$\mathbb{Z}_2^n$-SUPERMANIFOLDS

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The talk is based on a joint work with Tiffany Covolo and Norbert Poncin:

- The category of $\mathbb{Z}_2^n$-supermanifolds, *J. Math. Phys.* **57** (2016), 073503 (16pp).
- Splitting theorem for $\mathbb{Z}_2^n$-supermanifolds, *J. Geom. Phys.* **110** (2016), 393-401.

Batchelor-Gawędzki theorem

- A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n ,$$

  $A(x) \in \mathrm{GL}(n, \mathbb{R})$.
- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0 and 'linear coordinates' y have degree 1. Linearity in $y's$ is now equivalent to the fact that changes of coordinates respect the degrees.
- This implies that there is a well-defined homogeneity structure, i.e. an action of the multiplicative monoid of reals,

$$h : \mathbb{R} \times E \ni (t, v) \mapsto h_t(v) \in E , \quad h_t(x, y) = (x, ty) ,$$

  (multiplication by reals) and its infinitesimal generator $\nabla_E = y^a \partial_{y^a}$ (the Euler/Liouville vector field).

Batchelor-Gawędzki theorem

- Actually, one can prove (Grabowski-Rotkiewicz '09) that a vector bundle structure is just a homogeneity structure h on a manifold E which is regular, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} h_t(v) = 0 \;\Leftrightarrow\; v \in M = h_0(E)\,.$$

- A homogeneity structure defines $\mathbb{N}$-graded algebra generated by homogeneous functions: $f \in C^\infty(E)$ is homogeneous of degree $k \in \mathbb{N}$ if $f \circ h_t = t^k f$ ($\nabla_E(f) = kf$).
- If we replace the local fiber coordinates (y^i) of degree 1 with coordinates (ξ^i) which are not only of degree 1 but also odd, $\xi^i\xi^j = -\xi^j\xi^i$, then the coordinate transformations

$$(x,\xi) \mapsto (x, A(x)\xi)$$

remain consistent and define a supermanifold $\Pi E = E[1]$ with M being its body.

Batchelor-Gawędzki theorem

- Each coordinate neighbourhood $U \subset M$ is then a ringed space with the sheaf $\mathcal{O}_U$ of supercommutative rings

$$\mathcal{O}_U(V) = C^\infty(V)[\xi^1, \dots, \xi^n]$$

of Grassmann polynomials in variables (ξ^i) and coefficients in the algebra $C^\infty(V)$ of smooth functions on $V \subset U$.

Theorem (Gawędzki '77, Batchelor '79)

Any supermanifold $\mathcal{M}$ with the body M is (non-canonically) diffeomorphic with a supermanifold ΠE for a vector bundle $\tau : E \to M$.
The superalgebra $\mathcal{A}(\mathcal{M})$ of smooth (super)functions on $\mathcal{M}$ is then isomorphic with the Grassmann algebra

$$A^\bullet(E) = \bigoplus_{j=1}^{n} \mathrm{Sec}(\Lambda^j E)\,,$$

of multi-sections of E, where n is the rank of E.

$\mathbb{Z}_2$-grading is not enough

- Supermanifolds ΠE are special, because the $\mathbb{Z}_2$-grading in the structure sheaf comes from a $\mathbb{Z}$-grading (actually, $\mathbb{N}$-grading).
- Consider a supermanifold $\mathcal{M}$ with coordinates (even and odd) (x^a, ξ^i) and its tangent bundle $\mathsf{T}\mathcal{M}$ with coordinates $(x^a, \xi^i, \mathrm{d}x^b, \mathrm{d}\xi^j)$.
  We can consider $\mathsf{T}\mathcal{M}$ as a supermanifold, viewing $x^a, \mathrm{d}x^b$ as even and $\xi^i, \mathrm{d}\xi^j$ as odd, or, closer to the standard convention, viewing $x^a, \mathrm{d}\xi^j$ as even and $\mathrm{d}x^b, \xi^i$ as odd.
- Much more natural is to take advantage with the additional $\mathbb{N}$-grading on the vector bundle $\mathsf{T}\mathcal{M}$ and to consider the algebra of functions as $\mathbb{Z}_2 \times \mathbb{N}$ (thus also $\mathbb{Z}_2^2$)-graded. Hence the sign convention for homogeneous elements z^α, z^β of bi-degrees $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{Z}_2^2$ is (Deligne convention)
$$z^\alpha z^\beta = (-1)^{\langle \alpha, \beta \rangle} z^\beta z^\alpha\,,$$
  where $\langle \alpha, \beta \rangle = \alpha_1\beta_1 + \alpha_2\beta_2$ is the 'scalar product'. However, $\mathsf{T}\mathcal{M}$ is not a standard supermanifold but rather $\mathbb{Z}_2^2$-supermanifold.

Sign-rules and $\mathbb{Z}_2^n$-graded algebras

- let K be a commutative unital ring, $K^\times$ be the group of invertible elements of K, and let G be a commutative semigroup. A map $\varphi : G \times G \to K^\times$ is called a commutation factor on G if

$$\varphi(g,h)\varphi(h,g) = 1\,, \quad \varphi(f,g+h) = \varphi(f,g)\varphi(f,h)\,, \quad \varphi(g,g) = \pm 1\,,$$

for all $f,g,h \in G$. Note that these axioms imply that

$$\varphi(f+g,h) = \varphi(f,h)\varphi(g,h)$$

and that the condition $\varphi(g,g) = \pm 1$ follows automatically from the other two axioms if K is a field.

- Let $\mathcal{A}$ be a G-graded K-algebra $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}^g$. Elements x from $\mathcal{A}^g$ are called G*-homogeneous of degree* or *weight* $g =: \deg(x)$. The algebra $\mathcal{A}$ is said to be φ*-commutative* if

$$ab = \varphi(\deg(a), \deg(b))ba\,,$$

for all G-homogeneous elements $a, b \in \mathcal{A}$.

Sign-rules and $\mathbb{Z}_2^n$-graded algebras

- Homogeneous elements x with $p(\deg(x)) = p(g) := \varphi(g,g) = -1$ are odd, the other homogeneous elements are even. Graded algebras with commutation rules of this kind are known under the name color algebras. In this talk we will be interested in color associative algebras whose commutation factor is just a sign.
- In what follows, K will be $\mathbb{R}$ and φ will take the form

$$\varphi(g,h) = (-1)^{\langle g,h\rangle}\,,$$

  for a 'scalar product' $\langle -,-\rangle : G \times G \to \mathbb{Z}$. This means that we use the commutation factor as the sign rule. In this note we confine ourselves to $G = \mathbb{Z}_2^n$ and the standard 'scalar product' of $\mathbb{Z}_2^n$, what will lead to $\mathbb{Z}_2^n$-Supergeometry with nicer categorical properties than the standard Supergeometry. More precisely, we propose a generalization of differential $\mathbb{Z}_2$-Supergeometry to the case of a $\mathbb{Z}_2^n$-grading in the structure sheaf.

Sign-rules and $\mathbb{Z}_2^n$-graded algebras

Example

The real Clifford algebra $\mathrm{Cl}_{p,q}(\mathbb{R})$ is the associative $\mathbb{R}$-algebra generated by e_i, where $1 \leq i \leq n$ and $n = p + q$, of $\mathbb{R}^n$, modulo the relations

$$e_i e_j = -e_j e_i\,, \quad i \neq j\,,$$

$$e_i^2 = \begin{cases} +1\,, & i \leq p \\ -1\,, & i > p\,. \end{cases}$$

The pair of integers (p, q) is called the signature. Note that, as a vector space, $\mathrm{Cl}_{p,q}(\mathbb{R})$ is isomorphic to the Grassmann algebra $\bigwedge\langle e_1, \dots, e_n\rangle$ on the chosen generators. $\mathrm{Cl}_{p,q}(\mathbb{R})$ is often understood as quantization of the Grassmann algebra (in the same sense as the Weyl algebra is a quantization of the symmetric algebra).

The Clifford algebra $\mathrm{Cl}_{p,q}(\mathbb{R})$ is a $\mathbb{Z}_2^{p+q+1}$-commutative associative algebra with the degree of e_i being $(0, \dots, 0, 1, 0, \dots, 0, 1)$.

Sign-rules and $\mathbb{Z}_2^n$-graded algebras

Actually, from the scalar product on $\mathbb{Z}_2^n$ we can obtain arbitrary sign rule.

For, let S be a finite set, say $S = \{1, \ldots, m\}$, and let $\varphi : S \times S \to \{\pm 1\}$ be any symmetric function. We can understand φ as a sign rule for the associative algebra $\mathcal{A}$ generated freely by elements y^i, $i = 1, \ldots, m$, modulo the commutation identities

$$y^i y^j = \varphi(i,j) y^j y^i \,.$$

We then have the following.

Theorem

There is $n \leq 2m$ and a map $\sigma : S \to \mathbb{Z}_2^n$, $i \mapsto \sigma_i$, such that

$$\varphi(i,j) = (-1)^{\langle \sigma_i, \sigma_j \rangle_n} \,.$$

In other words, $\mathcal{A}$ can be made into a $\mathbb{Z}_2^n$-commutative associative algebra.

Note that there are non-nilpotent generators of non-zero degree.

$\mathbb{Z}_2^n$-supermanifolds

- The first idea is to define the function sheaf $\mathcal{O}_U$ of a $\mathbb{Z}_2^n$-superdomain $\mathcal{U} = (U, \mathcal{O}_U)$, over any open $V \subset U$, as the $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebra

$$\mathcal{O}_U(V) = C_U^\infty(V)[\xi^1, \ldots, \xi^q]$$

  of polynomials in the indeterminates ξ^a of degrees $\deg(\xi^a) \in \mathbb{Z}_2^n \setminus \{0\}$ with coefficients in smooth functions of V.
- However, for a proper development of differential calculus, we must be able to compose elements of degree 0 with smooth functions. But what is $F(x + \xi^2)$ for a 1-variable smooth function F, a variable x and a formal even variable ξ ?
- Since ξ is not nilpotent, the Taylor formula $F(x + \xi^2) = \sum_k \frac{1}{k!} F^{(k)}(x)\, \xi^{2k}$ leads to a formal power series.
- Hence we are forced to take

$$\mathcal{O}_U(V) = C_U^\infty(V)[[\xi^1, \ldots, \xi^q]]\,.$$

$\mathbb{Z}_2^n$-supermanifolds

Definition

Let $n, p, q_1, \ldots, q_{2^n-1} \in \mathbb{N}$ and set $\underline{q} = (q_1, \ldots, q_{2^n-1})$. Consider p coordinates $x^1, \ldots, x^p$ of degree $s_0 = 0$ (resp., q_1 coordinates $\xi^1, \ldots, \xi^{q_1}$ of degree s_1, q_2 coordinates $\xi^{q_1+1}, \ldots, \xi^{q_1+q_2}$ of degree s_2, ...), $\{s_i\} = \mathbb{Z}_2^n$. Assume that these coordinates (x, ξ) commute according to the $\mathbb{Z}_2^n$-commutation rule.

A $\mathbb{Z}_2^n$-superdomain (called also a color superdomain) of dimension $p|\underline{q}$ is a ringed space $\mathcal{U}^{p|\underline{q}} = (U, \mathcal{O}_U)$, where $U \subset \mathbb{R}^p$ is the open range of x, and where the structure sheaf is defined over any open $V \subset U$ as the $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebra

$$\mathcal{O}_U(V) = C^\infty_U(V)[[\xi^1, \ldots, \xi^q]], \quad q = q_1 + \cdots + q_{2^n-1},$$

of formal power series

$$f(x,\xi) = \sum_{|\mu|=0}^{\infty} f_{\mu_1 \ldots \mu_q}(x)\, (\xi^1)^{\mu_1} \ldots (\xi^q)^{\mu_q} = \sum_{|\mu|=0}^{\infty} f_\mu(x) \xi^\mu$$

in the formal variables $\xi^1, \ldots, \xi^q$ with coefficients in $C^\infty_U(V)$.

$\mathbb{Z}_2^n$-supermanifolds

Definition (Ringed space definition)

A (smooth) $\mathbb{Z}_2^n$-supermanifold (or a color supermanifold) $\mathcal{M}$ of dimension $p|\underline{q}$, $p \in \mathbb{N}$, $\underline{q} = (q_1, \ldots, q_{2^n-1}) \in \mathbb{N}^{2^n-1}$, is a locally $\mathbb{Z}_2^n$-ringed space $(M, \mathcal{O}_M)$ that is locally isomorphic to the $\mathbb{Z}_2^n$-superdomain $(\mathbb{R}^p, C^\infty_{\mathbb{R}^p}[[\xi^1, \ldots, \xi^q]])$, where $q = \sum_k q_k$, where $\xi^1, \ldots, \xi^q$ are $\mathbb{Z}_2^n$-commuting formal variables of which q_k have the k-th degree in $\mathbb{Z}_2^n \setminus \{0\}$, and where $C^\infty_{\mathbb{R}^p}$ is the function sheaf of the Euclidean space $\mathbb{R}^p$.

Roughly, a $\mathbb{Z}_2^n$-supermanifold can be viewed as a topological space M, which is covered by $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative coordinate systems (x, ξ) (x can be interpreted as a homeomorphism $x(m) \rightleftarrows m(x)$ between its Euclidean open range U and an open subset of M (which is often also denoted by U)) and is endowed with coordinate transformations that respect the $\mathbb{Z}_2^n$-degree and satisfy the cocycle condition.

Example. If $\mathcal{M}$ is a $\mathbb{Z}_2^n$-supermanifold, then $\mathsf{T}\mathcal{M}$ and $\mathsf{T}^*\mathcal{M}$ are canonically $\mathbb{Z}_2^{n+1}$-supermanifolds.

Double vector bundles

In geometry and applications one often encounters double vector bundles, i.e. manifolds equipped with two vector bundle structures which are compatible in a categorical sense. They were defined by Pradines and studied by Mackenzie, Grabowska and Urbański as vector bundles in the category of vector bundles. More precisely:

Definition

A double vector bundle $(D; A, B; M)$ is a system of four vector bundle structures

$$\begin{array}{ccc} D & \xrightarrow{q^D_B} & B \\ {\scriptstyle q^D_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow{q_A} & M \end{array}$$

in which D has two vector bundles structures, on bases A and B. The latter are themselves vector bundles on M, such that each of the four structure maps of each vector bundle structure on D (namely the bundle projection, zero section, addition and scalar multiplication) is a morphism of vector bundles with respect to the other structures.

The structure of double vector bundles

- In the above diagram, we refer to A and B as the side bundles of D, and to M as the double base.
- In the two side bundles, the addition and scalar multiplication are denoted by the usual symbols $+$ and juxtaposition, respectively.
- We distinguish the two zero-sections, writing $0^A : M \to A$, $m \mapsto 0^A_m$, and $0^B : M \to B$, $m \mapsto 0^B_m$.
- In the vertical bundle structure on D with base A, the vector bundle operations are denoted by $+_A$ and $\cdot_A$, with $\tilde{0}^A : A \to D$, $a \mapsto \tilde{0}^A_a$, for the zero-section.
- Similarly, in the horizontal bundle structure on D with base B we write $+_B$ and $\cdot_B$, with $\tilde{0}^B : B \to D$, $b \mapsto \tilde{0}^B_b$, for the zero-section.
- The two structures on D, namely (D, q^D_B, B) and (D, q^D_A, A) will also be denoted, respectively, by $\tilde{D}_B$ and $\tilde{D}_A$, and called the horizontal bundle structure and the vertical bundle structure.

Double vector bundles - compatibility conditions

The condition that each vector bundle operation in D is a morphism with respect to the other is equivalent to the following conditions, known as the interchange laws:

$$
\begin{aligned}
(d_1 \ +_B \ d_2) \ +_A \ (d_3 \ +_B \ d_4) &= (d_1 \ +_A \ d_3) \ +_B \ (d_2 \ +_A \ d_4), \\
t \cdot_A (d_1 \ +_B \ d_2) &= t \cdot_A d_1 \ +_B \ t \cdot_A d_2, \\
t \cdot_B (d_1 \ +_A \ d_2) &= t \cdot_B d_1 \ +_A \ t \cdot_B d_2, \\
t \cdot_A (s \cdot_B d) &= s \cdot_B (t \cdot_A d), \\
\tilde{0}^A_{a_1+a_2} &= \tilde{0}^A_{a_1} \ +_B \ \tilde{0}^A_{a_2}, \\
\tilde{0}^A_{ta} &= t \cdot_B \tilde{0}^A_a, \\
\tilde{0}^B_{b_1+b_2} &= \tilde{0}^B_{b_1} \ +_A \ \tilde{0}^A_{b_2}, \\
\tilde{0}^B_{tb} &= t \cdot_A \tilde{0}^B_b.
\end{aligned}
$$

Double vector bundles

- We can extend the concept of a double vector bundle of Pradines to n-fold vector bundles.
- However, thanks to our simple description in terms of a homogeneity structure, the 'diagrammatic' definition of Pradines can be substantially simplified.
- As two vector bundle structure on the same manifold are just two regular homogeneity structures, the obvious concept of compatibility leads to the following:

Definition (Grabowski-Rotkiewicz)

A double graded bundle is a manifold equipped with two homogeneity structures h^1, h^2 which are compatible in the sense that

$$h^1_t \circ h^2_s = h^2_s \circ h^1_t \quad \text{for all } s, t \in \mathbb{R}\,.$$

n-fold vector bundles

The above condition can also be formulated as commutation of the corresponding Euler vector fields, $[\nabla^1, \nabla^2] = 0$.

For vector bundles this is equivalent to the concept of a double vector bundle in the sense of Pradines and Mackenzie.

Theorem (Grabowski-Rotkiewicz)

The concept of a double vector bundle, understood as a particular double graded bundle in the above sense, coincides with that of Pradines.

All this can be extended to n-fold vector bundles in the obvious way:

Definition

A n-fold vector bundle is a manifold equipped with n regular homogeneity structures $h^1, \dots, h^n$ which are compatible in the sense that

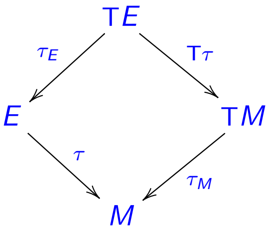
$$h^i_t \circ h^j_s = h^j_s \circ h^i_t \quad \text{for all } s, t \in \mathbb{R} \quad \text{and} \quad i, j = 1, \dots, n\,.$$

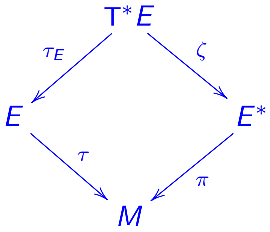
n-fold vector bundles - examples

- Any n-fold vector bundle is a polynomial bundle over $M = \cap_i h_0^i(E)$ with local coordinates (x^i, y^a_σ), where x^i is of degree 0, y^a_σ is of degree $\sigma \in \{0,1\}^n \setminus \{\underline{0}\}$, and transformation rules: $x'^i = x'^i(x)$ and

$$y'^a_\sigma = T^{a;\sigma}_b(x) y^b_\sigma + \sum_{\substack{1<j \\ \sigma^1+\cdots+\sigma^j=\sigma\neq 0}} T^{a;\sigma^1,\dots,\sigma^j}_{b_1,\dots,b_j}(x) y^{b_1}_{\sigma^1}\cdots y^{b_j}_{\sigma^j}\,.$$

  Here, the sum $\sigma^1+\sigma^2+\cdots+\sigma^j$ is in $\mathbb{N}^n$.
- If $\tau : E \to M$ is a vector bundle, then $\mathsf{T}E$ and T^*E are canonically double vector bundles:





n-fold vector bundles - splitting theorem

- Fundamental fact for applications: There is a canonical isomorphism of double vector bundles

$$\mathsf{T}^*E^* \simeq \mathsf{T}^*E\,.$$

- Split n-fold vector bundles. Let $\{E_\sigma\}_{\sigma\in\mathbb{Z}_2^n\setminus\{\underline{0}\}}$ be a family of vector bundles over M. Then $E=\oplus_{\sigma\in\mathbb{Z}_2^n\setminus\{\underline{0}\}}E_\sigma$ is canonically an n-fold vector bundle such that h^i_t is the multiplication by t in E_σ for those σ for which $\sigma_i=1$.
- For $n=2$, we have $E=E_{(1,0)}\oplus_M E_{(0,1)}\oplus_M E_{(1,1)}$ and

$$\begin{aligned} h^1_t\left(y_{(1,0)}+y_{(0,1)}+y_{(1,1)}\right) &= ty_{(1,0)}+y_{(0,1)}+ty_{(1,1)}\,,\\ h^2_t\left(y_{(1,0)}+y_{(0,1)}+y_{(1,1)}\right) &= y_{(1,0)}+ty_{(0,1)}+ty_{(1,1)}\,. \end{aligned}$$

Theorem

Any n-fold vector bundle is (non-canonically) isomorphic with a split n-fold vector bundle.

Superization of n-fold vector bundles

- The supports of the degrees of coordinates appearing in the coordinate transformations

$$y'^a_\sigma = T_b^{a;\sigma}(x)y^b_\sigma + \sum_{\substack{1<j \\ \sigma^1+\cdots+\sigma^j=\sigma\neq 0}} T^{a;\sigma^1,\dots,\sigma^j}_{b_1,\dots,b_j}(x)y^{b_1}_{\sigma^1}\cdots y^{b_j}_{\sigma^j}\,.$$

  of a $\mathbb{Z}_2^n$-tuple vector bundle E are pairwise disjoint, so the order in which $y^{b_i}_{\sigma^i}$ appear in the above formula is irrelevant if we assume that we replace y^a_σ with ξ^a_σ which(super)commute according to the $\mathbb{Z}_2^n$-rules of commutation, and these transformations correctly define an $\mathbb{Z}_2^n$-supermanifold.
- We denote the resulted supermanifold ΠE.
- We have the following colored version of Bachelor-Gawędzki theorem:

Colored Bachelor-Gawędzki theorem

Theorem

Any $\mathbb{Z}_2^n$-supermanifold is (non-canonically) isomorphic with a supermanifold of the form ΠE for an n-tuple vector bundle (thus a split n-fold vector bundle).

- This result is equivalent to the statement that any smooth $\mathbb{Z}_2^n$-supermanifold can noncanonically be equipped with an atlas, whose coordinates (x^i, ξ_σ^a) transform according to

$$x'^i = x'^i(x)\,, \quad \xi_\sigma^a = T_b^{a;\sigma}(x)\xi_\sigma^b\,.$$

- In other words, the coordinates of $\mathbb{Z}_2^n$-degree σ depend only on the old coordinates of the same degree σ.
- In the following, we consider sheafs $\mathcal{A}_M, C_M^\infty, \ldots$ over a smooth manifold M, but will, for simplicity, just write $\mathcal{A}, C^\infty, \ldots$.

Colored Bachelor-Gawędzki theorem

- Let $\mathcal{M} = (\mathcal{M}, \mathcal{A})$ be a $\mathbb{Z}_2^n$-supermanifold, $n \geq 1$, let $\varepsilon : \mathcal{A} \to C^\infty$ be the projection onto C^∞, let $\mathcal{J} = \ker \varepsilon$, and let $\mathcal{A} \supset \mathcal{J} \supset \mathcal{J}^2 \supset \ldots$ be the decreasing filtration of the structure sheaf by sheaves of $\mathbb{Z}_2^n$-graded ideals.
- The quotients $\mathcal{J}^{k+1}/\mathcal{J}^{k+2}$, $k \geq 0$, are locally finite free sheaves of modules over $C^\infty \simeq \mathcal{A}/\mathcal{J}$. In particular,

$$\mathcal{S} := \mathcal{J}/\mathcal{J}^2$$

  is a locally finite free sheaf of $\mathbb{Z}_2^n \setminus \{0\}$-graded C^∞-modules. Hence, there exists a $\mathbb{Z}_2^n \setminus \{0\}$-graded vector bundle $E \to M$ such that

$$\mathcal{S} \simeq \Gamma((\Pi E)^*)\,.$$

- Denote by $\odot$ the $\mathbb{Z}_2^n$-graded symmetric tensor product of $\mathbb{Z}_2^n$-graded C^∞-modules and of $\mathbb{Z}_2^n$-graded vector bundles. Then,

$$\Gamma(\odot^{k+1}(\Pi E)^*) \simeq \odot^{k+1}\mathcal{S} \simeq \mathcal{J}^{k+1}/\mathcal{J}^{k+2}\,.$$

Embedding $C^\infty_M \hookrightarrow \mathcal{A}_\mathcal{M}$

- Our goal is to show that

$$\mathcal{A}(\Pi E) := \prod_{k\geq -1} \Gamma(\odot^{k+1}(\Pi E)^*) = \prod_{k\geq -1} \odot^{k+1}\mathcal{S} \simeq \mathcal{A}$$

  as sheaf of $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebras.
- It is clear that locally the sheaves coincide. To prove that they are isomorphic, we will build a morphism $\prod_{k\geq -1} \odot^{k+1}\mathcal{S} \to \mathcal{A}$ of sheaves of $\mathbb{Z}_2^n$-superalgebras. The idea is to extend a morphism $\mathcal{S} \to \mathcal{A}$, or $\mathcal{J}/\mathcal{J}^2 \to \mathcal{J}$.
- The latter will be obtained as a splitting of the sequence $0 \to \mathcal{J}^2 \to \mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to 0$. One of the problems to solve is to show that this sequence can be viewed as a sequence of sheaves of C^∞-modules. Therefore, we need an embedding $C^\infty \to \mathcal{A}$ which, on the other hand, is a necessary condition.

Embedding $C^\infty_M \hookrightarrow \mathcal{A}_\mathcal{M}$

- We will actually construct a splitting of the short exact sequence $0 \to \mathcal{J} \to \mathcal{A} \xrightarrow{\varepsilon} C^\infty \to 0$, i.e., a morphism $\varphi : C^\infty \to \mathcal{A}$ of sheaves of $\mathbb{Z}_2^n$-superalgebras such that $\varepsilon \circ \varphi = \mathrm{id}$. More precisely, we build φ as the limit of an $\mathbb{N}$-indexed sequence of sheaf morphisms $\varphi_k : C^\infty \to \mathcal{A}/\mathcal{J}^{k+1}$.
- This sequence φ_k will be obtained by induction on k, starting from $\varphi_0 = \mathrm{id}$: we assume that we already got φ_{i+1} as an extension of φ_i for $0 \leq i \leq k-1$, and we aim at extending $\varphi_k : C^\infty \to \mathcal{A}/\mathcal{J}^{k+1}$ to

$$\varphi_{k+1} : C^\infty \to \mathcal{A}/\mathcal{J}^{k+2} \, ,$$

  in the sense that $f_{k,k+1} \circ \varphi_{k+1} = \varphi_k$, where $f_{k,k+1}$ is the canonical map $\mathcal{A}/\mathcal{J}^{k+2} \to \mathcal{A}/\mathcal{J}^{k+1}$.
- We build consistent extensions of the $\varphi_{k,U}$ by local (in the sense of (pre)sheaf morphisms) degree zero unital $\mathbb{R}$-algebra morphisms

$$\varphi_{k+1,U} : C^\infty(U) \to \mathcal{A}(U)/\mathcal{J}^{k+2}(U) \simeq C^\infty(U)[[\xi^1, \ldots, \xi^q]]_{\leq k+1}$$

  over a cover $\mathcal{U}$ by $\mathbb{Z}_2^n$-chart domains U.

Embedding $C^\infty_M \hookrightarrow \mathcal{A}_\mathcal{M}$

- Here, subscript $\leq k+1$ means that we confine ourselves to 'series' whose terms contain at most $k+1$ formal parameters. Further, 'consistent' means that, if U, V are two domains of the cover, we must have

$$\varphi_{k+1,U}|_{U\cap V} = \varphi_{k+1,V}|_{U\cap V}\ .$$

Lemma

Over any $\mathbb{Z}_2^n$-chart domain U, there exists an extension $\varphi_{k+1,U} : C^\infty(U) \to \mathcal{A}(U)_{\leq k+1} := C^\infty(U)[[\xi^1,\dots,\xi^q]]_{\leq k+1}$ of $\varphi_{k,U}$ as local degree zero unital $\mathbb{R}$-algebra morphism.

- Indeed, the association

$$\varphi_{k,U}(x^i) = x^i + \sum_{1\leq|\mu|\leq k} f^i_\mu(x)\xi^\mu \in \mathcal{A}(U)$$

uniquely define a degree zero unital $\mathbb{R}$-algebra morphism $\overline{\varphi}_{k,U} : C^\infty(U) \to \mathcal{A}(U)$.

Embedding $C^\infty_M \hookrightarrow \mathcal{A}_\mathcal{M}$

- To finalize the construction of the sheaf morphism $\varphi : C^\infty \to \mathcal{A}$, it now suffices to solve the consistency problem. Let U and V be $\mathbb{Z}_2^n$-chart domains and let $\varphi_{k+1,U}$ and $\varphi_{k+1,V}$ be the preceding extensions of $\varphi_{k,U}$ and $\varphi_{k,V}$, respectively.
- The difference

$$\omega_{k+1,UV}(f) := \varphi_{k+1,U}|_{U\cap V}(f) - \varphi_{k+1,V}|_{U\cap V}(f) \in \mathcal{A}(U\cap V)_{\leq k+1} \,,$$

for $f \in C^\infty(U\cap V)$, defines a derivation

$$\omega_{k+1,UV} : C^\infty(U\cap V) \to \mathcal{A}(U\cap V)_{=k+1} \,.$$

- Indeed, as

$$\varphi_{k+1,U}|_{U\cap V}(fg) = \varphi_{k+1,V}|_{U\cap V}(fg) + \omega_{k+1,UV}(fg)\,,$$

we finally get

$$\omega_{k+1,UV}(fg) = \omega_{k+1,UV}(f)\cdot g + f\cdot \omega_{k+1,UV}(g)\,.$$

Embedding $C^\infty_M \hookrightarrow \mathcal{A}_\mathcal{M}$

- Hence, $\omega_{k+1,UV}$ can be viewed as as a Čech 1-cocycle $\omega_{k+1} \in \mathrm{Sec}(U \cap V, \mathsf{T}M \otimes F)$ for a vector bundle F.
- In the smooth category, we have a partition of unity in M, so there exists a 0-cochain η_{k+1}, i.e. a family $\eta_{k+1,U} : C^\infty(U) \to \mathrm{Sec}(U \cap V, \mathsf{T}M \otimes F)$, such that

$$\varphi_{k+1,U}|_{U\cap V} - \varphi_{k+1,V}|_{U\cap V} = \omega_{k+1,UV} = \eta_{k+1,V}|_{U\cap V} - \eta_{k+1,U}|_{U\cap V}\,.$$

- It is now easily checked that the sum $\varphi'_{k+1,U} := \varphi_{k+1,U} + \eta_{k+1,U} : C^\infty(U) \to \mathcal{A}(U)_{\leq k+1}$ is a local degree zero unital $\mathbb{R}$-algebra morphism, which satisfies the consistency condition and extends $\varphi_{k,U}$. This proves the existence of the searched morphism $\varphi : C^\infty \to \mathcal{A}$ of sheaves of $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebras.
- By construction, $\varepsilon \circ \varphi = \mathrm{id}$.

Embedding $C^\infty_M \hookrightarrow \mathcal{A}_\mathcal{M}$

- We have proved the following.

Theorem

For any $\mathbb{Z}_2^n$-supermanifold $(M, \mathcal{A}_M)$, the short exact sequence

$$0 \to \mathcal{J}_M \to \mathcal{A}_M \xrightarrow{\varepsilon} C^\infty_M \to 0$$

of sheaves of $\mathbb{Z}_2^n$-commutative associative $\mathbb{R}$-algebras is noncanonically split.

- Due to the embedding $\varphi : C^\infty \to \mathcal{A}$, the short exact sequence of sheaves of $\mathcal{A}$-modules

$$0 \to \mathcal{J}^2 \to \mathcal{J} \to \mathcal{S} = \mathcal{J}/\mathcal{J}^2 \to 0 \tag{1}$$

can be viewed as a short exact sequence of sheaves of C^∞-modules.
- Although $\mathcal{J}^2$ and $\mathcal{J}$ are not locally finite free, we can find a splitting Φ^1 of (1).

Embedding $C^\infty_M \hookrightarrow \mathcal{A}_\mathcal{M}$

- We now extend Φ^1 to a morphism

$$\Phi : \mathcal{A}(\Pi E) = \prod_{k\geq 0} \odot^k \mathcal{S} \to \mathcal{A}$$

  of sheaves of $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebras, putting $\Phi := \varphi : C^\infty \to \mathcal{A}$ on C^∞, where φ is the above-constructed degree preserving unital algebra morphism, and

$$\Phi(\psi_1 \odot \ldots \odot \psi_k) := \Phi^1(\psi_1) \cdot \ldots \cdot \Phi^1(\psi_k) \in \mathcal{J}^k \subset \mathcal{A} \qquad (2)$$

  on $\odot^{k\geq 2}\mathcal{S}$, with the obvious extension to power series by Hausdorff continuity. This extension is well defined, since the RHS of (2) is $\mathbb{Z}_2^n$-commutative and C^∞-multilinear.
- This map $\Phi : \mathcal{A}(\Pi E) \to \mathcal{A}$ respects the degrees and the units, and is an $\mathbb{R}$-algebra morphism, what completes the proof of the colored Batchelor-Gawędzki theorem.

THANK YOU FOR YOUR ATTENTION!

Happy Birthday Alberto!