Integrability conditions of the Euler-Lagrange Equation


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Calculus of variations:

- \( E : TM \rightarrow \mathbb{R} \),
- \( I[\gamma] = \int_\gamma E(\gamma, \dot{\gamma}) \),
- \( \frac{d}{dt} \frac{\partial E}{\partial \dot{x}^\alpha} - \frac{\partial E}{\partial x^\alpha} = 0, \)
- \( \dot{x}^\beta \frac{\partial^2 E}{\partial x^\beta \partial \dot{x}^\alpha} + \ddot{x}^\beta \frac{\partial^2 E}{\partial \dot{x}^\beta \partial \dot{x}^\alpha} - \frac{\partial E}{\partial x^\alpha} = 0 \quad \Rightarrow \quad \ddot{x}^\alpha = f^\alpha(x, \dot{x}), \)

The inverse problem:

- Reg. Lagrangian \( \Rightarrow \) SODE

- Reg. Lagrangian \( \Leftarrow \) SODE
\[ \ddot{x}^i = f^i(x, \dot{x}) \quad \Rightarrow \quad y^j \frac{\partial^2 E}{\partial x^j \partial y^i} + f^\beta \frac{\partial^2 E}{\partial y^j \partial y^\beta} - \frac{\partial E}{\partial x^i} = 0, \]

\[ \Uparrow \quad g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j} \]

\[ \frac{\partial g_{ij}}{\partial y^k} - \frac{\partial g_{ik}}{\partial y^j} = 0, \]

\[ \Phi^k_j g_{ik} - \Phi^k_i g_{jk} = 0, \]

\[ g_{ij} - g_{ji} = 0, \]

\[ \det(g_{ij}) \neq 0, \]
Frölicher-Nijenhuis theory:

\( \Lambda(M) \): the \( C^\infty(M) \) modulus of the scalar forms.
\( \Psi(M) \): the \( C^\infty(M) \) modulus of the vector valued forms.

**Definition:**

1. A morphism \( D : \Lambda(M) \rightarrow \Lambda(M) \) is a derivation of \( \Lambda(M) \) of degree \( r \) if
   
   \[ D \Lambda^p(M) \subset \Lambda^{p+r}(M), \]
   \[ D(a \omega + b \omega') = a D\omega + b D\omega', \]
   \[ D(\omega \wedge \pi) = D\omega \wedge \pi + (-1)^r \deg \omega \omega \wedge D\pi. \]

2. The bracket of two derivations \( D_1 \) and \( D_2 \) is defined by
   \[ [D_1, D_2] = D_1 D_2 - (-1)^{(\deg D_1)(\deg D_2)} D_2 D_1 \]

3. A derivation is called of \( i_* \) type or algebraic, if its action is trivial on \( \Lambda^0(M) \), and of \( d_* \) type, if it commutes with the operator \( d \).
(4) An $i_\ast$ and a $d_\ast$ type derivation can be associated to $L \in \Psi^l(M)$:

- $i_\ast$ type derivation: $i_L$
  \[ i_L \omega(X_1, \ldots, X_l) = \omega(L(X_1, \ldots, X_l)), \quad \omega \in \Lambda^1(M) \]

- $d_\ast$ type derivation: $d_L$
  \[ d_L f(X_1, \cdots, X_l) = df(L(X_1, \cdots, X_l)), \quad f \in \Lambda^o(M). \]

(5) Let $L$ and $M$ be vector valued differential $l$- and $m$-forms. Then there exists a unique vector valued $(l + m)$-form (denoted by $[L, M]$) which satisfies the equation

\[ [d_L, d_M] = d_{[L,M]} . \]

(6) The $(\Psi(M), [ , ]) \text{ is a graded Lie algebra.}$
Geometric tools:

Vertical endomorphism $J$

$$J = dx^i \otimes \frac{\partial}{\partial y^i}$$

Liouville vector-field: $C$

$$C = y^i \frac{\partial}{\partial y^i},$$

Spray: $S$

$$S = y^i \frac{\partial}{\partial x^i} + f^i(x, y) \frac{\partial}{\partial y^i},$$

Path of the spray: $S_{\dot{\gamma}} = \ddot{\gamma}$

$$\frac{d^2 x^i}{dt^2} = f^i(x, \frac{dx}{dt}).$$

Connection: $\Gamma = [J, S]$

$$h = \frac{1}{2}(I + \Gamma), \quad v = \frac{1}{2}(I - \Gamma), \quad F = h[S, h] - J$$

Curvature: $R = -\frac{1}{2}[h, h]$,

$$R^i_{jk} = \frac{1}{2} \left( \frac{\partial \Gamma^i_k}{\partial x^j} - \frac{\partial \Gamma^i_j}{\partial x^k} + \Gamma^l_k \frac{\partial \Gamma^i_j}{\partial y^l} - \Gamma^l_j \frac{\partial \Gamma^i_k}{\partial y^l} \right),$$

Jacobi endomorphism: $\Phi = v[h, S]h$, $\Phi^i_j = y^k \frac{\partial \Gamma^i_j}{\partial x^k} + f^k \frac{\partial \Gamma^i_j}{\partial y^k} + \frac{\partial f^i}{\partial x^j} + \Gamma^k_j \Gamma^i_k$
The Euler-Lagrange PDE system

\[ \ddot{x}^i = f^i(x, \dot{x}) \Rightarrow y^j \frac{\partial^2 E}{\partial x^i \partial y^j} + f^j \frac{\partial^2 E}{\partial y^j \partial y^i} - \frac{\partial E}{\partial x^i} = 0, \]

\[ S \Rightarrow \omega_E = i_S dd_j E + d\mathcal{L}_CE - dE = 0 \]

\[ \omega_E = 0 \]

First compatibility condition: (with the notation \( \Omega_E = dd_j E \))

\[ i_\Gamma \Omega_E = 0 \iff i_F \Omega_E = 0 \iff \Omega_E (hX, hY) = 0 \]

Second compatibility conditions:

\[ i_\Phi \Omega_E = 0 \]
\[ i_R \Omega_E = 0 \]

\[ \Phi_i \frac{\partial^2 E}{\partial y^k \partial y^j} - \Phi_j \frac{\partial^2 E}{\partial y^j \partial y^i} = 0 \]
\[ R_{ij}^l \frac{\partial^2 E}{\partial y^l \partial y^k} + R_{jk}^l \frac{\partial^2 E}{\partial y^l \partial y^i} + R_{ki}^l \frac{\partial^2 E}{\partial y^l \partial y^j} = 0 \]

\[ \Phi_{ik}^k g_{kj} - \Phi_{jk}^k g_{ki} = 0 \]
\[ R_{ij}^l g_{lk} + R_{jk}^l g_{li} + R_{ki}^l g_{lj} = 0 \]
Definition: $L \in \Psi(TM)$

- The semi-basic derivation of $L$ with respect to the spray $S$ is

- The semi-basic derivation of $L$ with respect to $h$ is
  $d^h L := [h, L]$.

Definition: The graded Lie algebra $\mathcal{A}_S$ associated to the spray $S$ is the graded Lie sub-algebra of the vector-valued forms generated by
  - the vertical endomorphism $J$,
  - the Jacobi endomorphism $\Phi = v[h, S]$,
  - the action of the semi-basic derivation with respect to $S$,
  - the action of the derivation $d^h$,
  - and the Frölicher-Nijenhuis bracket $[\ , \ ]$. 
**Theorem:** Let $S$ be a variational spray and $E$ a Lagrangian associated to $S$.

1. For every element $L$ of $A_S$ we have
   \[ i_L \Omega_E = 0. \]
2. The elements of $A_S$ give algebraic conditions on the variational multiplier.

**Proof of the Theorem:**

- The vertical endomorphism:
  \[ i_J \Omega_E = i_J dd_J E = d^2_J E = \frac{1}{2} d_{[J,J]} E = 0, \]

- The Jacobi endomorphism:
  \[ i_\Phi \Omega_E = i_{[h,S]} \Omega_E + i_F \Omega_E = i_h \mathcal{L}_S \Omega_E - \mathcal{L}_S i_h \Omega_E \]
  \[ = i_h d\omega_E - \mathcal{L}_S (\Omega_E + \frac{1}{2} d_J \omega_E) = d_h \omega_E - \frac{1}{2} \mathcal{L}_S d_J \omega_E = 0 \]
\[ L' = h*v[S, L] = [S, L] + FL - L\wedge F, \]

If the equation \( i_L\Omega_E = 0 \) holds, then

\[
i_L\Omega = i_{[S,L]}\Omega + i_{FL}\Omega - i_{F\wedge L}\Omega = i_{[S,L]}\Omega + i_Fi_L\Omega - i_Li_F\Omega \\
= L_Si_L\Omega - d_L\omega + i_Fi_L\Omega - i_Li_F\Omega = 0
\]

\[ i_{[h,L]} = (-1)^l(i_hd_L - d_Li_h - d_L\wedge h) \]

If the equation \( i_L\Omega_E = 0 \) holds, then

\[
i_{d^hL}\Omega_E = (-1)^l(i_hd_{id}\Omega - d_Li_hd\Omega_E - l^d_Lid\Omega_E) = 0.
\]

\[ \text{Let } K \in A^k_S(TM), \ L \in A^l_S(TM) \text{ be such that } i_K\Omega_E = 0 \text{ and } i_L\Omega_E = 0. \]

\[
(-1)^l \ i_{[K,L]}\Omega_E = (i_Kd_L - (-1)^l(m-1)d_Li_K - d_L\wedge K) \Omega_E \\
= i_K(i_Ld - d_L) dd_J E - (-1)^l(m-1)d_Li_Kdd_J E - d_L\wedge Kdd_J E \\
= i_Kdi_L\Omega_E - (-1)^l(m-1)d_Li_K\Omega_E = 0.
\]
If $L \in \mathcal{A}^l \subset \Psi^l(TM)$, then

$$L = L_{i_1...i_l}^j \, dx^{i_1} \wedge ... \wedge dx^{i_l} \otimes \frac{\partial}{\partial y^j},$$

$$\Omega_E = \frac{1}{2} \left( \frac{\partial^2 E}{\partial x^i \partial y^j} - \frac{\partial^2 E}{\partial y^i \partial x^j} \right) \, dx^i \wedge dx^j - \frac{\partial^2 E}{\partial y^i \partial y^j} \, dx^i \wedge dy^j$$

$$i_L \Omega_E = \frac{1}{l!} \sum_{i \in \mathcal{S}_{l+1}} \text{sign}(i) L_{i_1...i_l}^j \frac{\partial^2 E}{\partial y^j \partial y^{i_{l+1}}} \, dx^{i_1} \wedge ... \wedge dx^{i_{l+1}},$$

$$\sum_{i \in \mathcal{S}_{l+1}} \text{sign}(i) \, L_{i_1...i_l}^j \, g_{ji_{l+1}} = 0$$
The graduation of $\mathcal{A}_S$ is given by $\mathcal{A}_S = \bigoplus_{k=1}^{n} \mathcal{A}_S^k$ where $\mathcal{A}_S^k = \mathcal{A}_S \cap \Psi^k(TM)$.

$\mathcal{A}_S^1 = \{ J, \Phi, \Phi', \Phi'', \Phi''', ... \}$

$\mathcal{A}_S^2 = \left\{ \begin{array}{l} [J, \Phi] = 3R, R', R'', R''', ... \\ [\Phi, \Phi], [\Phi, \Phi]', [\Phi, \Phi]'', ... \\ [h, \Phi], [h, \Phi]', [h, \Phi]''', ... \end{array} \right\}$

$\mathcal{A}_S^3 = \left\{ \begin{array}{l} [J, R], [J, R]', [J, R]'', [J, R]''', ... \\ [\Phi, R], [\Phi, R]', [\Phi, R]'', [\Phi, R]''', ... \\ [[\Phi, \Phi], h], [[\Phi, \Phi], h]', [[\Phi, \Phi], h]''', ... \\ [[[\Phi, \Phi], \Phi], [[\Phi, \Phi], \Phi]'], [[[\Phi, \Phi], \Phi], \Phi]'', [[[\Phi, \Phi], \Phi], \Phi]''', ... \end{array} \right\} ...$
Theorem 2: If
\[ \dim \{ J, \Phi, \Phi', \Phi'', \Phi''', \ldots \} > \frac{n(n+1)}{2}, \]
then \( S \) is not variational ([D], [AT], [S], [SCM]).

\[ \Leftrightarrow \]
If
\[ \dim \mathcal{A}_S^1 > \frac{n(n+1)}{2}, \]
then \( S \) is not variational ([D], [AT], [S], [SCM]).

Theorem 3: [M] If there exists an integer \( k \leq n \) for which
\[ \dim \mathcal{A}_S^k > k \left( \frac{n+1}{k+1} \right), \]
then the spray is not variational.
Theorem 4: [M] Let us consider the system of linear equations

\[ S = \left\{ \sum_{i \in S_{l+1}} \text{sign}(i) L_{i_1...i_l}^j x_{j_{i+1}} = 0 \mid L \in A_S \right\} \]

in the symmetric variables \( x_{ij} \) (\( x_{ij} = x_{ji} \)). If

\[ \text{rank } S \geq \frac{n(n + 1)}{2} \]

then \( S \) is non-variational.
Example 1:

- \{\ddot{x}_1 = 0, \, \ddot{x}_2 = 0\}
- \Gamma^i_j = 0, \, \Phi = 0,
- \mathcal{A}^1 = \{J\}, \, \mathcal{A}^2 = 0, \, \mathcal{A}^3 = 0, ...

Example 2:

- \left\{\ddot{x}_1 = -\frac{1+\dot{x}_1^2+\dot{x}_2^2}{x_1}, \, \ddot{x}_1 = \dot{x}_2^2\right\}
- \Gamma^1_1 = \frac{y_1}{x_1}, \, \Gamma^1_2 = \frac{y_2}{x_1}, \, \Gamma^2_1 = 0, \, \Gamma^2_2 = -y_2, \, \Phi = 0,
- \mathcal{A}^1 = \{J\}, \, \mathcal{A}^2 = 0, \, \mathcal{A}^3 = 0, ...

Example 3:

- \{\ddot{x}_1 = f^1(x_1, \dot{x}_1), \, \ddot{x}_2 = f^2(x_2, \dot{x}_2)\}
- \Gamma^2_1 = \Gamma^2_2 = 0, \, \Phi^2_1 = \Phi^2_2 = \Phi'_1 = \Phi'_2 = 0
- \mathcal{A}^1 = \{J, \Phi\},
- \dim \mathcal{A}^1 = 2
Example 4: [D]

- \( \{ \ddot{x}_1 = (x_1)^2 + (x_2)^2, \ \ddot{x}_2 = x_1 \} \) where \( \partial^3 f^1_{\dot{x}_2 \dot{x}_2 \dot{x}_2} \neq 0 \).
- \( \{ J, \Phi, \Phi', \Phi'', \Phi''' \} \) are independent
- \( \dim \mathcal{A}^1 \geq 4 \)
- No corresponding variation problem exists.
Example 5:

- Lie group: $G$
- Tangent space: $TG \cong G \times \mathfrak{g}$
- Coordinate system $(x, \alpha)$, where $\alpha$ is the Maurer-Cartan form
- SODE: associated to the canonical connection: $\ddot{x} = \dot{x} x^{-1} \dot{x}$
- $\Phi_{(x,\alpha)}(a^h) = \frac{1}{4}[[a, \alpha], \alpha]$, $\Phi' = \Phi'' = \ldots = 0$, $\mathcal{A}^1 = \{ J, \Phi \}$, $R'$
- $R_{(x,\alpha)}(a^h, b^h) = \frac{1}{4}[[a, b], \alpha]$, $R' = R'' = \ldots = 0$,
- $[\Phi, \Phi]_{(x,\alpha)}(a^h, b^h) = \frac{1}{16} \left( [[[b, \alpha], \alpha], a] - [[[a, \alpha], \alpha], b] - [[a, \alpha], [b, \alpha]] \right), \alpha$

- $G = ASO(n)$: $[\Phi, \Phi] = \frac{1}{4} R$
- $G = GL(2, \mathbb{R})$: $[\Phi, \Phi]$ and $R$ are independent
Variational principle for the canonical SODE:

**General case**: partial results, there is no a complete theory (G. Thompson: [GMT], [MT])

**Invariant case**: there is a complete answer, because the Euler-Lagrange system is integrable

\[ \omega_E(X_a) = [a, \alpha] \frac{\partial E}{\partial \alpha} + x \alpha a \frac{\partial^2 E}{\partial x \partial \alpha} - xa \frac{\partial E}{\partial x} \]

**Proposition**: [M] The canonical SODE is variational with respect a left-invariant Lagrangian if and only if there exists a regular function \( E : g \rightarrow \mathbb{R} \) such that

\[ [a, \alpha] \frac{\partial E}{\partial \alpha} = 0, \quad \forall a \in g. \]

**Theorem**: [M] There exists a left-invariant variational principle for the canonical flow of the Lie group \( G \) in a neighborhood of a generic element \( \alpha \in g \) if and only if the linear system

\[ C_{ij}^k \alpha_j x_k = 0, \quad i = 1, \ldots, n, \]
\[ C_{ij}^k x_k + C_{jm}^k \alpha_m x_{ik} = 0, \quad i, j = 1, \ldots, n, \]

has a solution \( \{x_i = \epsilon_i, x_{ij} = \epsilon_{ij}\} \) satisfying \( \det(\epsilon_{ij}) \neq 0 \).
Corollary 1: The canonical connection of a commutative Lie group is variational with respect to a left-invariant Lagrangian.

Corollary 2: If the derived Lie algebra is one dimensional, then there is no left-invariant variational principle for the canonical flow.

2-dimensional Lie groups

- $[ , ] = 0$: Corollary 1: there is an invariant variational principle
- $[ , ] \neq 0$: Corollary 2: there is no invariant variational principle

3-dimensional Lie groups

- $\dim(g^{(1)}) = 0$: there is an invariant variational principle
- $\dim(g^{(1)}) = 1$: (the Heisenberg algebra and the upper triangular matrices): there is no invariant variational principle
- $\dim(g^{(1)}) = 3$: $g$ is simple, the Killing form provides a metric.

4-dimensional Lie groups:

- Non-variational Lie algebra: $A_{4,7}, A_{4,9b}, A_{4,11a}$
- Variational Lie algebra:
  - There is an invariant variational principle: $A_{4,8}, A_{4,10}$;
  - There is no invariant variational principle: $A_{4,1}, A_{4,3}, A_{4,4}, A_{4,9}, A_{4,12}, A_{4,2a}, A_{4,5a}, A_{4,6a}$.


