\frac{i}{2}|\nu|^2 t)$, *and*

$$\dot{u}u^{-1} = -\frac{i}{2}\bar{\alpha}(\Re(\nu)+\Im(\nu))\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix} \in sb(2,\mathbb{C}). \tag{3.12}$$

Here, the velocity of a particle is rotating around 0 with the radius proportional to the momentum $\Re(\nu_0)+\Im(\nu_0)$ *and the frequency proportional to the energy* $H=\frac{1}{2}|\nu_0|^2$. *The velocities stay, however, in a commutative subalgebra of the unipotent Lie algebra* $sb(2,\mathbb{C})$, *so that*

$$u(t) = \begin{pmatrix} 1 & \alpha_0\frac{\Re(\nu_0)+\Im(\nu_0)}{|\nu_0|^2}(\exp(-\frac{i}{2}|\nu_0|^2 t)-1)\\ 0 & 1\end{pmatrix} u_0. \tag{3.13}$$

Let us end up with an example which shows that we can actually weaken the assumptions of Theorem 2. In fact, it is sufficient to assume that

$$g^{-1}\dot{g}(t) = \exp(tX)A(t)\exp(-tX) \tag{3.14}$$

for some $A(t)$, $X\in\mathcal{G}$, such that $X+A(t)$ lie in a commutative subalgebra of $\mathcal{G}$ for all t (e.g. $A(t)=const$), to assure that (3.7) is integrable by quadratures. Indeed, in the new variable

$$g_1(t) = \exp(-tX)g(t)\exp(tX) \tag{3.15}$$

the equation (3.7) reads

$$\dot{g}_1(t) = g_1(t)(X+A(t)) - Xg_1(t) \tag{3.16}$$

and, since the right- and the left-multiplications commute, we easily find that

$$g(t) = g_0 \exp\left(tX + \int_0^t A(s)\mathrm{d}s\right)\exp(-tX). \tag{3.17}$$

This procedure is similar to what is known as the Dirac interaction picture in the quantum evolution.

Example 5 *For our group double $SL(2,\mathbb{C})$, let us take the 1-form $\eta = F(r)\mathrm{d}H_0 + \lambda \mathrm{d}r$, $r \in \mathbb{R}$ on $SB(2,\mathbb{C})$ which is a perturbation of 2.30. For the dynamics of momenta, we get*

$$\begin{aligned} \dot{r} &= 0 && (3.18)\\ \dot{\gamma} &= -\frac{i}{2}\lambda r \gamma && (3.19) \end{aligned}$$

which can be easily integrated:

$$\begin{aligned} r(t) &= r_0 && (3.20)\\ \gamma(t) &= \gamma_0 \exp(-\frac{i}{2}\lambda r t). && (3.21) \end{aligned}$$

In particular, $r = const$ and $|\gamma| = const$. We have also

$$g^{-1}\dot{g} = -\frac{i}{4}\begin{pmatrix} F(r)(r^2 - r^{-2} + |\gamma|^2) - \lambda r & 2F(r)\gamma r^{-1} \\ 2F(r)\bar{\gamma} r^{-1} & F(r)(r^{-2} - r^2 - |\gamma|^2) + \lambda r \end{pmatrix}. \tag{3.22}$$

The velocities are no longer constant and rotate around the vector

$$C_0 = (F(r_0)(r_0^2 - r_0^{-2} + |\gamma_0|^2) - \lambda r_0)\begin{pmatrix} -\frac{i}{4} & 0 \\ 0 & \frac{i}{4} \end{pmatrix} \in su(2). \tag{3.23}$$

It can be interpreted as an effect of an interaction of moving charged particle with an external magnetic field, corresponding to the perturbation of the 'free system'. Since, as easily seen,

$$g^{-1}\dot{g}(t) = \exp(tX)A_0\exp(-tX), \tag{3.24}$$

with

$$X = \begin{pmatrix} -\frac{i}{4}\lambda r_0 & 0 \\ 0 & \frac{i}{4}\lambda r_0 \end{pmatrix} \tag{3.25}$$

and

$$A_0 = -\frac{i}{4}\begin{pmatrix} F(r_0)(r_0^2 - r_0^{-2} + |\gamma_0|^2) - \lambda r_0 & 2F(r_0)\gamma_0 r_0^{-1} \\ 2F(r_0)\bar{\gamma}_0 r_0^{-1} & F(r_0)(r_0^{-2} - r_0^2 - |\gamma_0|^2) + \lambda r_0 \end{pmatrix},$$

we can easily integrate (3.22):

$$\begin{aligned} g(t) &= g_0 \exp\left(\frac{-iF(r_0)}{4}t\begin{pmatrix} r_0^2 - r_0^{-2} + |\gamma_0|^2 & 2\gamma_0 r_0^{-1} \\ 2\bar{\gamma}_0 r_0^{-1} & r_0^{-2} - r_0^2 - |\gamma_0|^2 \end{pmatrix}\right) \times \\ & \begin{pmatrix} \exp\left(-\frac{i}{4}\lambda r_0 t\right) & 0 \\ 0 & \exp\left(\frac{i}{4}\lambda r_0 t\right) \end{pmatrix}. \end{aligned} \tag{3.26}$$

4 Final comments

Our identification of systems which can be integrated by quadratures with one-parameter subgroups of some Lie group G acting on the carrier space of the system gives us the possibility to dispose of the requirement that the system is Hamiltonian. We have to notice however that in some cases our system will turn out to be Hamiltonian with respect to a different symplectic structure (still invariant under the action of G) on the phase space. Indeed, if we consider $\eta = \nu_i \mathrm{d}I^i$ and make the assumption that

$$\mathrm{d}\nu_1 \wedge \ldots \wedge \mathrm{d}\nu_n \neq 0 \tag{4.27}$$

on some open-dense submanifold N, we can define on N the symplectic structure $\omega_N = \sum_i \mathrm{d}\nu_i \wedge \mathrm{d}\phi_i$ and $\Gamma = \nu_i \partial_{\phi_i}$ will be associated with the Hamiltonian $H = \frac{1}{2}\sum_i (\nu_i)^2$. In the general situation this procedure does not apply any more.

In any case, the importance of the Liouville-Arnold theorem relies on the fact that, in the hypothesis of the theorem, we can find the group G and its action on the manifold, and then show that our starting system is conjugated to the one written in the introduction (1.1,1.2) in terms of the action-angle variables. Our generalization is much more in the spirit of the Lie-Sheffers theorem [LS] and it consists of splitting our system along the orbits of an action of a Lie group and a 'transverse' component (which is either zero, or linear), so that the integration can be achieved easily.

What seems to us relevant is that completely integrable systems (in the Liouville sense) are a part of this more general scheme. In particular, we have shown that we can use the action of the group by non-canonical transformations, so that systems on group doubles can be cast in this generalization.

We are confident that this approach may be useful to quantize group doubles in the geometric quantization setting. Many of these questions are currently being investigated.

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