LINEARIZATION OF GRADED STRUCTURES AND WEIGHTED STRUCTURES

Janusz Grabowski

(Polish Academy of Sciences)

February 23, 2021

Plan of the talk

- Weighted algebroids
- Weighted structures
- Weighted Lie theory
- Holonomic vectors and linearization
- Functor of linearization
- Total linearization
- Variational calculus of statics and infinitesimal mechanics
- The classical Tulczyjew triple
- Euler-Lagrange equations

Weighted Lie groupoids and algebroids

Besides the compatibility of two graded bundle structures, we can consider a compatibility of a graded bundle structure with some other geometric structures, e.g. a Lie algebroid or a Lie groupoid structure.
Thanks to the fact that a graded bundle structure can be expressed in terms of an $(\mathbb{R}, \cdot)$-action, there is an obvious natural concept of such a compatibility.

Definition

A weighted algebroid of degree k is an algebroid equipped with a homogeneity structure h of degree k such that homotheties h_t act as algebroid morphisms for all $t \in \mathbb{R}$.

We use the name 'weighted', as the term graded is already used in various meanings.
Note that weighted Lie algebroids of degree 1 have already appeared in the literature under the name VB-algebroids.

Weighted structures

- Assuming the existence of a homogeneity structure h on a manifold equipped additionally with another structure, we can easily consider weighted objects for other than algebroid structures.

Definition

A weighted structure A (e.g. weighted groupoid structure) of degree k is a manifold G equipped as well with the structure A (e.g. Lie groupoid structure) as with a homogeneity structure h of degree k such that homotheties h_t, $t \in \mathbb{R}$, act as morphisms of the structure A (e.g. morphisms of Lie groupoid structure).

- Weighted Lie groupoids of degree 1 are called in the literature VB-groupoids.
- In these sense, weighted structures of degree 1 are VB-structures, e.g. VB-Poisson structures or VB-principal bundles.

Weighted Lie theory

- Example. If $\mathcal{G}$ is a Lie groupoid (algebroid), then $\mathsf{T}^k\mathcal{G}$ is canonically a weighted Lie groupoid (algebroid) of degree k.
- If $m \subset \mathcal{G} \times \mathcal{G} \times \mathcal{G}$ is the graph of the partial multiplication in the groupoid $\mathcal{G}$, then $\mathsf{T}^k m \subset \mathsf{T}^k\mathcal{G} \times \mathsf{T}^k\mathcal{G} \times \mathsf{T}^k\mathcal{G}$ is the graph of the partial multiplication in $\mathsf{T}^{\mathcal{G}} \rightrightarrows \mathsf{T}^k M$.

Theorem (Bruce-Grabowska-Grabowski)

There is a one-to-one correspondence between weighted Lie groupoids of degree k with simple-connected source fibers and integrable weighted Lie algebroids of degree k, i.e. compatible homogeneity structures can be differentiated and integrated.

- Example. Let G be a Lie groupoid with the Lie algebroid $\mathcal{G}$. The weighted Lie algebroid for $\mathsf{T}^k G$ is $\mathsf{T}^k\mathcal{G}$.

Holonomic vectors and linearization

- It is well know that $\mathsf{T}^k M$ is canonically embedded in $\mathsf{T}(\mathsf{T}^{k-1}M)$ as the set of "holonomic vectors". Obviously, $\mathsf{T}(\mathsf{T}^{k-1}M) \to \mathsf{T}^{k-1}M$ is a vector bundle and these features we wish to generalise to arbitrary graded bundles.
- Consider F_k equipped with local coordinates (x^A, y^a_w, z^i_k), where the weights are assigned as $\mathrm{w}(x) = 0$, $\mathrm{w}(y_w) = w$ $(1 \leq w < k)$ and $\mathrm{w}(z) = k$. The corresponding weight vector field is
$$\nabla_{F_k} = w\, y^a_w \partial_{y^a_w} + k\, z^i_k \partial_{z^i_k}\,.$$
- We can lift the graded structure to $\mathsf{T}F_k$. It is represented by the wight vector field
$$\mathrm{d}_{\mathsf{T}} \nabla_{F_k} = w\, y^a_w \partial_{y^a_w} + k\, z^i_k \partial_{z^i_k} + w\, \dot{y}^a_w \partial_{\dot{y}^a_w} + k\, \dot{z}^i_k \partial_{\dot{z}^i_k}\,.$$
- It is tangent to the submanifold $\dot{x}^A = 0$, i.e. it defines a graded bundle structure on the vertical bundle $\mathsf{V}F_k$ with coordinates
$$(\underbrace{x^A}_{(0)}, \underbrace{y^a_w}_{(w)}, \underbrace{z^i_k}_{(k)}; \underbrace{\dot{y}^b_w}_{(w)}, \underbrace{\dot{z}^j_k}_{(k)})\,.$$

The graded structure of the vertical bundle

- Consider the vertical bundle VF_k as a bi-graded subbundle of the tangent bundle $\mathsf{T}F_k$ with the other graded structure being the standard linear structure on $\mathsf{V}F_k$.
- We can shift the graded structure $\mathrm{d}_\mathsf{T}\nabla_{F_k}$ by subtracting $\nabla_{\mathsf{V}F_k}$, The corresponding weight vector field is $\nabla^1_{VF_k} = \mathrm{d}_\mathsf{T}\nabla_{F_k} - \nabla_{VF_k}$ (it is still a weight vector field) and employ homogeneous local coordinates with the bi-weights

$$\big(\underbrace{x^A}_{(0,0)},\ \underbrace{y^a_w}_{(w,0)},\ \underbrace{z^i_k}_{(k,0)};\ \underbrace{\dot{y}^b_w}_{(w-1,1)},\ \underbrace{\dot{z}^j_k}_{(k-1,1)}\big),$$

so that the vertical bundle itself a graded-linear bundle of bi-degree $(k,1)$.
- Finally, we can remove the highest degree variables for $\nabla^1_{VF_k}$, i.e. the variables z^i_k. We end-up with a graded linear bundle of bi-degree $(k-1,1)$– the linearization of F_k.

Linearization of graded bundles

Definition

The *linearization of a graded bundle* F_k is the graded-linear bundle of bi-degree $(k-1,1)$, defined as

$$\mathsf{l}(F_k) := \mathsf{V}F_k[\nabla^1_{\mathsf{V}F_k} \leq k-1]\,,$$

where $\nabla^1_{\mathsf{V}F_k} = \mathrm{d}_\mathsf{T}\nabla_{F_k} - \nabla_{\mathsf{V}F_k}$ and $\nabla_{\mathsf{V}F_k}$ is the Euler vector field of the vector bundle $\mathsf{V}F_k \to F_k$.

- Thus on $\mathsf{l}(F_k)$ we have local homogeneous coordinates

$$(\underbrace{x^A}_{(0,0)}, \underbrace{y^a_w}_{(w,0)}; \underbrace{\cancel{z^i_k}}_{(k,0)}, \underbrace{\dot{y}^b_w}_{(w-1,1)}, \underbrace{\dot{z}^j_k}_{(k-1,1)}).$$

- The natural projection $p^{\mathsf{V}F_k}_{\mathsf{l}(F_k)} : \mathsf{V}F_k \to \mathsf{l}(F_k)$ is just 'forgetting' the coordinates z^i_k.
- Let us observe that the weight vector field $\nabla_{F_k} : F_k \to \mathsf{V}F_k$ is a graded morphism of the graded bundle (F_k, ∇_{F_k}) into the vector bundle $(\mathsf{V}F_k, \nabla_{\mathsf{V}F_k})$ (the weight vector field is linear).

Holonomic embedding

- In coordinates
$$\nabla_{F_k}(x^A, y^a_w, z^i_k) = (x^A, y^a_w, z^i_k, w\, y^a_w, k\, z^i_k)\,.$$
- We can compose the map ∇_{F_k} with the projection $p^{VF_k}_{\mathsf{l}(F_k)}$ and obtain
$$\iota_{F_k} = p^{VF_k}_{\mathsf{l}(F_k)} \circ \nabla_F k : F_k \to \mathsf{l}(F_k)\,.$$
In coordinates,
$$\iota_{F_k}(x^A, y^a_w, z^i_k) = (x^A, y^a_w, w\, y^b_w, k\, z^j_k)\,.$$

Theorem

The map $\iota_{F_k} : F_k \to \mathsf{l}(F_k)$ is a graded embedding of F_k into its linearization equipped with the total degree represented by the total weight vector field
$$\nabla_{\mathsf{l}(F_k)} = w\, y^a_w \partial_{y^a_w} + w\, \dot{y}^b_w \partial_{\dot{y}^b_w} + k\, \dot{z}^j_k \partial_{\dot{z}^j_k}\,.$$

Linearization via 'time-derivative'

One can also understand the linearization as adding the 'time-derivative' of variables of non-zero degree. For instance, If (x^a, y^A, z^j) are coordinates on a graded bundle F_2 of degrees $0, 1, 2$, respectively. Then, the induced coordinate system on $\mathrm{l}(F_2)$ is

$$(x^A, y^a, \dot{y}^b, \dot{z}^j)\,,$$

where x^A, y^a, $\dot{y}^b$, and $\dot{z}^j$ are of bi-degree $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, respectively, so we deal with a double vector bundle. The transformation laws for the extra coordinates are obtained by differentiation of transition functions $y'^a = y^b T^a_b(x)$ and $z'^i = z^j T_j{}^i(x) + \frac{1}{2} y^b y^a T^i_{ab}(x)$:

$$\begin{aligned} \dot{y'}^a &= \dot{y}^b T^a_b(x), \\ \dot{z'}^i &= \dot{z}^j T_j{}^i(x) + \dot{y}^b y^a T^i_{ab}(x)\,. \end{aligned}$$

Thus,

$$(x^A, y^a, \dot{y}^b, \dot{z}^j) \mapsto (x^A, y^a)$$

is a linear fibration over F_1. The embedding $\iota : F_2 \hookrightarrow \mathrm{l}(F_2)$ reads

$$\iota(x^A, y^a, z^j) = (x^A, y^a, y^b, 2z^j)\,.$$

Functor of linearization

The described linearization procedure gives rise to a functor from the category of graded bundles into the category of GrL-bundles.

Theorem (Bruce-Grabowska-Grabowski)

There is a canonical linearization functor $\mathsf{l}:\mathsf{GrB}\to\mathsf{GrL}$ *from the category of graded bundles into the category of* GrL*-bundles which assigns, for an arbitrary graded bundle* F_k *of minimal degree* k*, a canonical* GrL*-bundle* $\mathsf{l}(F_k)$ *of bi-degree* $(k-1,1)$ *which is linear over* F_{k-1}*, called the linearization of* F_k*, together with a graded embedding* $\iota: F_k\hookrightarrow \mathsf{l}(F_k)$ *of* F_k *as an affine subbundle of the vector bundle* $\mathsf{l}(F_k)\to F_{k-1}$*.*

Elements of $\iota(F_k)\subset\mathsf{l}(F_k)$ may be viewed as 'holonomic vectors' in the linear-graded bundle $\mathsf{l}(F_k)$.
Example. We have $\mathsf{l}(\mathsf{T}^kM)\simeq\mathsf{T}\mathsf{T}^{k-1}M$ and

$$\iota:\mathsf{T}^kM\hookrightarrow\mathsf{l}(\mathsf{T}^kM)\simeq\mathsf{T}\mathsf{T}^{k-1}M$$

is the canonical embedding of T^kM as holonomic vectors in $\mathsf{T}\mathsf{T}^{k-1}M$.

Lie algebroid structures on graded bundles

Definition

The *linear dual of a graded bundle* F_k is the dual of the vector bundle $\mathrm{l}(F_k) \to F_{k-1}$, and we will denote this $\mathrm{l}^*(F_k)$.

Definition

We will say that a graded bundle F_k carries the structure of a weighted Lie algebroid if its linearization $\mathrm{l}(F_k)$ is equipped with a weighted Lie algebroid structure, i.e. if there exists a graded morphism

$$\varepsilon : \mathrm{T}^* \mathrm{l}(F_k) \to \mathrm{T}\, \mathrm{l}^*(F_k) \,,$$

such that $(\mathrm{l}(F_k), \varepsilon)$ is a weighted Lie algebroid.

In the above we view $\mathrm{T}^* \mathrm{l}(F_k)$ and $\mathrm{T}\, \mathrm{l}^*(F_k)$ as triple graded bundles. Note that F_k is canonically an affine subbundle in the vector bundle $\mathrm{l}(F_k) \to F_{k-1}$, so in an obvious sense, a double graded-affine bundle.

Total linerization of graded bundles

Applying the linearization functor consecutively to a graded bundle of minimal degree k, we arrive at a k-fold graded bundle od degree $(1, \ldots, 1)$, i.e. at a k-fold vector bundle. This functor from $\mathsf{GrB}[k]$ to $\mathsf{VB}[k]$ we call a total linearization. Its image consists of k-fold vector bundles equipped with an action of the symmetry group S_k permuting the order of vector bundle structures (symmetric k-fold vector bundles).

Theorem (Bruce-Grabowski-Rotkiewicz)

There is a canonical functor $\mathsf{L}[k] : \mathsf{GrB}[k] \to \mathsf{VB}[k]$ from the category of graded bundles of degree k into the category of k-fold vector bundles. It gives an equivalence of $\mathsf{GrB}[k]$ with the subcategory (not full) SymVB of symmetric k-fold vector bundles. There is a canonical graded embedding $\iota[k] : F_k \hookrightarrow \mathsf{L}(F_k)$ of F_k as a subbundle of symmetric (holonomic) vectors.

Example. We have $\mathsf{L}(\mathsf{T}^k M) \simeq \mathsf{T}^{(k)} M$, where $\mathsf{T}^{(k)} M = \mathsf{T}\mathsf{T} \cdots \mathsf{T} M$ is the iterated tangent bundle. The action of S_k comes from iterations of the canonical "flips" $\kappa : \mathsf{T}\mathsf{T}M \to \mathsf{T}\mathsf{T}M$ (see the homework).

Weighted Lie algebroids out of reductions

For a Lie groupoid $G \rightrightarrows M$, consider the subbundle $\mathsf{T}^k G^{\underline{s}} \subset \mathsf{T}^k G$ consisting of all higher order velocities tangent to source-leaves. The bundle

$$F_k = A^k(G) := \mathsf{T}^k G^{\underline{s}}\Big|_M ,$$

inherits graded bundle structure of degree k as a graded subbundle of $\mathsf{T}^k G$. Of course, $A = A^1(G)$ can be identified with the Lie algebroid of G.

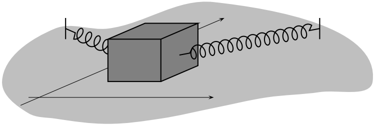
Theorem

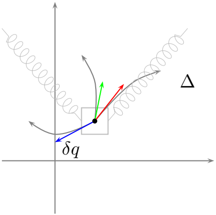
The linearization of $A^k(G)$ is given as

$$\mathsf{l}(A^k(G)) \simeq \{(Y, Z) \in A(G) \times \mathsf{T}A^{k-1}(G) | \quad \rho(Y) = \mathsf{T}\tau(Z)\},$$

viewed as a vector bundle over $A^{k-1}(G)$ with respect to the obvious projection of part Z onto $A^{k-1}(G)$, where $\rho : A(G) \to \mathsf{T}M$ is the standard anchor of the Lie algebroid and $\tau : A^{k-1}(G) \to M$ is the obvious projection. Moreover, the above bundle is canonically a weighted Lie algebroid, a Lie algebroid prolongation in the sense of Popescu and Martínez.

Variational calculus in statics





- Q - manifold of configurations
- Γ - admissible processes, i.e., one-dimensional oriented submanifolds with boundary (sometimes, however, we use a parametrization)
- $\mathcal{W} : \Gamma \to \mathbb{R}$ - the cost function

$$\mathcal{W}(\gamma) = \int_\gamma W\,,$$

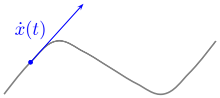
- for W being a positively homogeneous function on the set $\Delta \subset \mathsf{T}Q$ of vectors δq tangent to admissible processes.

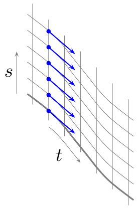
Mechanics: infinitesimal version

Let M be a manifold of positions of mechanical system. We will use first jets of smooth curves in M and first-order Lagrangians:

- Configurations: $Q = \mathsf{T}M$, $q = (x, \dot{x})$
- Functions: $S(q) = L(x, \dot{x})$
- Curves in Q come from homotopies: $\chi : \mathbb{R}^2 \to M$
- Tangent vectors: $\mathsf{T}Q = \mathsf{T}\mathsf{T}M$, i.e, equivalence classes of curves in $\mathsf{T}M$, $\delta q = \delta\dot{x}$.

  Additionally, $\kappa_M : \mathsf{T}\mathsf{T}M \to \mathsf{T}\mathsf{T}M$, $\kappa(\chi)(s,t) = \chi(t,s)$.
- Covectors: $\mathsf{T}^*Q = \mathsf{T}^*\mathsf{T}M$

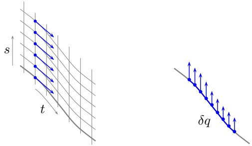




Canonical isomorphisms

- Tangent vectors $\delta\dot{x}$ are in one-to-one correspondence with vectors tangent to curves $t \mapsto \delta x(t)$ in $\mathsf{T}M$

$$\kappa_M : \mathsf{TT}M \ni \delta\dot{x} \mapsto (\delta x)^{\cdot} \in \mathsf{TT}M$$



- We get also the tangent evaluation between TT^*M and $\mathsf{TT}M$ defined on elements $\dot{p}$ and $(\delta x)^{\cdot}$ with the same tangent projection δx on $\mathsf{T}M$:

$$\langle\langle \dot{p}, (\delta x)^{\cdot} \rangle\rangle = \left.\frac{d}{dt}\right|_{t=0} \langle p(t), \delta x(t) \rangle.$$

- The map dual to κ,

$$\alpha_M : \mathsf{TT}^*M \longrightarrow \mathsf{T}^*\mathsf{T}M$$

gives us an identification of covectors from $\mathsf{T}^*\mathsf{T}M$ with elements of TT^*M.

Dynamics

- By (usually implicit) first-order dynamics on a manifold N we will understand a submanifold D in $\mathsf{T}N$.
- A curve $\gamma : \mathbb{R} \to N$ satisfies this dynamics (is a solution), if its tangent prolongation belongs to D, $\mathrm{t}(\gamma) : \mathbb{R} \to D \subset \mathsf{T}N$.

Example

A vector field X on N, i.e. a section of the tangent bundle $X : N \to \mathsf{T}N$, defines the dynamics $D = X(N) \subset \mathsf{T}N$.

- In local coordinates, for the vector field $X = f_a(q)\frac{\partial}{\partial q^a}$, we have

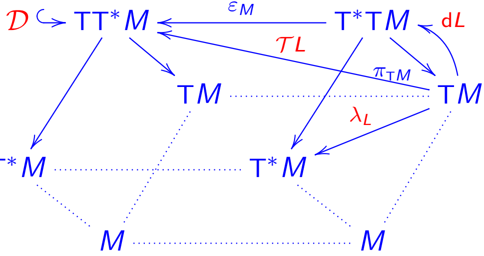
$$D = \{(q^a, \dot{q}^b) \in \mathsf{T}N : \dot{q}^b = f_b(q)\}$$

and the explicit dynamical equations $\frac{\mathrm{d}q^a}{\mathrm{d}t}(t) = f_a(q(t))$ are the equations for trajectories of this vector field.

The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot{\gamma} \in \mathcal{D}$. For the Lagrangian phase equations:

- M - positions,
- $\mathsf{T}M$ - (kinematic) configurations,
- $L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
- T^*M - phase space



$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x}\right\}\,,$$

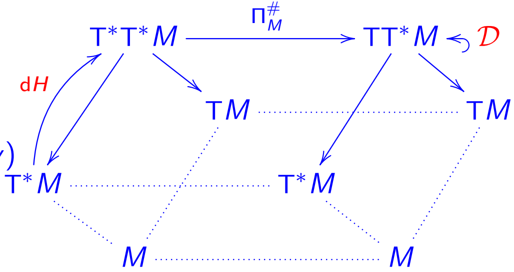
whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

The Tulczyjew triple - Hamiltonian side

Hamiltonian side of the triple

canonical isomorphism
$\mathsf{T}^*\mathsf{T}M \simeq \mathsf{T}^*\mathsf{T}^*M$,

$E : \mathsf{T}^*M \times_M \mathsf{T}M \to \mathbb{R}$
$\widetilde{H}(p,v) = \langle\, p,\ v \,\rangle - L(v)$
$H : \mathsf{T}^*M \to \mathbb{R}$

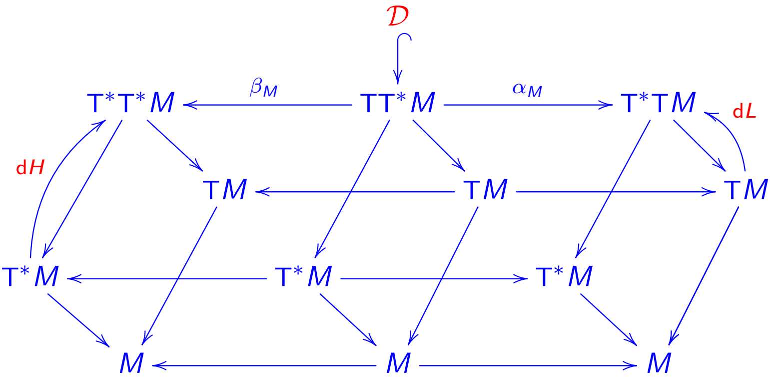


$$\mathcal{D} = \Pi^{\#}_M(\mathrm{d}H(\mathsf{T}^*M))$$

$$\mathcal{D} = \left\{(x,p,\dot{x},\dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}\right\},$$

whence the Hamilton equations.

Tulczyjew triple in mechanics



The dynamics is in the middle, the right-hand side is Lagrangian, the left-hand side – Hamiltonian.

The Legendre transform

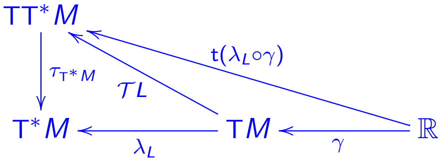
- The Legendre transform is a pass from the Lagrange to the Hamilton description of the dynamics:
  we try to describe the Lagrangian phase dynamics as a Hamiltonian phase dynamics.
- It is easy in the case of hyperregular Lagrangians (the Legendre map $(q,p)\mapsto\lambda_L(q,\dot{q})=(q,p)$ is a diffeomorhism).
- In this case the Lagrangian phase dynamics D_L is simultaneously Hamiltonian with the Hamiltonian function

$$\begin{aligned} H(q,p) &= \dot{q}^a p_a - L(q,\dot{q})\,, \\ (q,\dot{q}) &= \lambda_L^{-1}(q,p)\,. \end{aligned}$$

- In other words, the Lagrangian submanifolds $\mathrm{d}L(\mathrm{T}M)\subset\mathrm{T}^*\mathrm{T}M$ and $\mathrm{d}H(\mathrm{T}^*M)\subset\mathrm{T}^*\mathrm{T}^*M$ are related by the canonical isomorphism $\mathcal{R}_{\tau_M}$.

Euler-Lagrange equations

- The Euler-Lagrange equation for a curve $\underline{\gamma}:\mathbb{R}\to M$ takes in this model the form
$$\mathsf{t}(\lambda_L\circ\gamma)=\mathcal{T}L\circ\gamma\,,$$
where $\mathcal{T}L=\varepsilon\circ\mathrm{d}L$ is the Tulczyjew differential and $\gamma=\mathsf{t}(\underline{\gamma})$ is the tangent prolongation of $\underline{\gamma}$.
- In this sense, the Euler-Lagrange equation can be viewed as a first-order differential equation on curves γ in $\mathsf{T}M$:



- The equation just tells that the curve $\mathcal{T}L\circ\gamma$ is admissible, i.e. that it is a tangent prolongation of a curve (it must be $\lambda_L\circ\gamma$) on the phase space, $\mathcal{T}L\circ\gamma=\mathsf{t}(\lambda_L\circ\gamma)$.

Euler-Lagrange equations (continued)

- In local coordinates,

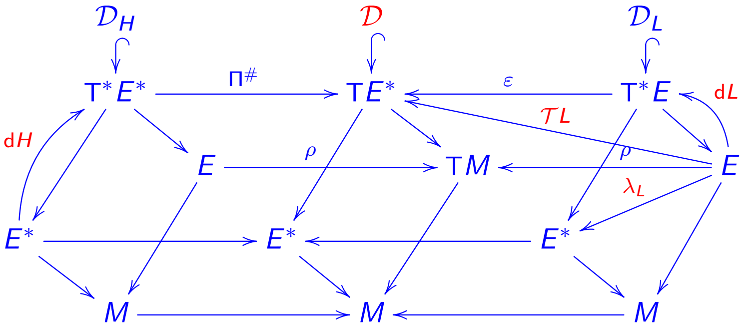
$$\mathcal{T}L(q,\dot q)=(q,\frac{\partial L}{\partial \dot q}(q,\dot q),\dot q,\frac{\partial L}{\partial q}(q,\dot q))\,.$$

For $\gamma(t)=(q(t),\dot q(t))$ this implies the equations

$$\dot q(t)=\frac{\mathrm{d}q}{\mathrm{d}t}(t)\,,\quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q}(q(t),\dot q(t))=\frac{\partial L}{\partial q}(q(t),\dot q(t))\,.$$

- These equations are second-order equations for curves $q=q(t)$ in M.
- Regularity of the Lagrangian is completely irrelevant for this formalism. Singular Lagrangians just produce complicated and implicit dynamics, but the geometric model is the same.

Algebroid setting



$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \mathcal{T}L(E)$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathsf{T}^*E^*$ $\qquad$ $\mathcal{D} = \Pi^\#(\mathrm{d}H(E^*))$ $\qquad$ $\mathcal{D}_L \subset \mathsf{T}^*E$

The Euler-Lagrange equations read $\mathcal{T}L \circ \gamma = \mathrm{t}(\lambda_L \circ \gamma)$.

Euler-Lagrange equations for algebroids

If (q^a) are local coordinates in M,
(y^i) i (ξ_i) are linear coordinates in fibers of, respectively, E and E^*, and

$$\Pi = c^k_{ij}(q)\xi_k\partial_{\xi_i}\otimes\partial_{\xi_j} + \rho^b_i(q)\partial_{\xi_i}\otimes\partial_{q^b} - \sigma^a_j(q)\partial_{q^a}\otimes\partial_{\xi_j}\,,$$

then the Euler-Lagrange equations read

$$(1)\qquad \frac{\mathrm{d}q^a}{\mathrm{d}t} = \rho^a_k(q)y^k\,,$$

$$(2)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^j}\right)(q,y) = c^k_{ij}(q)y^i\frac{\partial L}{\partial y^k}(q,y) + \sigma^a_j(q)\frac{\partial L}{\partial q^a}(q,y)\,.$$

They are first-order differential equations (!) but for admissible curves in E, i.e. for curves satisfying (1). For $E = \mathsf{T}M$, they are exactly the tangent prolongations of curves in M, for which the equation is second-order.

Euler-Poincaré equations

A particular example of the equation (2) is not only the classical Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^a}(q,\dot{q}) = \frac{\partial L}{\partial q^a}(q,\dot{q})\,.$$

but also the Lagrange-Poincaré equation for G-invariant Lagrangians on principal G-bundle

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a}\right)(q,\dot{q},v) - \left(B^k_{ba}(q)\dot{q}^b + D^k_{ia}(q)v^i\right)\frac{\partial L}{\partial v^k}(q,\dot{q},v) = 0\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^j}(q,\dot{q},v) - \left(D^k_{aj}(q)\dot{q}^a + C^k_{ij}v^i\right)\frac{\partial L}{\partial v^k}(q,\dot{q},v) = 0\,,$$

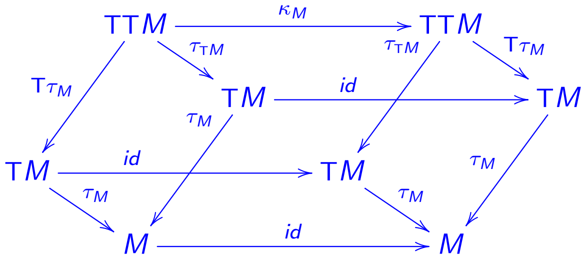
and the Euler-Poincaré equations, for instance the rigid body equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^j}(v) - C^k_{ij}v^i\frac{\partial L}{\partial v^k}(v) = 0\,.$$

THANK YOU FOR YOUR ATTENTION!

Homework

- Problem 1. As tangent vectors are ‘infinitesimal curves’, elements of the iterated tangent bundle $\mathsf{T}\mathsf{T}M$ are represented by homotopies $f : \mathbb{R}^2 \ni (s,t) \to f(s,t) \in M$. Show that the transposition $(\kappa f)(s,t) = f(t,s)$ induces an automorphism of the double vector bundle $\mathsf{T}\mathsf{T}M$:



- Problem 2. Prove that holonomic vectors in $\mathsf{T}\mathsf{T}M$ are described as those $v \in \mathsf{T}\mathsf{T}M$ which are invariant with respect to κ, $\kappa(v) = v$.