Measurable entanglement measures

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- construct quantities measurable in experiments

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- C locally unitary invariant, i.e. invariant under the direct product of the unitary groups $U(n_i) = U(\mathcal{H}_i)$

$$(U(n_1) \times \cdots \times U(n_N), \mathcal{B}^+) \to \mathcal{B}^+$$

$$\rho = \rho_1 \otimes \cdots \otimes \rho_N \mapsto \rho' = U_1 \rho_1 U_1^\dagger \otimes \cdots \otimes U_N \rho_N U_N^\dagger \Leftrightarrow C(\rho) = C(\rho')$$

$$U_i \in U(n_i)$$

Entanglement measures

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i.e. $C(\rho) < C(\rho') \Rightarrow \forall_{F\in\mathcal{O}} F(\rho) \neq \rho'$ i.e. ρ' can not be reached from ρ *via* operations from $\mathcal{O}$

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- 'allowed operations'
  - local completely positive maps: $\Lambda_i : \mathcal{B}_i \rightarrow \mathcal{B}_i$

    $\Lambda : Hom(\mathcal{H}) \rightarrow Hom(\mathcal{H})$ is completely positive iff
    $(\mathbb{I}_k \otimes \Lambda) : \mathbb{C}^{k\times k} \otimes Hom(\mathcal{H}) \rightarrow \mathbb{C}^{k\times k} \otimes Hom(\mathcal{H})$
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- Experimentally measurable (e.g. $C(\rho) = \mathrm{Tr} A\rho$, $A = A^\dagger$)

Two subsystems

- Pure states: $\rho = P_{|\Psi\rangle} = |\Psi\rangle\langle\Psi|$, $\langle\Psi|\Psi\rangle = 1$.

$$|\Psi\rangle = \sum_{ij} \beta_{ij} |e_i\rangle \otimes |f_j\rangle$$

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- $\beta\beta^\dagger$ and $\beta^\dagger\beta$ - two hermitian positive semidefinte matrices, diagonalizable by unitary W and $W^\dagger$

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$$|e'_k\rangle = \sum_i W_{ik} |e_i\rangle, \quad |f'_k\rangle = \sum_i \overline{V}_{ik} |f_i\rangle$$

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- ρ - separable iff $rank(\Lambda) = 1$ (i.e. only one λ_k does not vanish).

Concurrence

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- hence

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- A is proportional to a projection on $\left(\mathcal{H}_1 \wedge \mathcal{H}_1 \right) \otimes \left(\mathcal{H}_2 \wedge \mathcal{H}_2 \right)$, $A^2 = 4A$, $A = A^{\dagger}$

Experimental determination

- The original definition of the concurrence (for two-qubit systems, i.e. $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$ involved an unphysical operation of complex conjugation (not expressible in the form of a completely positive map),

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- L. Aolita and F. Mintert, *Measuring multipartite concurrence with a single factorizable observable*, quant-ph/0608012

Mixed states

- An arbitrary mixed state, spectral decomposition:

$$\rho = \sum_{i=1}^{r} |\phi_i\rangle\langle\phi_i|$$

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- Concurrence of a mixed state

$$C(\rho) := \inf_{decompositions} \sum_{i=1}^{M} C(\psi_i) \geq 0$$

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$$End(V) = V \otimes V^*, \qquad \mathcal{J}^{-1} : \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}^* \otimes \mathcal{H}^* \to \mathcal{H} \otimes \mathcal{H}^* \otimes \mathcal{H} \otimes \mathcal{H}^*$$

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- define: $T_{ij}^\alpha := \langle \phi_i | \tau_\alpha | \phi_j \rangle$, $T := \sum_\alpha z_\alpha T^\alpha$, $\sum_\alpha |z_\alpha|^2 = 1$

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- then $C(\rho) \geq \max\{0, \lambda_M - \sum_{i=1}^{M-1} \lambda_i\}$,
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Mixed states

- Choi-Jamiołkowski-isomorphism

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$$End(V) = V \otimes V^*, \qquad \mathcal{J}^{-1} : \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}^* \otimes \mathcal{H}^* \to \mathcal{H} \otimes \mathcal{H}^* \otimes \mathcal{H} \otimes \mathcal{H}^*$$

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- for each choice of z_i we obtain a measure (partially) discriminating between separable and entangled mixed states.

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- ... or in terms of partial traces:

$$C_A(|\psi\rangle) = \sqrt{\sum_{s_1,\ldots,s_N} p_{s_1,\ldots,s_N} \sum_{S \in 2^{\{1,\ldots,N\}}} \mathrm{Tr}\left(\mathrm{Tr}_S(|\psi\rangle\langle\psi|)^2\right) \prod_{i \in S} s_i},$$

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- **Theorem.** Concurrence C_A is an entanglement monotone if, for all $S \in 2^{\{1,\ldots,N\}}$, $S \neq \{1,\ldots,N\}$, $S \neq \varnothing$)

$$\sum_{s_1,\ldots,s_N} p_{s_1,\ldots,s_N} \prod_{i \in S} s_i \leq 0.$$

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- ad(b,c): $A \in End(\mathcal{H}^{\otimes k})$, $k = 3, 4$ constructed in an anologous way ((anti)-symmetrized projections)

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