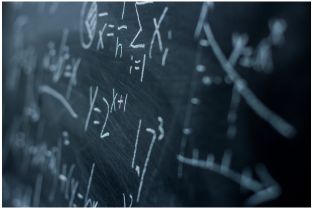
# ON A LIE'S THEOREM ABOUT INTEGRABILITY BY QUADRATURES

**Janusz Grabowski**
(Polish Academy of Sciences)



GROUP ANALYSIS OF DIFFERENTIAL EQUATIONS
AND INTEGRABLE SYSTEMS

*Larnaca, 10-14 June, 2018*

# Contents

- Integrability by quadratures
- Rectification
- Integrability by quadratures - examples
- Solvable Lie algebras
- Lie's Theorem
- Main result
- Abelian Lie ideals
- Rectification of Abelian algebras of vector fields
- Sketch of the proof and example

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

# Integrability by quadratures

- An autonomous system of differential equations,

$$\dot{x}^i = f^i(x^1, \dots, x^n)\ , \qquad i = 1, \dots, n, \tag{1}$$

  is geometrically interpreted in terms of a vector field $\Gamma$ in a $n$-dimensional manifold $M$ with a local expression

$$\Gamma = \sum_{i=1}^{n} f^i(x^1, \dots, x^n)\partial_{x^i}\ .$$

- The integral curves of $\Gamma$ are the solutions of (1). Integrating the system amounts to determine its general solution.
- In particular, we speak about the integrability by quadratures if you can determine the solutions by means of a finite number of algebraic operations and integrations of known functions. Historically, this is the first concept of integrability.

# Rectification

- Note, however, that the concept of integrability by quadratures is computational and not geometric, as it depends on coordinates in which we work.
- The result of the straightening out (rectification) Theorem asserts the existence of coordinates $(y^1, \ldots, y^n)$ in a neighbourhood of a point where $\Gamma$ is different from zero such that

$$\Gamma = \partial_{y^n} .$$

- The new coordinates $y^1, \ldots, y^{n-1}$, are constants of motion and therefore we cannot find easily such coordinates in a general case.
- It is clear that the if we use such rectifying coordinates for $\Gamma$ the integration is immediate, the solution being

$$y^k(t) = y_0^k, \quad k = 1, \ldots, n-1, \qquad y^n(t) = y_0^n + t.$$

- This proves that the concept of integrability by quadratures depends on the choice of initial coordinates, because using these rectifying coordinates the system is always integrable by quadratures.

# Integrability by quadratures - examples

- Consider first the non-autonomous inhomogeneous linear differential equation in dimension one,

$$\dot{x} = c_0(t) + c_1(t)\,x,$$

which is well known to be integrable in terms of two quadratures:

$$x(t) = \exp\left(\int_0^t c_1(t')\,dt'\right)\left[x_0 + \int_0^t \exp\left(-\int_0^{t'} c_1(t'')\,dt''\right) c_0(t')\,dt'\right].$$

- Another example is given by the nonautonomous system of differential equations

$$\dot{x}^i = \sum_{j=1}^n H^i{}_j\,x^j + b^i(t), \quad i = 1, \dots, n,$$

where $H^i{}_j$ are real numbers. Then, the solution starting from the point $\boldsymbol{x}_0$ is given by

$$\boldsymbol{x}(t) = \exp(Ht)\left[\boldsymbol{x}_0 + \int_0^t \exp(-Ht')\;\boldsymbol{b}(t')dt'\right].$$

# Solvable Lie algebras

- Recall that the derived algebra of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ is the subalgebra $\mathfrak{g}^1$ of $\mathfrak{g}$, defined by $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, while the derived series is the sequence of Lie subalgebras defined by $\mathfrak{g}^0 = \mathfrak{g}$ and

$$\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k], \quad k \in \mathbb{N}\,.$$

- Such a sequence satisfies $\mathfrak{g}^{k+1} \subset \mathfrak{g}^k$, and the Lie algebra $\mathfrak{g}$ is said to be solvable if the derived series eventually arrives at the zero subalgebra, i.e. there exists the smallest natural number $m$ such that $\mathfrak{g}^{m+1} = \{0\}$ or, in other words, $\mathfrak{g}^m$ is Abelian.
- Example. The Lie algebra of upper-triangular matrices is solvable.

$$\begin{pmatrix} * & & \ast \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

# Lie's Theorem

- A classical example is the celebrated result due to Lie, who established the following theorem :

**Theorem**

*If $n$ vector fields, $X_1,\ldots,X_n$, which are linearly independent at each point of an open set $U \subset \mathbb{R}^n$, span a solvable Lie algebra and satisfy $[X_1, X_i] = \lambda_i\, X_1$ with $\lambda_i \in \mathbb{R}$, then $X_1$ is integrable by quadratures in $U$.*

- A different result is due to Kozlov.

**Theorem**

*Let vector fields, $X_1,\ldots,X_n$, be linearly independent at each point of an open set $U \subset \mathbb{R}^n$ and span a Lie algebra $L$ such that the corresponding operators of the adjoint representation $ad_{X_i} = [X_i, \cdot]$ have a common triangular form*

$$[X_i, X_j] = \sum_{k=1}^{i} C_{ij}{}^{k}\, X_k, \qquad C_{ij}{}^{k} \in \mathbb{R}\,.$$

*Then, all the vector fields $X_i$, $i = 1, \ldots, n$, are integrable by quadratures.*

# Sketch of the Lie's proof for $n = 2$

- The differential equation can be integrated if we are able to find a first integral $F$ for $X_1$, i.e. $X_1 F = 0$, such that $dF \neq 0$ in $U$.
- As $X_1$ and $X_2$ are two linearly independent vector fields such that $[X_1, X_2] = \lambda_2 X_1$, there exists a 1-form $\alpha_0$ such that $i(X_1)\alpha_0 = 0$, $i(X_2)\alpha = 1$.
- We can see that $\alpha$ is then closed, because $X_1$ and $X_2$ generate $\mathfrak{X}(\mathbb{R}^2)$ and

$$d\alpha(X_1, X_2) = X_1\alpha(X_2) - X_2\alpha(X_1) + \alpha([X_1, X_2]) = \alpha([X_1, X_2]) = \lambda_2\, \alpha(X_1) = 0$$

- The (locally defined) function $F$ such that

$$F(x^1, x^2) = \int_{\gamma_{(x^1, x^2)}} \alpha,$$

where $\gamma_{(x^1,x^2)}$ is any curve joining a reference point $(x_0^1, x_0^2) \in U$ with the point $(x^1, x^2)$, is the first integral we were looking for.

# Example

- The dynamics is given by the vector field $X_1$, defined in $M = \mathrm{T}^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates $(x, y, p_x, p_y)$, by

$$X_1 = p_x\partial_x + p_y\partial_y - \frac{k_2}{y^{2/3}}\partial_{p_x} + \frac{2}{3}\frac{k_2\,x + k_3}{y^{5/3}}\partial_{p_y}\,,$$

  where $k_2$ and $k_3$ are arbitrary constants.
- Now, with $X_i$, $i = 2, 3, 4$, we denote the vector fields

$$\begin{aligned} X_2 &= \left(6\,p_x^2 + 3\,p_y^2 + k_2\frac{6x}{y^{2/3}} + k_3\frac{6}{y^{2/3}}\right)\partial_x + (6\,p_x p_y + 9\,k_2 y^{1/3})\partial_y \\ &- k_2\frac{6}{y^{2/3}}\,p_x\partial_{p_x} + \left(4k_2\frac{x}{y^{5/3}} - 3\frac{1}{y^{2/3}}\,p_y\right)\partial_{p_y}\,, \end{aligned}$$

# Example

$$
\begin{aligned}
X_3 &= \left(4\,p_x^3 + 4\,p_x p_y^2 + \frac{8(k_2 x + k_3)}{y^{2/3}} p_x + 12 k_2\, y^{1/3}\, p_y\right)\partial_x \\
&+ \left(4 p_x^2\, p_y + 12 k_2\, y^{1/3}\, p_x\right)\partial_y - 4 k_2 \frac{1}{y^{2/3}} p_x^2\,\partial_{p_x} \\
&+ \left(\frac{8}{3}\,\frac{k_2 x + k_3}{y^{5/3}} p_x^2 - 4 k_2 \frac{1}{y^{2/3}} p_x p_y - 12\, k_2^2 \frac{1}{y^{1/3}}\right)\partial_{p_y}\,,
\end{aligned}
$$

and

$$
\begin{gathered}
X_4 = \\
\left(6 p_x^5 + 12\, p_x^3 p_y^2 + 24 \frac{k_3 + k_2 x}{y^{2/3}} p_x^3 + 108\, k_2 y^{1/3} p_x^2 p_y + 324\, k_2^2 y^{2/3} p_x\right)\partial_x \\
+\left(6\, p_x^4 p_y + 36\, k_2 y^{1/3} p_x^3\right)\partial_y - 6\,\left(\frac{k_2}{y^{2/3}} p_x^4 - 972 k_2^3\right)\partial_{p_x} \\
+\left(4\,\frac{k_3 + k_2 x}{y^{5/3}} p_x^4 - 12 \frac{k_2}{y^{2/3}} - 108\, k_2^2 \frac{1}{y^{1/3}} p_x^2\right)\partial_{p_y}\,.
\end{gathered}
$$

# Example

- Then, we have
$$[X_1, X_i] = 0\,, \quad i = 2,3,4.$$
- $$[X_2, X_3] = 0\,, \qquad [X_2, X_4] = 1944\, k_2^3\, X_1\,, \qquad [X_3, X_4] = 432\, k_2^3\, X_2\,.$$
- Therefore, $X_1, X_2, X_3, X_4$ generate a four-dimensional solvable real Lie algebra $L$ and are linearly independent in $\mathbb{R}^4$.
- In view of Lie's Theorem,
$$X_1 = p_x\partial_x + p_y\partial_y - \frac{k_2}{y^{2/3}}\partial_{p_x} + \frac{2}{3}\frac{k_2\, x + k_3}{y^{5/3}}\partial_{p_y}\,,$$
is integrable by quadratures
- and in view of Kozlov's Theorem all $X_1, X_2, X_3, X_4$ are integrable by quadratures.

# Main result

- We want to generalize the mentioned theorems of Lie and Kozlow on a finite-dimansional solvable Lie algebra $L$ of vector fields on $M$ by
- skipping the assumption that the dimension of $L$ equals $\dim(M)$,
- skipping the triangularizability assumption.
- Hence, our main result can be formulated as follows.

**Theorem**

*If $L$ is a finite-dimensional solvable and transitive real Lie algebra of vector fields on a manifold $M$, then each vector field $\Gamma \in L$ is integrable by quadratures.*

- We will proceed by induction on $n = \dim(M)$ using the following lemma.

**Lemma**

*Any solvable finite-dimensional real Lie algebra $L$ contains an Abelian Lie ideal $A$ of dimension $1$ or $2$.*

# Abelian Lie ideals

Proof.

- Another important Lie theorem ensures that every finite-dimensional representation of a solvable Lie algebra over an algebraically closed field has an eigenvector common to all the operators of the representation.
- If we consider the adjoint representation, the theorem implies that any finite-dimensional complex, solvable Lie algebra has a one dimensional ideal.
- Therefore, we can consider the complexified Lie algebra $L^{\mathbb{C}} = L \oplus iL$ and its adjoint representation for which we can use the standard Lie theorem. As there is a common eigenvector $\nu = \nu_1 + i\nu_2$, the vectors $\nu_1, \nu_2 \in L$ span an Abelian Lie ideal $A$ of dimension 1 or 2.
- Indeed, $A$ is clearly a Lie ideal, $[x, \nu] = [x, \nu_1] + i[x, \nu_2] = \lambda(x)\nu$, and if $\dim(A) = 2$, then $[\nu_2, \nu_1] = [\nu_2, \nu] = \lambda\nu$, so $\lambda = 0$.

□

# Rectification of Abelian algebras of vector fields

## Definition

We say that an Abelian subalgebra of vector fields $A$ is straightened out, (or rectified) by quadratures in an open set $U$ if, by quadratures, we can find local coordinates $(Q^1, \ldots, Q^n)$ in $U$ such that the set $\{\partial_{Q^1}, \ldots, \partial_{Q^r}\} \subset A$ and it generates the same distribution as $A$.

## Proposition

*Any Abelian ideal of a transitive finite-dimensional solvable Lie algebra of vector fields, can be straightened out by quadratures.*

## Example

Let $L = \langle \partial_x, \partial_y, x\partial_x, y\partial_y, y^2\partial_x, y\partial_x \rangle$ . be a solvable and transitive Lie algebra of vector fields on $\mathbb{R}^2$. The associated descending series is
$L^1 = \langle \partial_x, \partial_y, y^2\partial_x, y\partial_x \rangle$, $A = L^2 = \langle \partial_x, y\partial_x \rangle$, $L^3 = \{0\}$.
$A$ is a rectified Abelian ideal with respect to coordinates $x, y$.

# Proof of the main theorem

- We shall use induction on the dimension $n$ of the manifold $M$, but as the considerations are local, we can as well assume that $M = \mathbb{R}^n$.
- The case $n = 0$ is trivial, so assume that $n \geq 1$ and let us pick up an Abelian ideal $A \subset L$ of dimension one or two, whose existence is granted for real solvable finite-dimensional Lie algebras.
- Due to the fact that $L$ is transitive, we know that the distribution $\mathcal{D}_A$ spanned by $A$ is regular, say of rank $r \leq 2$. As it is also involutive, it generates a foliation $\mathcal{F}_A$.
- Moreover, one can obtain by quadratures a coordinate system $Q^1, \ldots, Q^n$ such that $\mathcal{D}_A$ is generated by $\partial_{Q^1}, \ldots, \partial_{Q^r} \in A$ and leaves of $\mathcal{F}_A$ are the level sets of the functions $Q^{r+1}, \ldots, Q^n$.
- We will first consider the case in which the dimension of the Abelian Lie algebra $A$ coincides with the dimension of the integral leaves of the foliation $\mathcal{F}_A$, i.e. $\dim(A) = r$.

# Proof of the main theorem

For $\Gamma \in L$, we first conclude that $[\partial_{Q^i}, \Gamma] = \sum_{j=1}^{r} H_i^j \partial_{Q^j}$ implies that

$$\Gamma = \sum_{j=1}^{r} \left( \sum_{i=1}^{r} H_i^j Q^i + b^j(Q^{r+1}, \ldots, Q^n) \right) \partial_{Q^j} + \bar{\Gamma} \,,$$

where $H_i^j \in \mathbb{R}$, and $b^j$ as well as the vector field $\bar{\Gamma} = \sum_{s=r+1}^{n} \gamma^s(Q^{r+1}, \ldots, Q^n) \partial_{Q^s}$ depend on coordinates $Q^{r+1}, \ldots, Q^n$ only. This leads to a system which in coordinates reads

$$\dot{Q}^j = \sum_{i=1}^{r} H_i^j Q^i + b^j(Q^{r+1}, \ldots, Q^n) \,, \quad j = 1, \ldots, r \,, \tag{2}$$

$$\dot{Q}^s = \gamma^s(Q^{r+1}, \ldots, Q^n) \,, \quad s = r+1, \ldots, n \,. \tag{3}$$

Solving (3) by the inductive assumption, we end up with

$$\dot{Q}^j = \sum_{i=1}^{r} H_i^j Q^i + b^j(Q^{r+1}(t), \ldots, Q^n(t)) \,, \quad j = 1, \ldots, r \,,$$

which can be integrated by quadratures.

# Proof of the main theorem

- Now, we should still consider the possibility that the dimension of $A$ is two, but the dimension of the integral leaves of the foliation $\mathcal{D}_A$ is one.
- In this case, we chose a one-dimensional subspace $A_1 \subset A$, whose generator $X_1$ spans $\mathcal{D}_A$. As we already know, $X_1$ can be integrated by quadratures and can be taken as $\partial_{Q^1}$ in our system of coordinates.
- From $[\partial_{Q^1}, \Gamma] \in \mathcal{F}_A$ we get that $\Gamma$ must be of the form

$$\Gamma = \left(f(Q^2, \ldots, Q^n)Q^1 + w(Q^2, \ldots, Q^n)\right)\partial_{Q^1} + \sum_{s=2}^{n} \gamma_s(Q^2, \ldots, Q^n)\partial_{Q^s}\,.$$

- We can first solve, by inductive assumption,

$$\dot{Q}^s = \gamma_s(Q^2, \ldots, Q^n)\,, \quad s = 2, \ldots, n\,,$$

  and so reduce to

$$\dot{Q}^1 = f(t)Q^1 + w(t)$$

  which also can be solved by quadratures.

# Example

- To see our method in action, consider the Lie algebra of vector fields in $\mathbb{R}^2$ spanned by

$$X_1 = \partial_x, \quad X_2 = y\partial_x, \quad J = xy\partial_x + (1 + y^2)\partial_y.$$

The Lie algebra $L$ is solvable and $A = \langle \partial_x, y\partial_x \rangle$ is its only non trivial ideal.

- If we take $\Gamma = J$ as the dynamical vector field, we immediately see that the Lie's procedure cannot be applied, as $J$ is not an element of any commutative ideal in $L$.
- Also the mentioned Kozlov's result is not applicable, since the algebra is not triangular and the vector fields are not independent at every point.
- Take $A_1 = \langle \partial_x \rangle$. The equation for the coordinate $x$ in the fibre is

$$\dot{x} = xy\,.$$

# Example

- The differential equation corresponding to the projection $\bar{\Gamma}$ of the dynamical vector field in coordinate $y$ is

$$\dot{y} = 1 + y^2 \,.$$

- This can be immediately integrated to give $y(t) = y_0 + \tan t$.
- Substituting into the equation in the fibre, we get

$$\dot{x} = (y_0 + \tan t)x \,,$$

whose solution can be expressed as $x(t) = x_0 \exp(y_0 t)/\cos t$.

**THANK YOU FOR YOUR ATTENTION!**