Integrability conditions of the Euler-Lagrange Equation

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Zoltán Muzsnay

Calculus of variations:

- $E : TM \longrightarrow \mathbb{R}$,
- $I[\gamma] = \int_\gamma E(\gamma, \dot\gamma)$,
- $\dfrac{d}{dt}\dfrac{\partial E}{\partial \dot x^\alpha} - \dfrac{\partial E}{\partial x^\alpha} = 0$,
- $\dot x^\beta \dfrac{\partial^2 E}{\partial x^\beta \partial \dot x^\alpha} + \ddot x^\beta \dfrac{\partial^2 E}{\partial \dot x^\beta \partial \dot x^\alpha} - \dfrac{\partial E}{\partial x^\alpha} = 0 \qquad \Rightarrow \qquad \ddot x^\alpha = f^\alpha(x, \dot x)$,

$$\boxed{\text{Reg. Lagrangian} \quad \Rightarrow \quad \text{SODE}}$$

The inverse problem:

$$\boxed{\text{Reg. Lagrangian} \quad \stackrel{?}{\longleftarrow} \quad \text{SODE}}$$

$$\ddot{x}^i = f^i(x, \dot{x}) \qquad \Rightarrow \qquad \boxed{y^j \frac{\partial^2 E}{\partial x^j \partial y^i} + f^\beta \frac{\partial^2 E}{\partial y^j \partial y^i} - \frac{\partial E}{\partial x^i} = 0,}$$

$$\Updownarrow \quad g_{ij} = \tfrac{\partial^2 E}{\partial y^i \partial y^j}$$

$$\ddot{x}^i = f^i(x, \dot{x}) \qquad \Rightarrow \qquad \boxed{\begin{aligned} \frac{d}{dt} g_{ij} + \frac{1}{2}\frac{\partial f^k}{\partial y^j} g_{ik} + \frac{1}{2}\frac{\partial f^k}{\partial y^i} g_{jk} &= 0, \\ \frac{\partial g_{ij}}{\partial y^k} - \frac{\partial g_{ik}}{\partial y^j} &= 0, \\ \Phi^k_j g_{ik} - \Phi^k_i g_{jk} &= 0, \\ g_{ij} - g_{ji} &= 0, \\ \det(g_{ij}) &\neq 0, \end{aligned}}$$

Frölicher-Nijenhuis theory:

$\Lambda(M)$: the $C^\infty(M)$ modulus of the scalar forms.
$\Psi(M)$: the $C^\infty(M)$ modulus of the vector valued forms.

Definition:

(1) A morphism $D : \Lambda(M) \longrightarrow \Lambda(M)$ is a derivation of $\Lambda(M)$ of degree r if

$$a)\quad D\Lambda^p(M) \subset \Lambda^{p+r}(M),$$

$$b)\quad D(a\,\omega + b\,\omega') = a\,D\omega + b\,D\omega',$$

$$c)\quad D(\omega \wedge \pi) = D\omega \wedge \pi + (-1)^{r\,\mathrm{deg}\omega}\omega \wedge D\pi.$$

(2) The bracket of two derivations D_1 and D_2 is defined by

$$[D_1, D_2] = D_1 D_2 - (-1)^{(\mathrm{deg}D_1)(\mathrm{deg}D_2)} D_2 D_1$$

(3) A derivation is called of i_* type or algebraic, if its action is trivial on $\Lambda^o(M)$, and of d_* type, if it commutes with the operator d.

(4) An i_* and a d_* type derivation can be associated to $L \in \Psi^l(M)$:

- i_* type derivation: i_L

$$i_L\omega(X_1, ..., X_l) = \omega(L(X_1, ..., X_l)), \qquad \omega \in \Lambda^1(M)$$

- d_* type derivation: d_L

$$d_L f(X_1, \cdots, X_l) = df(L(X_1, \cdots, X_l)), \qquad f \in \Lambda^o(M).$$

(5) Let L and M be vector valued differential l- and m-forms. Then there exists a unique vector valued $(l+m)$-form (denoted by $[L, M]$) which satisfies the equation

$$[d_L, d_M] = d_{[L,M]}.$$

(6) The $\big(\Psi(M), [\ ,\ ]\big)$ is a graded Lie algebra.

Geometric tools:

Vertical endomorphism	J	$J = dx^i \otimes \dfrac{\partial}{\partial y^i}$
Liouville vector-field:	C	$C = y^i \dfrac{\partial}{\partial y^i},$
Spray:	S	$S = y^i \dfrac{\partial}{\partial x^i} + f^i(x,y) \dfrac{\partial}{\partial y^i},$
Path of the spray:	$S_{\dot\gamma} = \ddot\gamma$	$\dfrac{d^2x^i}{dt^2} = f^i\big(x, \dfrac{dx}{dt}\big).$
Connection:	$\Gamma = [J,S]$	$h = \frac{1}{2}(I+\Gamma), \quad v = \frac{1}{2}(I-\Gamma), \quad F = h[S,h] - J$
Curvature:	$R = -\frac{1}{2}[h,h],$	$R^i_{jk} = \dfrac{1}{2}\left(\dfrac{\partial \Gamma^i_k}{\partial x^j} - \dfrac{\partial \Gamma^i_j}{\partial x^k} + \Gamma^l_k \dfrac{\partial \Gamma^i_j}{\partial y^l} - \Gamma^l_j \dfrac{\partial \Gamma^i_k}{\partial y^l}\right),$
Jacobi endomorphism:	$\Phi = v[h,S]h,$	$\Phi^i_j = y^k \dfrac{\partial \Gamma^i_j}{\partial x^k} + f^k \dfrac{\partial \Gamma^i_j}{\partial y^k} + \dfrac{\partial f^i}{\partial x^j} + \Gamma^k_j \Gamma^i_k$

The Euler-Lagrange PDE system

$$\begin{aligned} \ddot{x}^i = &f^i(x, \dot{x}) && \Rightarrow && y^j \frac{\partial^2 E}{\partial x^j \partial y^i} + f^j \frac{\partial^2 E}{\partial y^j \partial y^i} - \frac{\partial E}{\partial x^i} = 0, \\ &S && \Rightarrow && \omega_E = i_S d d_J E + d\mathcal{L}_C E - dE = 0 \end{aligned}$$

$$\boxed{\omega_E = 0}$$

First compatibility condition: (with the notation $\Omega_E = dd_J E$)

$$\boxed{i_\Gamma \Omega_E = 0} \Leftrightarrow \boxed{i_F \Omega_E = 0} \Leftrightarrow \boxed{\Omega_E(hX, hY) = 0}$$

Second compatibility conditions:

$$\boxed{\begin{aligned} i_\Phi \Omega_E = 0 \\ i_R \Omega_E = 0 \end{aligned}} \Leftrightarrow \boxed{\begin{aligned} \Phi_i^k \frac{\partial^2 E}{\partial y^k \partial y^j} - \Phi_j^k \frac{\partial^2 E}{\partial y^k \partial y^i} &= 0 \\ R_{ij}^l \frac{\partial^2 E}{\partial y^l \partial y^k} + R_{jk}^l \frac{\partial^2 E}{\partial y^l \partial y^i} + R_{ki}^l \frac{\partial^2 E}{\partial y^l \partial y^j} &= 0 \end{aligned}} \Leftrightarrow \boxed{\begin{aligned} \Phi_i^k g_{kj} - \Phi_j^k g_{ki} = 0 \\ R_{ij}^l g_{lk} + R_{jk}^l g_{li} + R_{ki}^l g_{lj} = 0 \end{aligned}}$$

Definition: $L \in \Psi(TM)$

- The semi-basic derivation of L with respect to the spray S is

$$L' := h^* v[S, L].$$

- The semi-basic derivation of L with respect to h is

$$d^h L := [h, L].$$

Definition: The graded Lie algebra $\mathcal{A}_S$ associated to the spray S is the graded Lie sub-algebra of the vector-valued forms generated by

· the vertical endomorphism J,
· the Jacobi endomorphism $\Phi = v[h, S]$,
· the action of the semi-basic derivation with respect to S,
· the action of the derivation d^h,
· and the Frölicher-Nijenhuis bracket $[\ ,\ ]$.

Theorem: *Let S be a variational spray and E a Lagrangian associated to S.*

(1) For every element L of $\mathcal{A}_S$ we have

$$i_L \Omega_E = 0.$$

(2) The elements of $\mathcal{A}_S$ give algebraic conditions on the variational multiplier.

Proof of the Theorem:

- The vertical endomorphism:

$$i_J \Omega_E = i_J d d_J E = d_J^2 E = \frac{1}{2} d_{[J,J]} E = 0,$$

- The Jacobi endomorphism:

$$\begin{aligned} i_\Phi \Omega_E &= i_{[h,S]} \Omega_E + i_F \Omega_E = i_h \mathcal{L}_S \Omega_E - \mathcal{L}_S i_h \Omega_E \\ &= i_h d\omega_E - \mathcal{L}_S(\Omega_E + \frac{1}{2} d_J \omega_E) = d_h \omega_E - \frac{1}{2} \mathcal{L}_S d_J \omega_E = 0 \end{aligned}$$

- $L' = h^* v[S, L] = [S, L] + FL - L \barwedge F$,

  If the equation $i_L \Omega_E = 0$ holds, then

$$i_{L'} \Omega = i_{[S,L]} \Omega + i_{FL} \Omega - i_{F \barwedge L} \Omega = i_{[S,L]} \Omega + i_F i_L \Omega - i_L i_F \Omega$$
$$= \mathcal{L}_S i_L \Omega - d_L \omega + i_F i_L \Omega - i_L i_F \Omega = 0$$

- $i_{[h,L]} = (-1)^l (i_h d_L - d_L i_h - d_{L \barwedge h})$

  If the equation $i_L \Omega_E = 0$ holds, then

$$i_{d^h L} \Omega_E = (-1)^l (i_h d i_L d d_J E - d_L i_h d d_J E - l \, d i_L d d_J E) = 0.$$

- Let $K \in \mathcal{A}^k_S(TM)$, $L \in \mathcal{A}^l_S(TM)$ be such that $i_K \Omega_E = 0$ and $i_L \Omega_E = 0$.

$$(-1)^l \, i_{[K,L]} \Omega_E = \left(i_K d_L - (-1)^{l(m-1)} d_L i_K - d_{L \barwedge K} \right) \Omega_E$$
$$= i_K (i_L d - d i_L) \, d d_J E - (-1)^{l(m-1)} d_L i_K d d_J E - d_{L \barwedge K} d d_J E$$
$$= i_K d i_L \Omega_E - (-1)^{l(m-1)} d_L i_K \Omega_E = 0.$$

If $L \in \mathcal{A}^l \subset \Psi^l(TM)$, then

$$L = L^j_{i_1 \dots i_l} dx^{i_1} \wedge \dots \wedge dx^{i_l} \otimes \frac{\partial}{\partial y^j},$$

$$\Omega_E = \frac{1}{2}\left(\frac{\partial^2 E}{\partial x^i \partial y^j} - \frac{\partial^2 E}{\partial y^i \partial x^j}\right) dx^i \wedge dx^j - \frac{\partial^2 E}{\partial y^i \partial y^j} dx^i \wedge dy^j$$

$$i_L \Omega_E = \frac{1}{l!} \sum_{i \in \mathfrak{S}_{l+1}} \operatorname{sign}(i) L^j_{i_1 \dots i_l} \frac{\partial^2 E}{\partial y^j \partial y^{i_{l+1}}} dx^{i_1} \wedge \dots \wedge dx^{i_{l+1}},$$

$$\boxed{\sum_{i \in \mathfrak{S}_{l+1}} \operatorname{sign}(i)\, L^j_{i_1 \dots i_l}\, g_{j i_{l+1}} = 0}$$

The graduation of $\mathcal{A}_S$ is given by $\mathcal{A}_S = \oplus_{k=1}^{n} \mathcal{A}_S^k$ where $\mathcal{A}_S^k = \mathcal{A}_S \cap \Psi^k(TM)$.

$$\mathcal{A}_S^1 = \{J,\ \Phi,\ \Phi',\ \Phi'',\ \Phi''',\ \ldots\}$$

$$\mathcal{A}_S^2 = \left\{ \begin{array}{l} [J,\Phi] = 3R,\ R',\ R'',\ R''',\ \ldots \\ [\Phi,\Phi],\ [\Phi,\Phi]',\ [\Phi,\Phi]'',\ \ldots \\ [h,\Phi],\ [h,\Phi]',\ [h,\Phi]'',\ \ldots \end{array} \right\}$$

$$\mathcal{A}_S^3 = \left\{ \begin{array}{l} [J,R],\ [J,R]',\ [J,R]'',\ [J,R]''',\ \ldots \\ [\Phi,R],\ [\Phi,R]',\ [\Phi,R]'',\ [\Phi,R]''',\ \ldots \\ \big[[\Phi,\Phi],h\big],\ \big[[\Phi,\Phi],h\big]',\ \big[[\Phi,\Phi],h\big]''',\ \ldots \\ \big[[\Phi,\Phi],\Phi\big],\ \big[[\Phi,\Phi],\Phi\big]',\ \big[[\Phi,\Phi],\Phi\big]''',\ \ldots \end{array} \right\} \qquad \ldots$$

Theorem 2: If

$$\dim \left\{ J,\ \Phi,\ \Phi',\ \Phi'',\ \Phi''', ... \right\} > \frac{n(n+1)}{2},$$

then S is not variational ([D], [AT], [S], [SCM]).

$\Leftrightarrow$

If

$$\dim \mathcal{A}_S^1 \ > \ \frac{n(n+1)}{2},$$

then S is not variational ([D], [AT], [S], [SCM]).

Theorem 3: [M] If there exists an integer $k \leq n$ for which

$$\dim \mathcal{A}_S^k \ > \ k \binom{n+1}{k+1},$$

then the spray is not variational.

Theorem 4: [M] Let us consider the system of linear equations

$$\mathcal{S} = \left\{ \sum_{i \in \mathfrak{S}_{l+1}} \operatorname{sign}(i)\, L^{j}_{i_1 \dots i_l}\, x_{j i_{l+1}} = 0 \;\middle|\; L \in \mathcal{A}_S \right\}$$

in the symmetric variables x_{ij} $(x_{ij} = x_{ji})$. If

$$\operatorname{rank} \mathcal{S} \;\geq\; \frac{n(n+1)}{2}$$

then S is non-variational.

Example 1 :

- $\{\ddot{x}_1 = 0, \quad \ddot{x}_2 = 0\}$
- $\Gamma^i_j = 0, \quad \Phi = 0,$
- $\mathcal{A}^1 = \{J\}, \quad \mathcal{A}^2 = 0, \quad \mathcal{A}^3 = 0, \, ...$

Example 2 :

- $\left\{\ddot{x}_1 = -\frac{1+\dot{x}_1^2+\dot{x}_2^2}{x_1}, \ \ddot{x}_1 = \dot{x}_2^2\right\}$
- $\Gamma^1_1 = \frac{y_1}{x_1}, \, \Gamma^1_2 = \frac{y_2}{x_1}, \, \Gamma^2_1 = 0, \, \Gamma^2_2 = -y_2, \quad \Phi = 0,$
- $\mathcal{A}^1 = \{J\}, \quad \mathcal{A}^2 = 0, \quad \mathcal{A}^3 = 0, \, ...$

Example 3 :

- $\{\ddot{x}_1 = f^1(x_1, \dot{x}_1), \ \ddot{x}_2 = f^2(x_2, \dot{x}_2)\}$
- $\Gamma^2_1 = \Gamma^1_2 = 0, \, \Phi^2_1 = \Phi^1_2 = {\Phi'}^2_1 = {\Phi'}^1_2 = 0$
- $\mathcal{A}^1 = \{J, \Phi\},$
- $\dim \mathcal{A}^1 = 2$

Example 4: [D]

- $\{\ddot{x}_1 = (x_1)^2 + (x_2)^2,\ \ddot{x}_2 = x_1\}$ where $\partial^3 f^1_{\dot{x}_2\dot{x}_2\dot{x}_2} \neq 0.$
- $\{J,\ \Phi,\ \Phi',\ \Phi'',\ \Phi'''\}$ are independent
- $\dim \mathcal{A}^1 \geq 4$
- No corresponding variation problem exists.

Example 5 :

- Lie group: G
- Tangent space: $TG \simeq G \times \mathfrak{g}$
- Coordinate system (x, α), where α is the Maurer-Cartan form
- SODE: associated to the canonical connection: $\ddot{x} = \dot{x}x^{-1}\dot{x}$
- $\Phi_{(x,\alpha)}(a^h) = \frac{1}{4}\big[[a, \alpha], \alpha\big]$, $\Phi' = \Phi'' = ... = 0$, $\mathcal{A}^1 = \{J, \Phi\}$,
- $R_{(x,\alpha)}(a^h, b^h) = \frac{1}{4}\big[[a, b], \alpha\big]$, $R' = R'' = ... = 0$,

$$[\Phi, \Phi]_{(x,\alpha)}(a^h, b^h) = \frac{1}{16}\Big[\Big([[[b, \alpha], \alpha], a] - [[[a, \alpha], \alpha], b] - [[a, \alpha], [b, \alpha]]\Big), \alpha\Big]$$

- $G = ASO(n)$: $[\Phi, \Phi] = \frac{1}{4}R$
- $G = GL(2, \mathbb{R})$: $[\Phi, \Phi]$ and R are independent

Variational principle for the canonical SODE:

General case : partial results, there is no a complete theory (G. Thompson: [GMT], [MT])

Invariant case : there is a complete answer, because the Euler-Lagrange system is integrable

$$\omega_E(X_a) = [a, \alpha]\frac{\partial E}{\partial \alpha} + x\alpha a\frac{\partial^2 E}{\partial x \partial \alpha} - xa\frac{\partial E}{\partial x}$$

Proposition: [M] The canonical SODE is variational with respect a left-invariant Lagrangian if and only if there exists a regular function $\mathcal{E} : \mathfrak{g} \longrightarrow \mathbb{R}$ such that

$$[a, \alpha]\,\frac{\partial \mathcal{E}}{\partial \alpha} = 0, \qquad \forall a \in \mathfrak{g}.$$

Theorem: [M] There exists a left-invariant variational principle for the canonical flow of the Lie group G in a neighborhood of a generic element $\alpha \in \mathfrak{g}$ if and only if the linear system

$$C_{ij}^k \alpha_j x_k = 0, \qquad i = 1, ..., n,$$

$$C_{ij}^k x_k + C_{jm}^k \alpha_m x_{ik} = 0, \qquad i, j = 1, ..., n,$$

has a solution $\{x_i = \epsilon_i, x_{ij} = \epsilon_{ij}\}$ satisfying $\det(\epsilon_{ij}) \neq 0$.

Corollary 1: The canonical connection of a commutative Lie group is variational with respect to a left-invariant Lagrangian.

Corollary 2: If the derived Lie algebra is one dimensional, then there is no left-invariant variational principle for the canonical flow.

2-dimensional Lie groups

- $[\, ,\,] = 0$: Corollary 1: there is an invariant variational principle
- $[\, ,\,] \neq 0$: Corollary 2: there is no invariant variational principle

3-dimensional Lie groups

- $\dim(g^{(1)}) = 0$: there is an invariant variational principle
- $\dim(g^{(1)}) = 1$: (the Heisenberg algebra and the upper triangular matrices): there is no invariant variational principle
- $\dim(g^{(1)}) = 3$: g is simple, the Killing form provides a metric.

4-dimensional Lie groups:

- Non-variational Lie algebra: $A_{4,7}$, $A_{4,9b}$, $A_{4,11a}$
- Variational Lie algebra:
  - There is an invariant variational principle: $A_{4,8}$, $A_{4,10}$;
  - There is no invariant variational principle: $A_{4,1}$, $A_{4,3}$, $A_{4,4}$, $A_{4,9}$, $A_{4,12}$, $A_{4,2a}$, $A_{4,5ab}$, $A_{4,6ab}$.

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