Geometry of the Legendre transformation

Paweł Urbański

Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski
Hoża 74, 00-682 Warszawa
email: urbanski@fuw.edu.pl

1. Introduction.

Lagrangian systems and Hamiltonian systems offer different representations of the same object - the dynamics of a mechanical system. The Legendre transformation is the passage from one of these representations to the other. To get rough idea what is the story about, let us consider the case of a hyperregular lagrangian. Lagrangian is a function

$$L\colon \mathsf{T}M \longrightarrow \mathbb{R},$$

with the vertical derivative

$$\mathrm{d}_{\mathsf{v}}L\colon \mathsf{T}M \longrightarrow \mathsf{T}^*M.$$

Lagrangian L is called *hyperregular* if $\mathrm{d}_{\mathsf{v}}L$ is a diffeomorphism. If it is the case, then we can define Hamiltonian by

$$H\colon \mathsf{T}^*M \longrightarrow \mathbb{R}\colon p \mapsto \langle p, \mathrm{d}_{\mathsf{v}}^{-1}L(p)\rangle - L \circ \mathrm{d}_{\mathsf{v}}^{-1}L(p).$$

We use this Hamiltonian to generate a vector field X_H on T^*M:

$$-\langle \mathrm{d}H, Y\rangle = \omega(X_H, Y).$$

This vector field can be obtained directly from the Lagrangian by the canonical diffeomorphism given by the equivalence of functors TT^* and $\mathsf{T}^*\mathsf{T}$

$$\alpha_M\colon \mathsf{TT}^*M \longrightarrow \mathsf{T}^*\mathsf{T}M.$$

We have for $p = \mathrm{d}_{\mathsf{v}}L(v)$

$$\alpha_M(X_H(p)) = \mathrm{d}L(v).$$

We can summarize this example with a picture

$$\begin{array}{ccccccc}
\mathsf{T}^*\mathsf{T}^*M & \xleftarrow{\beta_M} & & \mathsf{TT}^*M & & \xrightarrow{\alpha_M} & \mathsf{T}^*\mathsf{T}M \\
& \searrow^{\pi_{\mathsf{T}^*M}} & \swarrow_{\tau_{\mathsf{T}^*M}} & & \searrow^{\mathsf{T}\pi_M} & \swarrow_{\pi_{\mathsf{T}M}} & \\
& \mathsf{T}^*M & & & & \mathsf{T}M & \\
& & \searrow^{\pi_M} & & \swarrow_{\tau_M} & & \\
& & & M & & &
\end{array} \tag{1}$$

The dynamics is in the center, Langrangian system is on the right, Hamiltonian system on the left. This diagram is known as Tulczyjew triple.

It follows from this picture that each of the top objects is a symplectic manifold and also has two (compatible) vector bundle structures, i.e. they are double vector bundles. Thus the following structures are involved in the Legendre transformation:

(1) Lagrangian submanifolds and generating objects,
(2) Iterated tangent functors and their equivalence,
(3) Double vector bundles.

2. Generating objects of Lagrangian submanifolds.

In Physics, we have to accept more general dynamics then vector fields, more general lagrangian submanifolds in T^*N then sections of π_N and more general generating objects (Lagrangian, Hamiltonian systems) then functions on N.

2.1. Morse and regular families.

Let

$$\begin{array}{c} R \\ \rho \Big\downarrow \\ X \end{array} \tag{2}$$

be a differential fibration. We denote by $\mathsf{V}R$ the *bundle of vertical vectors* defined by

$$\mathsf{V}R = \{v \in \mathsf{T}R;\ \mathsf{T}\rho(v) = 0\} \tag{3}$$

We denote by $\mathsf{V}^\circ R$ the *polar* of the vertical bundle defined by

$$\mathsf{V}^\circ R = \big\{s \in \mathsf{T}^*R;\ \forall_{v\in\mathsf{V}R}\tau_R(v) = \pi_R(s) \Rightarrow \langle s, v\rangle = 0\big\} \tag{4}$$

A function $F\colon R \to \mathbb{R}$ can be considered a *family of functions* defined on fibres of the fibration ρ. We will represent a family of functions by a diagram

$$\begin{array}{ccc} R & \xrightarrow{F} & \mathbb{R} \\ \rho \Big\downarrow & & \\ X & & \end{array} \tag{5}$$

The *critical set* for a family of functions $F\colon R \to \mathbb{R}$ is the set

$$S(F,\rho) = \Big\{r \in R;\ \forall_{v\in\mathsf{V}_rR}\langle \mathrm{d}F, v\rangle = 0\Big\}. \tag{6}$$

At each point $r \in S(F,\rho)$ we define a bilinear mapping

$$\begin{aligned} W(F,r)\colon & \mathsf{V}_rR \times \mathsf{T}_rR \to \mathbb{R} \\ & \colon (v,w) \mapsto \mathrm{D}^{(1,1)}(F\circ\chi)(0,0), \end{aligned} \tag{7}$$

where χ is a mapping from $\mathbb{R}^2$ to R such that $v = \mathsf{t}\chi(\cdot,0)(0)$ and $w = \mathsf{t}\chi(0,\cdot)(0)$. The family F is said to be *regular* if $S(F,\rho)$ is a submanifold and the rank of $W(F,r)$ at each $r \in S(F,\rho)$ is equal to the codimension of $S(F,\rho)$. The family F is called a *Morse family* if the rank of $W(F,r)$ is maximal at each $r \in S(F,\rho)$. Morse family is regular.

It is known that if F is a regular family, then the set

$$\begin{aligned} N = \Big\{p \in \mathsf{T}^*X;\ & \exists_{r\in R}\, \rho(r) = q = \pi_X(p) \\ & \forall_{v\in\mathsf{T}_qX}\forall_{w\in\mathsf{T}_rR}\ \mathsf{T}\rho(w) = v \Rightarrow \langle p, v\rangle = \langle \mathrm{d}F, w\rangle\Big\} \end{aligned} \tag{8}$$

is an immersed Lagrangian submanifold of the symplectic space $(\mathsf{T}^*X, \vartheta_X)$. This Lagrangian submanifold is said to be *generated* by the family (5).

For each regular family F we have the mapping $\kappa\colon S(F,\rho) \to \mathsf{T}^*X$ characterized by

$$\langle \kappa(r), v\rangle = \langle \mathrm{d}F, w\rangle \tag{9}$$

for each $v \in \mathsf{T}X$ such that $\tau_X(v) = \rho(r)$ and each $w \in \mathsf{T}R$ such that $\mathsf{T}\rho(w) = v$. The mapping κ is a subimmersion. If F is a Morse family, then κ is an immersion. The immersed Lagrangian submanifold (8) is the image $\mathrm{im}(\kappa)$ of the mapping κ.

There is a surjective submersion $\lambda : \mathsf{V}^\circ R \to \mathsf{T}^*X$ characterized by

$$\langle \lambda(q), v\rangle = \langle q, w\rangle \tag{10}$$

for each $v \in \mathsf{T}X$ such that $\tau_X(v) = \rho(\pi_R(q))$ and each $w \in \mathsf{T}R$ such that $\mathsf{T}\rho(w) = v$. The intersection $\mathrm{im}(\mathrm{d}F) \cap \mathsf{V}^\circ R$ is clean if and only if F is a regular family. This intersection is transverse if and only if F is a Morse family. The immersed Lagrangian submanifold (8) is the set $\lambda(\mathrm{im}(\mathrm{d}F) \cap \mathsf{V}^\circ R)$.

If the fibration ρ is the identity morphism $1_X\colon X \to X$, then the family of functions is a function

$$F\colon X \to \mathbb{R} \tag{11}$$

and the Lagrangian submanifold N is the set

$$\begin{aligned} N &= \left\{ p \in \mathsf{T}^*X;\ \pi_X(p) = q,\ \forall_{v \in \mathsf{T}_qX} \langle p, v\rangle = \langle \mathrm{d}F, v\rangle \right\} \\ &= \mathrm{im}(\mathrm{d}F). \end{aligned} \tag{12}$$

We have to consider even more general generating object. It is when we generate a submanifold of T^*Q and the manifold X of the family is not the whole Q, but an immersed submanifold of Q only.

$$\begin{array}{ccccc} \mathsf{T}^*Q & & R & \xrightarrow{F} & \mathbb{R} \\ \Big\downarrow{\scriptstyle \pi_Q} & & \Big\downarrow{\scriptstyle \rho} & & \\ Q & \xleftarrow{\iota_X} & X & & \end{array} \tag{13}$$

2.2. Reduction of generating objects. Let

$$\begin{array}{ccc} R & \xrightarrow{F} & \mathbb{R} \\ \Big\downarrow{\scriptstyle \rho} & & \\ X & & \end{array} \tag{14}$$

be a family of functions generating a Lagrangian submanifold N of $(\mathsf{T}^*X, \mathrm{d}\vartheta_x)$.

Proposition 1. *If N is generated by a family (14) such that the fibration ρ is the composition $\widetilde{\rho} \circ \rho'$ of fibrations $\rho'\colon R \to \widetilde{R}$ and $\widetilde{\rho}\colon \widetilde{R} \to X$ and the critical set $S(F, \rho')$ is the image of a section $\sigma\colon \widetilde{R} \to R$ of ρ', then N is generated by the family*

$$\begin{array}{ccc} \widetilde{R} & \xrightarrow{\widetilde{F}} & \mathbb{R} \\ \Big\downarrow{\scriptstyle \widetilde{\rho}} & & \\ X & & \end{array} \tag{15}$$

with $\widetilde{F} = F \circ \sigma$.

2.3. Lagrangian system. Let M be the configuration manifold of a mechanical system.

A generating object

$$\begin{array}{ccccc} \mathsf{T}\mathsf{T}^*M & & Y & \xrightarrow{L} & \mathbb{R} \\ {\scriptstyle \mathsf{T}\pi_M}\big\downarrow & & \big\downarrow{\scriptstyle \eta} & & \\ \mathsf{T}M & \xleftarrow{\iota_C} & C & & \end{array} \tag{16}$$

is called a *Lagrangian system*. The family of functions

$$\begin{array}{ccc} Y & \xrightarrow{L} & \mathbb{R} \\ {\scriptstyle \eta}\big\downarrow & & \\ C & & \end{array} \tag{17}$$

is called a *Lagrangian family*.

2.4. Hamiltonian system.

Let (P, ω) be a symplectic manifold representing the phase space of a mechanical system. A generating object

$$\begin{array}{ccccc} \mathsf{T}P & & Z & \xrightarrow{-H} & \mathbb{R} \\ {\scriptstyle \tau_P}\big\downarrow & & \big\downarrow{\scriptstyle \zeta} & & \\ P & \xleftarrow{\iota_K} & K & & \end{array} \tag{18}$$

is called a *Hamiltonian system*. The family of functions

$$\begin{array}{ccc} Z & \xrightarrow{H} & \mathbb{R} \\ {\scriptstyle \zeta}\big\downarrow & & \\ K & & \end{array} \tag{19}$$

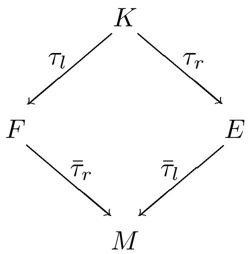
is called a *Hamiltonian family*. A Hamiltonian system

$$\begin{array}{ccccc} \mathsf{T}^*P & & & & \\ {\scriptstyle \pi_P}\big\downarrow & & & & \\ P & \xleftarrow{\iota_K} & K & \xrightarrow{-H} & \mathbb{R} \end{array} \tag{20}$$

is called a *Dirac system*. A more general Hamiltonian system (18) is called a *generalized Dirac system*.

3. Double vector bundles.

Let $\mathbf{K}$ be a system $(\mathbf{K}_r, \mathbf{K}_l, \mathbf{E}, \mathbf{F})$ of vector bundles, where $\mathbf{K}_r = (K, \tau_r, E)$, $\mathbf{K}_l = (K, \tau_l, F)$, $\mathbf{E} = (E, \bar{\tau}_l, M)$ and $\mathbf{F} = (F, \bar{\tau}_r, M)$, such that the diagram



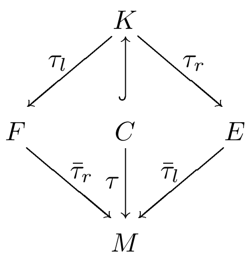
(21)

is commutative. By m_r, m_l, $\bar{m}_r$, and $\bar{m}_l$ we denote the operations of addition in $\mathbf{K}_r$, $\mathbf{K}_l$, $\mathbf{E}$ and $\mathbf{F}$ respectively. A system $\mathbf{K}$ is called *double vector bundle* if the diagram (21) is commutative and the following conditions are satisfied:

(1) pairs $(\tau_l, \bar{\tau}_l)$, $(\tau_r, \bar{\tau}_r)$ are vector bundle morphisms,
(2) pairs of additions $(m_l, \bar{m}_l)$ and $(m_r, \bar{m}_r)$ are vector bundle morphisms $\mathbf{K}_r \times_F \mathbf{K}_r \to \mathbf{K}_r$ and $\mathbf{K}_l \times_E \mathbf{K}_l \to \mathbf{K}_l$ respectively,
(3) zero sections $\mathbf{0}_r\colon \mathbf{E} \to \mathbf{K}_l$, $\mathbf{0}_l\colon \mathbf{F} \to \mathbf{K}_r$ are vector bundle morphisms.

PROPOSITION 1. *The vector bundle structures of* $\mathbf{K}_r$ *and* $\mathbf{K}_l$ *coincide on the intersection* C *of* $\ker \tau_l$ *and* $\ker \tau_r$.

Thus,$\mathbf{C} = (C, \tau, M)$, where $\tau = \bar{\tau}_l \tau_r = \bar{\tau}_r \tau_l$ is a vector bundle is called *the core* of $\mathbf{K}$. We have the diagram



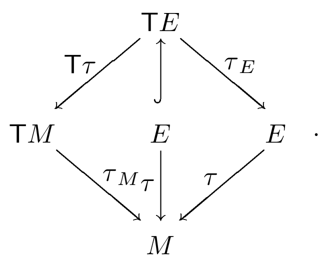
(22)

which will be used instead of the diagram (21) to represent the double vector bundle.

PROPOSITION 2.

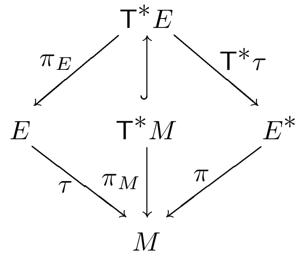
(1) $\ker \tau_r$ *with the vector bundle structure induced from* $\mathbf{K}_r$ *is canonically isomorphic to the Whitney sum* $\mathbf{F} \oplus_M \mathbf{C}$.
(2) $\ker \tau_r$ *with the vector bundle structure induced from* $\mathbf{K}_l$ *is canonically isomorphic to the vector bundle* $F \times_M \mathbf{C}$, *i.e., to the pull-back of* $\mathbf{C}$ *by the projection* $\bar{\tau}_r$.
(3) $\ker \tau_l$ *with the vector bundle structure induced from* $\mathbf{K}_l$ *is canonically isomorphic to the Whitney sum* $\mathbf{E} \oplus_M \mathbf{C}$.
(4) $\ker \tau_l$ *with the vector bundle structure induced from* $\mathbf{K}_r$ *is canonically isomorphic to the vector bundle* $E \times_M \mathbf{C}$, *i.e., to the pull-back of* $\mathbf{C}$ *by the projection* $\bar{\tau}_l$.

3.1. Examples. Let (E, τ, M) be a vector bundle. The tangent manifold $\mathsf{T}M$ is in a canonical way a double vector bundle with the diagram



(23)

Also its dual with respect to the canonical projection, i.e the cotangent bundle T^*E is a double vector bundle with the diagram



, (24)

where $\mathsf{T}^*\tau$ is dual to the vertical lift $E \times_M E \to \mathsf{T}M$.

This is an example of a general situation.

3.2. Duality. Let $\mathbf{K}^*_r$be the vector bundle, dual to $\mathbf{K}_r$. We denote by K^{*_r} the total fiber bundle space and by π_l the projection

$$\pi_l\colon K^{*_r} \to E.$$

Let $a \in K^{*_r}$ and $k \in C$ satisfy $\tau(k) = \bar{\tau}_l(\pi_l(a))$. We can evaluate a on a vector $(\pi_l(a), k)$ of $\ker \tau_l$. We define a mapping $\pi_r\colon K^{*_r} \to C^*$ by the formula

$$\langle k, \pi_r(a)\rangle = \langle(\pi_l(a), k), a\rangle. \tag{25}$$

It follows directly from this construction that

Proposition 3. *The mapping $\pi_r\colon K^{*_r} \to C^*$ is a morphism of vector bundles*

$$\pi_r\colon \mathbf{K}^*_r \to \mathbf{C}^*.$$

We define a relation

$$P_r(m_l)\colon K^{*_r} \times K^{*_r} \to K^{*_r}$$

in the following way: $c \in P_r(m_l)(a, b)$ if

(1) $\pi_l(c) = \pi_l(a) + \pi_l(b)$,
(2) $\langle w, c\rangle = \langle v, a\rangle + \langle v', b\rangle$ for each $w, v, v' \in K$ such that $\tau_r(w) = \pi_l(c)$, $\tau_r(v) = \pi_l(a)$, $\tau_r(v') = \pi_l(b)$ and $w = m_l(v, v')$.

Local coordinates. Let $(x^i, e^a, f^A, c^\alpha)$ be an adapted coordinate system on K and let $(x^i, e^a, p_A, q_\alpha)$ be the adapted coordinate system on the dual bundle K^{*_r}, i. e., the canonical evaluation is given by the formula

$$\langle v, a\rangle = \sum_A p_A(a) f^A(v) + \sum_\alpha q_\alpha(a) c^\alpha(v). \tag{26}$$

We use (x^i, c^α) as a coordinate system in C and (x^i, q_α) as a coordinate system on C^*. In these coordinate sytems we have

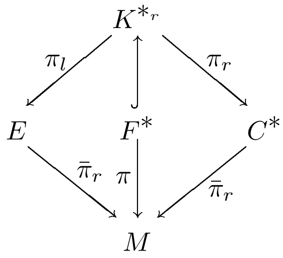
$$\begin{aligned} x_i \circ \pi_r(a) &= x_i(a), \\ q_\alpha \circ \pi_r(a) &= q_\alpha(a) \end{aligned} \tag{27}$$

and

$$\begin{aligned} x^i(a +_r b) &= x^i(a) = x^i(b), \\ e^a(a +_r b) &= e^a(a) + e^a(b), \\ p_A(a +_r b) &= p_A(a) + p_A(b), \\ q_\alpha(a +_r b) &= q_\alpha(a) = q_\alpha(b). \end{aligned} \tag{28}$$

It follows that (K^{*_r}, π_r, C^*) is a vector bundle. We denote it by $\mathbf{K}^{*_r}_r$. The vector bundle (K^{*_r}, π_l, E) we denote by $\mathbf{K}^{*_r}_l$.

Theorem 4. *The system* $\mathbf{K}^{*_r} = (\mathbf{K}^{*_r}_r, \mathbf{K}^{*_l}_l, \mathbf{C}^*, \mathbf{E})$ *is a double vector bundle with the diagram*



. (29)

In the following, we show that there is a canonical isomorphism of double vector bundles $\mathbf{K}$ and $((\mathbf{K}^{*_r})^{*_r})^{*_r}$. Through this section we denote by $\tau_r, \pi_r, \xi_r, \vartheta_r$ and by $\tau_l, \pi_l, \xi_l, \vartheta_l$ the right and left projections in $\mathbf{K}$, $\mathbf{K}^{*_r}$, $(\mathbf{K}^{*_r})^{*_r}$, $((\mathbf{K}^{*_r})^{*_r})^{*_r}$ respectively. Identifying vector bundles with their second duals, we have

$$\xi_r \colon (K^{*_r})^{*_r} \to F \qquad \xi_l \colon (K^{*_r})^{*_r} \to C^*$$
$$\vartheta_r \colon ((K^{*_r})^{*_r})^{*_r} \to E \qquad \vartheta_l \colon ((K^{*_r})^{*_r})^{*_r} \to F.$$

The core of $(\mathbf{K}^{*_r})^{*_r}$ is $\mathbf{E}^*$ and the core of $((\mathbf{K}^{*_r})^{*_r})^{*_r}$ is $(\mathbf{C}^*)^* = \mathbf{C}$.

We define a relation $\mathcal{R}_K \subset K \times ((K^{*_r})^{*_r})^{*_r}$ in the following way:

Let $v \in K$, $\phi \in ((K^{*_r})^{*_r})^{*_r}$ be such that $\bar{\tau}_l(\tau_r(v)) = \bar{\vartheta}_l(\vartheta_r(\phi))$. We say that $(v, \phi) \in \mathcal{R}_K$ if for each $a \in K^{*_r}$, $\alpha \in (K^{*_r})^{*_r}$ such that

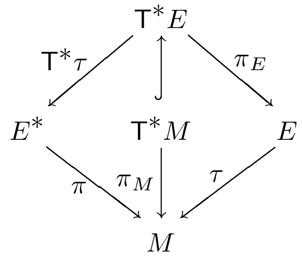
$$\tau_r(v) = \pi_l(a),\ \pi_r(a) = \xi_l(\alpha),\ \xi_r(\alpha) = \vartheta_l(\phi)$$

we have

$$\langle a, \alpha \rangle = \langle v, a \rangle + \langle \alpha, \phi \rangle. \tag{30}$$

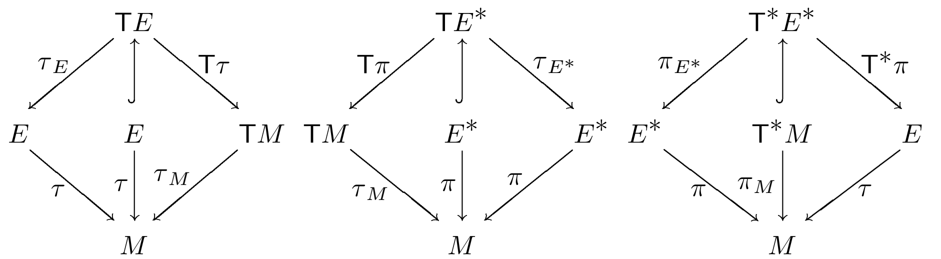
Theorem 5. *The relation* $\mathcal{R}_K$ *is an isomorphism of double vector bundles.*

Example. 2. Let $\mathbf{K} = \mathsf{T}^*\mathbf{E}$:



. (31)

Then the first, second and third right duals can be identified with double vector bundles $\mathsf{T}E$, $\mathsf{T}E^*$ and T^*E^*, represented by the diagrams



. (32)

In this case the formula (30) means that the relation $\mathcal{R}_K$, interpreted as a submanifold of $\mathsf{T}^*(E \times E^*$, is a lagrangian submanifold generated by the canonical pairing viewed as a function $E \times_M E^* \to \mathbb{R}$.

If we replace the right-hand side of (30) by a different combination of $\langle v, a\rangle$ and $\langle \alpha, \phi\rangle$, we obtain another isomorphism. The isomorphism corresponding to $\langle v, a\rangle - \langle \alpha, \phi\rangle$ we denote by $\mathcal{R}_K^{\pm}$, the isomorphism corresponding to $-\langle v, a\rangle + \langle \alpha, \phi\rangle$ we denote by $\mathcal{R}_K^{\mp}$ and the isomorphism corresponding to $-\langle v, a\rangle - \langle \alpha, \phi\rangle$ we denote by $\mathcal{R}_K^{=}$.

Thus the canonical isomorphisms $\mathcal{R}_K$, $\mathcal{R}_K^{\pm}$, $\mathcal{R}_K^{\mp}$, $\mathcal{R}_K^{=}$ define diffeomorphisms from T^*E^* to T^*E. For $\mathcal{R}_K$, $\mathcal{R}_K^{=}$ these diffeomorphisms are antisymplectomorphisms with respect to the canonical symplectic structure of the cotangent bundle and for $\mathcal{R}_K^{\pm}$, $\mathcal{R}_K^{\mp}$ we obtain symplectomorphisms.

4. Iterated tangent functors.

The Lagrange formulation of the dynamics is based on the canonical diffeomorphism $\alpha_M\colon \mathsf{T}\mathsf{T}^*M \longrightarrow \mathsf{T}^*\mathsf{T}M$. It can be obtained by by the composition of $\beta_M\colon \mathsf{T}\mathsf{T}^*M \to \mathsf{T}^*\mathsf{T}^*M$ and the canonical diffeomorphism (see previous sections) $\gamma_M\colon \mathsf{T}^*\mathsf{T}^*M \to \mathsf{T}^*\mathsf{T}M$. It is known that the canonical symplectic structure on a tangent bundle is linear, i.e. the corresponding map β_M is a morphism of double vector bundles. It follows that $\gamma_M \circ \beta_M$ is also a morphism of double vector bundles.

However, to avoid discussion concerning proper choice of sign in (30), it is better to have α_M defined independently.

Elements of the iterated bundle $\mathsf{T}\mathsf{T}M$ are equivalence classes of curves in a set of equivalence classes of curves in M. A simpler representation of these elements is needed. Let $\chi\colon \mathbb{R}^2 \to M$ be a differentiable mapping. For each $s \in \mathbb{R}$ we denote by $\mathsf{t}^{(0,1)}\chi(s,0)$ the vector $\mathsf{t}\chi(s,\cdot)(0) \in \mathsf{T}M$. For each $t \in \mathbb{R}$ we denote by $\mathsf{t}^{(1,0)}\chi(0,t)$ the vector $\mathsf{t}\chi(\cdot,t)(0) \in \mathsf{T}M$. We have curves

$$\mathsf{t}^{(0,1)}\chi(\cdot,0)\colon \mathbb{R} \to \mathsf{T}M \tag{33}$$

and

$$\mathsf{t}^{(1,0)}\chi(0,\cdot)\colon \mathbb{R} \to \mathsf{T}M. \tag{34}$$

Vectors $\mathsf{tt}^{(0,1)}\chi(\cdot,0)(0) \in \mathsf{T}\mathsf{T}M$ and $\mathsf{tt}^{(1,0)}\chi(0,\cdot)(0) \in \mathsf{T}\mathsf{T}M$ will be denoted by $\mathsf{tt}^{(0,1)}\chi(0,0)$ and $\mathsf{tt}^{(1,0)}\chi(0,0)$ respectively. For each $w \in \mathsf{T}\mathsf{T}M$ there is a mapping $\chi\colon \mathbb{R}^2 \to M$ such that $w = \mathsf{tt}^{(0,1)}\chi(0,0)$. The mapping specified by coordinate relations

$$(q^\kappa \circ \chi)(s,t) = (q^\kappa(w) + \dot{q}^\kappa(w)t + q'^\kappa(w)s + \dot{q}'^\kappa(w)st) \tag{35}$$

has the required property. We consider mappings $\chi\colon \mathbb{R}^2 \to M$ and $\chi'\colon \mathbb{R}^2 \to M$ equivalent if

$$\mathsf{tt}^{(0,1)}\chi'(0,0) = \mathsf{tt}^{(0,1)}\chi(0,0). \tag{36}$$

These mappings are equivalent if

$$\chi'(0,0) = \chi(0,0), \tag{37}$$

$$\mathrm{D}^{(1,0)}\chi'(0,0) = \mathrm{D}^{(1,0)}\chi(0,0), \tag{38}$$

$$\mathrm{D}^{(0,1)}\chi'(0,0) = \mathrm{D}^{(0,1)}\chi(0,0), \tag{39}$$

and

$$\mathrm{D}^{(1,1)}\chi'(0,0) = \mathrm{D}^{(1,1)}\chi(0,0). \tag{40}$$

We have obtained an efficient representation of elements of $\mathsf{T}\mathsf{T}M$. In terms of this representation we define the *canonical involution*

$$\begin{aligned} \kappa_M\colon &\mathsf{T}\mathsf{T}M \to \mathsf{T}\mathsf{T}M \\ &: \mathsf{tt}^{(0,1)}\chi(0,0) \mapsto \mathsf{tt}^{(1,0)}\chi(0,0) = \mathsf{tt}^{(0,1)}\widetilde{\chi}(0,0), \end{aligned} \tag{41}$$

with

$$\begin{aligned}\widetilde{\chi}\colon \mathbb{R}^2 &\to M \\ : (s,t) &\mapsto \chi(t,s).\end{aligned} \tag{42}$$

The coordinate expression of this involution is given by

$$(q^\kappa, \dot{q}^\lambda, q'^\mu, \dot{q}'^\nu) \circ \kappa_M = (q^\kappa, q'^\lambda, \dot{q}^\mu, \dot{q}'^\nu). \tag{43}$$

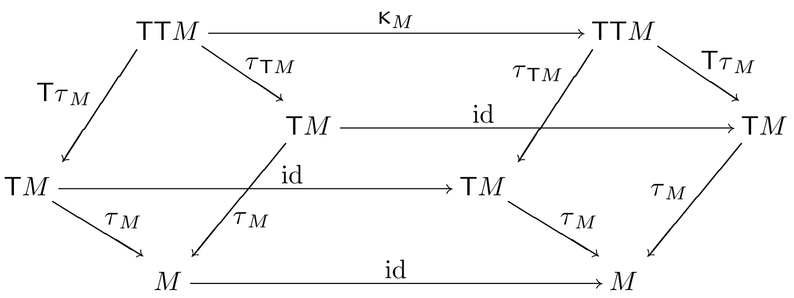
The commutative diagram

$$\begin{array}{ccc} \mathsf{T}\mathsf{T}M & \xrightarrow{\ \kappa_M\ } & \mathsf{T}\mathsf{T}M \\ {\scriptstyle \mathsf{T}\tau_M}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{\mathsf{T}M}} \\ \mathsf{T}M & = & \mathsf{T}M \end{array} \tag{44}$$

is a vector fibration isomorphism. The diagram

$$\begin{array}{ccc} \mathsf{T}\mathsf{T}M & \xrightarrow{\ \kappa_M\ } & \mathsf{T}\mathsf{T}M \\ {\scriptstyle \tau_{\mathsf{T}M}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{T}\tau_M} \\ \mathsf{T}M & = & \mathsf{T}M \end{array} \tag{45}$$

is the inverse isomorphism. It follows that κ_M is a morphism of double vector bundles.



. (46)

For a differentiable mapping $\alpha\colon M \to N$ we have

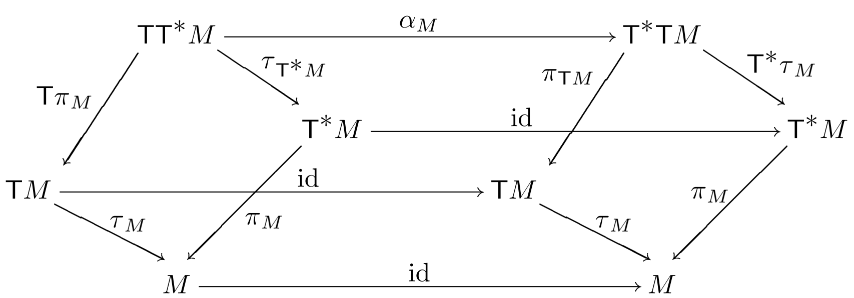
$$\mathsf{T}\mathsf{T}\alpha(\mathsf{tt}^{\{(0,1)\}}\chi(0,0)) = \mathsf{tt}^{\{(0,1)\}}(\alpha \circ \chi)(0,0) \tag{47}$$

and

$$\kappa_M \circ \mathsf{T}\mathsf{T}\alpha = \mathsf{T}\mathsf{T}\alpha \circ \kappa_N, \tag{48}$$

i.e. κ is a natural equivalence of functors.

The morphism α_M is defined as the dual to (45).



. (49)

5. The Legendre transformation.

Let M be the configuration manifold of a mechanical system. The phase space of the system is the symplectic manifold $(\mathsf{T}^*M, \mathrm{d}\vartheta_M)$.

The identity morphism $1_{\mathsf{T}\mathsf{T}^*M}$ is a symplectic relation generated by the generating object

$$\begin{array}{ccccc} \mathsf{T}\mathsf{T}^*M \times \mathsf{T}\mathsf{T}^*M & & & & \\ {\scriptstyle \tau_{\mathsf{T}^*M} \times \mathsf{T}\pi_M}\Big\downarrow & & & & \\ \mathsf{T}^*M \times \mathsf{T}M & \xleftarrow{\iota_{\mathsf{T}^*M \times_M \mathsf{T}M}} & \mathsf{T}^*M \times_M \mathsf{T}M & \xrightarrow{-\langle\,,\rangle} & \mathbb{R} \end{array} \tag{50}$$

and by the generating object

$$\begin{array}{ccccc} \mathsf{T}\mathsf{T}^*M \times \mathsf{T}\mathsf{T}^*M & & & & \\ {\scriptstyle \mathsf{T}\pi_M \times \tau_{\mathsf{T}^*M}}\Big\downarrow & & & & \\ \mathsf{T}M \times \mathsf{T}^*M & \xleftarrow{\iota_{\mathsf{T}M \times_M \mathsf{T}^*M}} & \mathsf{T}M \times_M \mathsf{T}^*M & \xrightarrow{\langle\,,\rangle} & \mathbb{R} \end{array} \tag{51}$$

where $\langle\,,\rangle$ is the canonical pairing

$$\begin{aligned} \langle\,,\rangle^{\sim}\colon \mathsf{T}M \times_M \mathsf{T}^*M &\to \mathbb{R} \\ : (v,p) &\mapsto \langle p, v\rangle. \end{aligned} \tag{52}$$

Starting with a Lagrangian system

$$\begin{array}{ccccc} \mathsf{T}\mathsf{T}^*M & & Y & \xrightarrow{L} & \mathbb{R} \\ {\scriptstyle \mathsf{T}\pi_M}\Big\downarrow & & \Big\downarrow{\scriptstyle \eta} & & \\ \mathsf{T}M & \xleftarrow{\iota_C} & C & & \end{array} \tag{53}$$

we construct a Hamiltonian system

$$\begin{array}{ccccc} \mathsf{T}\mathsf{T}^*M & & Z & \xrightarrow{-H} & \mathbb{R} \\ {\scriptstyle \tau_{\mathsf{T}^*M}}\Big\downarrow & & \Big\downarrow{\scriptstyle \zeta} & & \\ \mathsf{T}^*M & \xleftarrow{\iota_K} & K & & \end{array} \tag{54}$$

with

$$\begin{aligned} K &= {\pi_M}^{-1}(\tau_M(C)) \\ &= \left\{ p \in \mathsf{T}^*M;\ \exists_{v \in C} \pi_M(p) = \tau_M(v) \right\}, \end{aligned} \tag{55}$$

$$Z = \{(p, v, y) \in \mathsf{T}^*M \times_M \mathsf{T}M \times Y;\ v \in C, \eta(y) = v\}, \tag{56}$$

$$\begin{aligned} \zeta\colon Z &\to K \\ : (p, v, y) &\mapsto p, \end{aligned} \tag{57}$$

and

$$\begin{aligned} H\colon Z &\to \mathbb{R} \\ : (p, v, y) &\mapsto \langle p, v\rangle - L(y). \end{aligned} \tag{58}$$

This Hamiltonian system is obtained by composing the Lagrangian system with the generating object (50). The two systems generate the same dynamics since the generating object (50) generates the identity relation. The passage from the Lagrangian system (53) to the Hamiltonian system (54) is called the *Legendre transformation*.

Conversely, given a Hamiltonian system

$$\begin{array}{ccccc} \mathsf{T}\mathsf{T}^*M & & Z & \xrightarrow{-H} & \mathbb{R} \\ \Big\downarrow \tau_{\mathsf{T}^*M} & & \Big\downarrow \zeta & & \\ \mathsf{T}^*M & \xleftarrow{\iota_K} & K & & \end{array} \tag{59}$$

we construct a Lagrangian system

$$\begin{array}{ccccc} (\mathsf{T}\mathsf{T}^*M, \mathrm{d}_T\vartheta_M) & & Y & \xrightarrow{L} & \mathbb{R} \\ \Big\downarrow \mathsf{T}\pi_M & & \Big\downarrow \eta & & \\ \mathsf{T}M & \xleftarrow{\iota_C} & C & & \end{array} \tag{60}$$

with

$$\begin{aligned} C &= {\tau_M}^{-1}(\pi_M(K)) \\ &= \Big\{ v \in \mathsf{T}^*M;\ \exists_{p\in K} \pi_M(p) = \tau_M(v) \Big\}, \end{aligned} \tag{61}$$

$$Y = \{(p, v, z) \in \mathsf{T}^*M \times_M \mathsf{T}M \times Z;\ p \in K, \zeta(z) = p\}, \tag{62}$$

$$\begin{aligned} \eta &\colon Y \to C \\ &: (p, v, z) \mapsto v, \end{aligned} \tag{63}$$

and

$$\begin{aligned} L &\colon Y \to \mathbb{R} \\ &: (p, v, z) \mapsto \langle p, v\rangle - H(z) \end{aligned} \tag{64}$$

by composing the Hamiltonian system with the generating object (51). This passage is called the *inverse Legendre transformation*.

In addition to the Legendre transformations we have *Legendre relations*. Given a Lagrangian system (53) we have the *first Legendre relation*

$$\Lambda_1(L)\colon Y \to \mathsf{T}^*M, \tag{65}$$

whose graph is the set

$$\begin{aligned} \operatorname{graph}(\Lambda_1(L)) = \Big\{ &(p, y) \in \mathsf{T}^*M \times Y;\ y \in S(L, \eta), \pi_M(p) = q = \tau_M(\eta(y)) \\ &\forall_{u \in C \cap \mathsf{T}_qM} \forall_{z \in \mathsf{T}_yY} \mathsf{T}\eta(z) = \chi_M(\eta(y), u) \Rightarrow \langle p, u\rangle = \langle \mathrm{d}L, z\ \rangle \Big\} \end{aligned} \tag{66}$$

and the *second Legendre relation*

$$\Lambda_2(L)\colon \mathsf{T}M \to \mathsf{T}^*M, \tag{67}$$

whose graph is the set

$$\operatorname{graph}(\Lambda_2(L)) = \Big\{ (p, v) \in \mathsf{T}^*M \times \mathsf{T}M;\ \exists_{y\in Y} \eta(y) = v, (p, y) \in \operatorname{graph}(\Lambda_1(L)) \Big\}. \tag{68}$$

If $D \subset \mathsf{T}\mathsf{T}^*M$ is the dynamics generated by the Lagrangian system, then

$$\operatorname{graph}(\Lambda_2(L)) = (\tau_{\mathsf{T}^*M}, \mathsf{T}\pi_M)(D). \tag{69}$$

A Lagrangian system is said to be *hyperregular* if the second Legendre relation $\Lambda_2(L)$ is a diffeomorphism.

6. Lie algebroids.

A Lie algebroid structure on a vector bundle $\tau\colon E \to M$ is usually defined in terms of a Lie bracket on sections of τ. However, it can be characterized (and defined) in terms of associated structures

(1) linear Poisson structure on the dual bundle E^*,
(2) a morphism $\varepsilon\colon \mathsf{T}^*E \to \mathsf{T}E^*$ of double vector bundle,
(3) a relation $\kappa\colon \mathsf{T}E \to \mathsf{T}E$, dual to ε,
(4) complete lift of multisections of E to multivector fields on E.

For the Lie algebroid of a tangent bundle $\mathsf{T}M$, the linear Poisson structure is given by the canonical symplectic structure on T^*M, $\varepsilon = \alpha_M^{-1}$, and $\kappa = \mathsf{\kappa}_M$. The complete lift of a Lie algebroid is the well known complete lift d_T of multivector fields. Unlike for general Lie algebroid, it can be defined also on differential forms. In particular, $\mathrm{d}_\mathsf{T}\omega_M$ is a symplectic form on $\mathsf{T}\mathsf{T}^*M$. With this form α_M and ezb_M become symplectomorphisms.

We see, that it is Lie algebroid structure of the tangent bundle, which makes possible Lagrangian formulation of the dynamics and consequently, the Legendre transformation.

7. Homogeneous systems.

Here, we consider an important for physics class of singular Lagrangian systems, which appear in the calculus of variations for non-parameterized curves (space-time trajectories).

7.1. Homogeneous Lagrangian systems. Let M be a differential manifold and let κ be the action

$$\begin{aligned} \boldsymbol{\kappa}\colon \mathbb{R}_+ \times \mathsf{T}M &\to \mathsf{T}M \\ \colon (k, v) &\mapsto kv. \end{aligned} \tag{70}$$

Let

$$\overset{\circ}{\mathsf{T}}M = \{v \in \mathsf{T}M;\ v \neq 0\} \tag{71}$$

be the tangent bundle of M with the image of the zero section removed. The set $\overset{\circ}{\mathsf{T}}M$ is an homogeneous open submanifold of $\mathsf{T}M$.

A Lagrangian system

$$\begin{array}{ccccc} (\mathsf{T}\mathsf{T}^*M, \mathrm{d}_T\vartheta_M) & & Y & \xrightarrow{\ L\ } & \mathbb{R} \\ \Big\downarrow{\scriptstyle \mathsf{T}\pi_M} & & \Big\downarrow{\scriptstyle \eta} & & \\ \mathsf{T}M & \xleftarrow{\ \iota_C\ } & C & & \end{array} \tag{72}$$

is said to be *homogeneous* with respect to an action

$$\begin{array}{ccc} \mathbb{R}_+ \times Y & \xrightarrow{\ \rho\ } & Y \\ \Big\downarrow{\scriptstyle 1_{\mathbb{R}_+} \times \eta} & & \Big\downarrow{\scriptstyle \eta} \\ \mathbb{R}_+ \times C & \xrightarrow{\ \kappa\ } & C \end{array} \tag{73}$$

if C is an homogeneous submanifold of $\overset{\circ}{\mathsf{T}}M$,

$$\begin{aligned} \kappa\colon \mathbb{R}_+ \times C &\to C \\ \colon (k, v) &\mapsto kv. \end{aligned} \tag{74}$$

is the restriction of the action $\boldsymbol{\kappa}$ to C, and

$$L(\rho(k, y)) = kL(y) \tag{75}$$

for each $y \in Y$ and each $k \in \mathbb{R}_+$. The Lagrangian family

$$\begin{array}{ccc} Y & \xrightarrow{\ L\ } & \mathbb{R} \\ {\scriptstyle \eta}\big\downarrow & & \\ C & & \end{array} \tag{76}$$

is said to be a *homogeneous Lagrangian family.*

The action $\boldsymbol{\kappa}$ is lifted to the action

$$\begin{array}{ccc} \mathbb{R}_+ \times \mathsf{T}\mathsf{T}^*M & \xrightarrow{\quad \widehat{\kappa} \quad} & \mathsf{T}\mathsf{T}^*M \\ {\scriptstyle 1_{\mathbb{R}_+} \times \mathsf{T}\pi_M}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{T}\pi_M} \\ \mathbb{R}_+ \times \mathsf{T}M & \xrightarrow{\quad \boldsymbol{\kappa} \quad} & \mathsf{T}M \end{array} \tag{77}$$

with the action

$$\widehat{\kappa}\colon \mathbb{R}_+ \times \mathsf{T}\mathsf{T}^*M \to \mathsf{T}\mathsf{T}^*M \tag{78}$$

defined by

$$\widehat{\kappa}(k, \cdot) = \alpha_M^{-1} \circ \overline{\boldsymbol{\kappa}}(k, \cdot) \circ \alpha_M \tag{79}$$

for each $k \in \mathbb{R}_+$. The relation

$$\widehat{\kappa}(k, w) = kw \tag{80}$$

holds for each $w \in \mathsf{T}\mathsf{T}^*M$ and each $k \in \mathbb{R}_+$.

The dynamics D generated by a homogeneous Lagrangian system is homogeneous.

7.2. Homogeneous Hamiltonian systems.

Let (P, ω) be a symplectic manifold representing the phase space of a mechanical system and let

$$\begin{array}{ccccc} (\mathsf{T}P, \mathrm{i}_T\omega) & & Z & \xrightarrow{\ -H\ } & \mathbb{R} \\ {\scriptstyle \tau_P}\big\downarrow & & {\scriptstyle \zeta}\big\downarrow & & \\ P & \xleftarrow{\quad \iota_K \quad} & K & & \end{array} \tag{81}$$

be a Hamiltonian system. Let

$$\sigma\colon \mathbb{R}_+ \times Z \to Z \tag{82}$$

be an action of the group $(\mathbb{R}_+, \cdot)$ such that $\zeta \circ \sigma(k, \cdot) = \zeta$ for each $k \in \mathbb{R}_+$. The Hamiltonian system (81) is said to be *homogeneous* with respect to the group action σ if

$$H(\sigma(k, z)) = kH(z) \tag{83}$$

for each $z \in Z$ and each $k \in \mathbb{R}_+$. The Hamiltonian family

$$\begin{array}{ccc} Z & \xrightarrow{\ H\ } & \mathbb{R} \\ {\scriptstyle \zeta}\big\downarrow & & \\ K & & \end{array} \tag{84}$$

is said to be a *homogeneous Hamiltonian family.*

The same concepts of homogeneity apply to a Hamiltonian system

$$
\begin{array}{ccc}
(\mathsf{T}\mathsf{T}^*M, \mathrm{i}_T \mathrm{d}\vartheta_M) & & Z \xrightarrow{-H} \mathbb{R} \\
\Big\downarrow {\scriptstyle \tau_{\mathsf{T}^*M}} & & \Big\downarrow {\scriptstyle \zeta} \\
\mathsf{T}^*M & \xleftarrow{\ \iota_K\ } & K
\end{array}
\tag{85}
$$

based on a phase space $(\mathsf{T}^*M, \mathrm{d}\vartheta_M)$.

The dynamics D generated by a homogeneous Hamiltonian system is homogeneous.

8. Relativistic mechanics.

Let (Q, V, σ, g) be the affine space-time of special relativity. The triple (Q, V, σ) is an affine space and $g\colon V \to V^*$ is the Minkowski metric.

8.1. Relativistic particle. The dynamics of a free particle of mass m is generated by the Lagrangian system

$$
\begin{array}{ccc}
(\mathsf{T}\mathsf{T}^*Q, \mathrm{d}_T \vartheta_Q) & & \\
\Big\downarrow {\scriptstyle \mathsf{T}\pi_Q} & & \\
\mathsf{T}Q & \xleftarrow{\ \iota_C\ } & C \xrightarrow{L} \mathbb{R}
\end{array}
\tag{86}
$$

where

$$
C = Q \times \{\dot{q} \in V;\ \langle g(\dot{q}), \dot{q} \rangle > 0\}
\tag{87}
$$

and

$$
\begin{aligned}
L\colon & C \to \mathbb{R} \\
&: (q, \dot{q}) \mapsto m\sqrt{\langle g(\dot{q}), \dot{q} \rangle}.
\end{aligned}
\tag{88}
$$

The dynamics is the set

$$
D = \left\{ (q, p, \dot{q}, \dot{p}) \in \mathsf{T}\mathsf{T}^*Q;\ \langle g(\dot{q}), \dot{q} \rangle > 0, p = \frac{m g(\dot{q})}{\sqrt{\langle g(\dot{q}), \dot{q} \rangle}}, \dot{p} = 0 \right\}.
\tag{89}
$$

The set C is a homogeneous submanifold of $\overset{\circ}{\mathsf{T}}Q$ and $L(q, k\dot{q}) = kL(q, \dot{q})$. It follows that the Lagrangian system (86) is homogeneous.

The Legendre transformation applied to the Lagrangian system (86) leads to a Hamiltonian system

$$
\begin{array}{ccc}
(\mathsf{T}\mathsf{T}^*Q, \mathrm{i}_T \mathrm{d}\vartheta_Q) & & Z \xrightarrow{-H} \mathbb{R} \\
\Big\downarrow {\scriptstyle \tau_{\mathsf{T}^*Q}} & & \Big\downarrow {\scriptstyle \zeta} \\
\mathsf{T}^*Q & \Longleftrightarrow\!\!= & \mathsf{T}^*Q
\end{array}
\tag{90}
$$

with

$$
Z = \mathsf{T}^*Q \times \{v \in V;\ \langle g(v), v \rangle > 0\},
\tag{91}
$$

$$
\begin{aligned}
\zeta\colon & Z \to \mathsf{T}^*Q \\
&: (q, p, v) \mapsto (q, p),
\end{aligned}
\tag{92}
$$

and

$$\begin{aligned} H\colon Z &\to \mathbb{R} \\ &: (q, p, v) \mapsto \langle p, v\rangle - m\sqrt{\langle g(v), v\rangle}. \end{aligned} \tag{93}$$

The Hamiltonian system (90) can be simplified. The fibration ζ can be represented as a composition $\zeta'' \circ \zeta'$ of fibrations

$$\begin{aligned} \zeta'\colon Z &\to Z' \\ &: (q, p, v) \mapsto (q, p, \|v\|), \end{aligned} \tag{94}$$

and

$$\begin{aligned} \zeta''\colon Z' &\to \mathsf{T}^*Q \\ &: (q, p, \lambda) \mapsto (q, p), \end{aligned} \tag{95}$$

where

$$Z' = \mathsf{T}^*Q \times \mathbb{R}_+ \tag{96}$$

and

$$\|v\| = \sqrt{\langle g(v), v\rangle}. \tag{97}$$

Equating to zero the derivative of H along the fibres of ζ' we obtain the relation

$$p = \mu g(v) \tag{98}$$

valid for some values of the Lagrange multiplier μ. It follows from this relation that the covector p is in the set

$$\left\{p \in V^*;\ \langle p, g^{-1}(p)\rangle > 0\right\} \tag{99}$$

and that

$$\{(q, p, v) \in Z;\ \|v\|p = \pm\|p\|g(v)\} \tag{100}$$

is the critical set $S(H, \zeta')$. We have denoted by $\|p\|$ the norm $\sqrt{\langle p, g^{-1}(p)\rangle}$ of p. The images $\widetilde{K} = \zeta(S(H, \zeta'))$ and $\widetilde{Z} = \zeta'(S(H, \zeta'))$ of the critical set $S(H, \zeta')$ are the sets

$$\widetilde{K} = Q \times \left\{p \in V^*;\ \langle p, g^{-1}(p)\rangle > 0\right\} \tag{101}$$

and

$$\widetilde{Z} = \widetilde{K} \times \mathbb{R}_+. \tag{102}$$

The mapping

$$\begin{aligned} \widetilde{\zeta}\colon \widetilde{Z} &\to \widetilde{K} \\ &: (q, p, \lambda) \mapsto (q, p). \end{aligned} \tag{103}$$

is a differential fibration. The critical set $S(H, \zeta')$ is the union of images of two local sections

$$\begin{aligned} \xi_+\colon \widetilde{Z} &\to Z \\ &: (q, p, \lambda) \mapsto \left(q, p, \frac{\lambda}{\|p\|} g^{-1}(p)\right) \end{aligned} \tag{104}$$

and

$$\begin{aligned} \xi_-\colon \widetilde{Z} &\to Z \\ &: (q, p, \lambda) \mapsto \left(q, p, -\frac{\lambda}{\|p\|} g^{-1}(p)\right) \end{aligned} \tag{105}$$

of the fibration ζ'. We have two families of functions

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\widetilde{H}_-} & \mathbb{R} \\ {\scriptstyle\widetilde{\zeta}}\big\downarrow & & \\ \widetilde{K} & & \end{array} \qquad\qquad \begin{array}{ccc} \widetilde{Z} & \xrightarrow{\widetilde{H}_+} & \mathbb{R} \\ {\scriptstyle\widetilde{\zeta}}\big\downarrow & & \\ \widetilde{K} & & \end{array} \tag{106}$$

The functions

$$\begin{aligned} \widetilde{H}_-\colon \widetilde{Z} &\to \mathbb{R} \\ \colon (q,p,\lambda) &\mapsto -\lambda\left(\|p\| + m\right) \end{aligned} \tag{107}$$

and

$$\begin{aligned} \widetilde{H}_+\colon \widetilde{Z} &\to \mathbb{R} \\ \colon (q,p,\lambda) &\mapsto \lambda\left(\|p\| - m\right) \end{aligned} \tag{108}$$

are the compositions $H \circ \xi_-$ and $H \circ \xi_+$ respectively. It is easily seen that the family (106) generates an empty set. The dynamics of the particle is generated by the reduced Hamiltonian system

$$\begin{array}{ccccc} (\mathsf{T}\mathsf{T}^*Q, \mathrm{i}_T\mathrm{d}\vartheta_Q) & & \widetilde{Z} & \xrightarrow{-\widetilde{H}_+} & \mathbb{R} \\ {\scriptstyle\tau_{\mathsf{T}^*Q}}\big\downarrow & & {\scriptstyle\widetilde{\zeta}}\big\downarrow & & \\ \mathsf{T}^*Q & \xleftarrow{\iota_{\widetilde{K}}} & \widetilde{K} & & \end{array} \tag{109}$$

The critical set $S(\widetilde{H}_+, \widetilde{\zeta})$ is the submanifold

$$\left\{(q,p,\lambda) \in \widetilde{Z};\ \|p\| = m\right\} \tag{110}$$

and the image $\widetilde{\zeta}(S(\widetilde{H}_+, \widetilde{\zeta}))$ is the mass shell

$$K_m = \left\{(q,p) \in \widetilde{K};\ \|p\| = m\right\}. \tag{111}$$

The mapping

$$\begin{aligned} \widehat{\zeta}\colon S(\widetilde{H}_+, \widetilde{\zeta}) &\to K_m \\ \colon (q,p,\lambda) &\mapsto (q,p) \end{aligned} \tag{112}$$

is a differential fibration with connected fibres.

8.2. Space-time formulation of geometric optics. The dynamics of light rays in affine Minkowski space-time (Q, V, σ, g) is generated by the Lagrangian system

$$\begin{array}{ccccc} (\mathsf{T}\mathsf{T}^*Q, \mathrm{d}_T\vartheta_Q) & & Y & \xrightarrow{L} & \mathbb{R} \\ {\scriptstyle\mathsf{T}\pi_Q}\big\downarrow & & {\scriptstyle\eta}\big\downarrow & & \\ \mathsf{T}Q & \xleftarrow{\iota_C} & C & & \end{array} \tag{113}$$

with

$$C = \overset{\circ}{\mathsf{T}}Q, \tag{114}$$

$$Y = C \times \mathbb{R}_+, \tag{115}$$

$$\begin{aligned} \eta\colon Y &\to C \\ &: (q, \dot{q}, \mu) \mapsto (q, \dot{q}), \end{aligned} \tag{116}$$

and

$$\begin{aligned} L\colon Y &\to \mathbb{R} \\ &: (q, \dot{q}, \mu) \mapsto \frac{1}{2\mu}\langle g(\dot{q}), \dot{q}\rangle. \end{aligned} \tag{117}$$

The dynamics is the set

$$D = \left\{ (q, p, \dot{q}, \dot{p}) \in \mathsf{T}\mathsf{T}^*Q;\ \langle g(\dot{q}), \dot{q}\rangle = 0, \exists_{\mu \in \mathbb{R}_+} p = \frac{1}{\mu} g(\dot{q}), \dot{p} = 0 \right\}. \tag{118}$$

The Lagrangian system (113) is homogeneous.

The result of the Legendre transformation applied to the Lagrangian system (113) is the Hamiltonian system

$$\begin{array}{ccc} (\mathsf{T}\mathsf{T}^*Q, \mathrm{i}_T \mathrm{d}\vartheta_Q) & & Z \xrightarrow{\ -H\ } \mathbb{R} \\ \downarrow{\scriptstyle \tau_{\mathsf{T}^*Q}} & & \downarrow{\scriptstyle \zeta} \\ \mathsf{T}^*Q & = & \mathsf{T}^*Q \end{array} \tag{119}$$

with

$$Z = \mathsf{T}^*Q \times \overset{\circ}{V} \times \mathbb{R}_+, \tag{120}$$

$$\begin{aligned} \zeta\colon Z &\to \mathsf{T}^*Q \\ &: (q, p, v, \mu) \mapsto (q, p), \end{aligned} \tag{121}$$

and

$$\begin{aligned} H\colon Z &\to \mathbb{R} \\ &: (q, p, v, \mu) \mapsto \langle p, v\rangle - \frac{1}{2\mu}\langle g(v), v\rangle. \end{aligned} \tag{122}$$

The Hamiltonian system (119) can be simplified as in the case of a relativistic particle with mass $m > 0$. The reduced system is the Hamiltonian system

$$\begin{array}{ccc} (\mathsf{T}\mathsf{T}^*Q, \mathrm{i}_T \mathrm{d}\vartheta_Q) & & \widetilde{Z} \xrightarrow{\ -\widetilde{H}\ } \mathbb{R} \\ \downarrow{\scriptstyle \tau_{\mathsf{T}^*Q}} & & \downarrow{\scriptstyle \widetilde{\zeta}} \\ \mathsf{T}^*Q & \xleftarrow{\ \iota_{\widetilde{K}}\ } & \widetilde{K} \end{array} \tag{123}$$

with

$$\widetilde{K} = \left\{ (q, p) \in \mathsf{T}^*Q;\ p \neq 0 \right\}, \tag{124}$$

$$\begin{aligned} \widetilde{\zeta}\colon \widetilde{Z} &\to \widetilde{K} \\ &: (q, p, \mu) \mapsto (q, p), \end{aligned} \tag{125}$$

and

$$\begin{aligned} \widetilde{H}\colon \widetilde{Z} &\to \mathbb{R} \\ &: (q, p, \mu) \mapsto \frac{\mu}{2}\langle p, g^{-1}(p)\rangle. \end{aligned} \tag{126}$$

No further simplification is possible.

9. Dirac systems.

In [1] Dirac proposed the following procedure of deriving Hamiltonian formulation of a system with singular Lagrangian. Let $L\colon \mathsf{T}M \rightarrow \mathbb{R}$ be the Lagrangian of a system. Suppose that the Legendre mapping $\mathrm{d}_v L\colon \mathsf{T}M \rightarrow C \subset \mathsf{T}^*M$ is a fibration over a submanifold C with connected fibres. It is easy task to verify that $\langle p, v\rangle - L(v)$ is constant on fibres. Consequently, we have a function H on C and a Dirac system

$$\begin{array}{ccccc} \mathsf{TT}^*M & & & & \\ \Big\downarrow{\scriptstyle \tau_{\mathsf{T}^*M}} & & & & \\ \mathsf{T}^*M & \xleftarrow{\ \iota_C\ } & K_m & \xrightarrow{\ -H\ } & \mathbb{R} \end{array} \tag{127}$$

The dynamics $\widehat{D}$ of this system includes the dynamics D of the original Lagrangian system. It may happen that they are not equal. An example is given by the relativistic particle of previous section. The corresponding Dirac system is

$$\begin{array}{ccccc} \mathsf{TT}^*M, & & & & \\ \Big\downarrow{\scriptstyle \tau_{\mathsf{T}^*M}} & & & & \\ \mathsf{T}^*M & \xleftarrow{\ \iota_{K_m}\ } & K_m & \xrightarrow{\ 0\ } & \mathbb{R} \end{array}$$

and $\widehat{D} \supsetneq D$.

10. Legendre transformation and convex analysis.

Not yet ready.

11. References.

[1] P. A. M. Dirac, *Generalized Hamiltonian Dynamics*, Canad. J. Math. **2** (1950), 129–148.

[2] P. Libermann, M.-C. Marle, *Symplectic Geometry and Analytical Mechanics*, D. Reidel Publishing Company, Dordrecht, 1987.

[3] W. M. Tulczyjew, *Hamiltonian Systems, Lagrangian Systems and the Legendre Transformation*, Symposia Mathematica **16** (1974), 247-258.

[4] W. M. Tulczyjew, *The Legendre transformation*, Ann. Inst. Henri Poincaré, **27** (1977), 101–114.

[5] G. Pidello and W. M. Tulczyjew, *Derivations of differential forms on jet bundles*, Ann. Mat. Pura Appl., **147** (1987).

[6] W. M. Tulczyjew, P. Urbanski, *Homogeneous Lagrangian systems*, in *Gravitation, Electromagnetism and Geometric Structures*, Pitagora Editrice, 1996.

[7] W. M. Tulczyjew, P. Urbanski, *A slow and careful Legendre transformation for singular Lagrangians*, Acta Phys. Pol. B **30** (1999).