REMARKS ON CONTACT GEOMETRY

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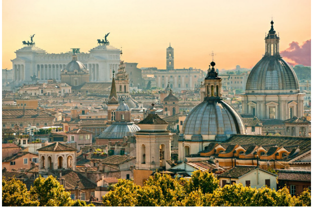

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Contact structures

- A contact form on a manifold M (necessary of an odd dimension $\dim(M) = 2n+1$) is a traditionally defined as a 1-form α such that $\alpha \wedge (\mathrm{d}\alpha)^n$ is a volume form, i.e.

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0\,. \tag{1}$$

- If α is a contact form and the function f is nowhere vanishing, then $f\alpha$ is again a contact form, so one can distinguish rank 1 distributions C in the vector bundle T^*M generated locally by contact forms. They are called contact structures.
- Sometimes as a contact structure one defines annihilators $K = C^0 \subset \mathsf{T}M$ which are distributions of rank $2n$. Then, (1) is equivalent to the fact that K are 'maximally nonintegrable',

$$[K, K] = \mathsf{T}M\,.$$

- Note that (1) does not make much sense on a supermanifold.

Contact structures

- Looking for the most useful definition of a contact structure we will use the following observation.
- For a vector bundle $\tau : L \to M$ denote with $L^\times$ the manifold L with the 0-section removed.

Proposition

A line bundle $L \subset \mathsf{T}^*M$ is a contact structure on M if and only if $L^\times$ is a symplectic submanifold of T^*M.

- Indeed, if α is a 1-form locally generating L over $U \subset M$, then

$$\Phi : \mathbb{R}^\times \times U \ni (t, x) \mapsto t\alpha(x) \in L \subset \mathsf{T}^*M$$

is a local dyfeomorphism and the pullback of the canonical symplectic form ω_M on T^*M is

$$\Phi^*(\omega_M)(x) = \mathrm{d}t \wedge \alpha(x) + t\,\mathrm{d}\alpha(x)\,.$$

The latter is symplectic if and only if α is contact.

Symplectic principal $\mathbb{R}^\times$-bundles

- Note that, for a line bundle $\tau : L \to M$, the bundle $L^\times \to M$ is an $\mathfrak{gl}(1;\mathbb{R}) = \mathbb{R}^\times$-principal bundle.
- Conversely, if $P \to M$ is an $\mathbb{R}^\times$-principal bundle and $L = \hat{P}$ is the associated line bundle, then $P = L^\times$.
- As the canonical symplectic form ω_M is linear, it is homogeneous of degree 1:

$$h_t^* \left(\omega_M(q, tp)\right) = t \cdot \omega_M(q, p)\,, \quad t \in \mathbb{R}\,.$$

- Here, h_t is the multiplication by t. Hence, if $C \subset \mathrm{T}^*M$ is a contact structure, then the $\mathbb{R}^\times$-principal bundle $C^\times$ is equipped with a canonical symplectic form ω which is homogeneous:

$$h_t^* \left(\omega\right) = t \cdot \omega\,, \quad t \in \mathbb{R}^\times\,.$$

- We will call structures $(C^\times, h_t, \omega)$ symplectic principal $\mathbb{R}^\times$-bundles. This is more general than symplectizations $C = \mathbb{R} \times M$ of contact manifolds (M, α).

New definition

Proposition

The assignment $C \mapsto C^\times$ establishes a canonical one-to-one correspondence between contact structures and symplectic principal $\mathbb{R}^\times$-bundles.

Proof.

One has to prove that any symplectic principal $\mathbb{R}^\times$-bundle (P, h_t, ω) over M, $\tau : P \to M = P/\mathbb{R}^\times$, is canonically identified with $C^\times$ for a contact structure C on M.

Let ∇ be the fundamental vector field of the $\mathbb{R}^\times$-action on P. It is easy to see that the one-form $\tilde{\alpha} = i_\nabla \omega$ is semi-basic, i.e.

$$\tilde{\alpha}(y) = \tau^*(\alpha(y))$$

for some $\alpha(y) \in \mathsf{T}^*_{\tau(y)}M$. It defines an $\mathbb{R}^\times$-equivariant embedding

$$\Phi : P \ni y \mapsto \alpha(y) \in \mathsf{T}^*M$$

of (P, ω) onto $(C^\times, \omega_M)$ for some contact structure $C \subset \mathsf{T}^*M$ on M. □

Jet bundles

- Let now $\tau : L \to M$ be a line bundle and $P = L^\times$ the corresponding $\mathbb{R}^\times$-principal bundle.
- The principal action $h : \mathbb{R}^\times \times L^\times \ni (t, y) \mapsto h_t(y) \in L^\times$ can be lifted to an $\mathbb{R}^\times$-principal action T^*h on the cotangent bundle $\mathsf{T}^*(L^\times)$ by

$$(\mathsf{T}^*h)_t = t \cdot (\mathsf{T}h_{t^{-1}})^*\,.$$

- Locally, the action $h_t(x, s) = (x, ts)$ is lifted to

$$(\mathsf{T}^*h)_t(x, s, p, z) = (x, ts, tp, z)\,.$$

- It is easy to see that the symplectic form $\omega_{L^\times}$ is homogeneous with respect to this action, so $\mathsf{T}^*(L^\times)$ is canonically a symplectic principal $\mathbb{R}^\times$-bundle, thus a contact structure. This is a contact structure on

$$\mathsf{T}^*(L^\times)/\mathbb{R}^\times \simeq \mathsf{J}^1 L^*\,.$$

- This proves that first jet bundles $\mathsf{J}^1 L$ of line bundles are canonically equipped with contact structures (Cartan distributions).

Another example

- We have additionally the following commutative diagram

$$\begin{array}{ccc} C_L^\times \simeq \mathsf{T}^*(L^\times) & \longrightarrow & \mathsf{J}^1 L^* \\ \downarrow & & \downarrow \\ L^\times & \longrightarrow & M \end{array} .$$

  and the canonical contact form $\alpha = \mathrm{d}z - p_a\,\mathrm{d}x^a$ on $\mathsf{J}^1(M \times \mathbb{R})$.

- Another example we get considering $(\mathsf{T}^*M)^\times$. With respect to the multiplication by reals it is canonically an $\mathbb{R}^\times$-principal bundle. As the canonical symplectic form is homogeneous with respect to the multiplication by reals, we get a canonical symplectic principal $\mathbb{R}^\times$-bundle, i.e. a contact structure.

- This is a contact structure on

$$(\mathsf{T}^*M)^\times/\mathbb{R}^\times \simeq \mathcal{P}(\mathsf{T}^*M)\,,$$

  the projectivization of T^*M.

Contact brackets

- If (P, h_t, ω) is a symplectic principal $\mathbb{R}^\times$-bundle, then the Poisson bracket $\{\cdot,\cdot\}_\omega$ is closed on homogeneous functions of degree 1.
- Homogeneous functions of degree 1 on $\tau : P = L^\times \to M$ can be naturally identified with sections σ of the dual line bundle L^* *via*:
$$\iota_\sigma(y) = \langle \sigma(\tau(y)), y \rangle \,.$$
- The bracket $\{\cdot,\cdot\}$ on $\mathrm{Sec}(L^*)$ defined by
$$\iota_{\{\sigma_1,\sigma_2\}} = \{\iota_{\sigma_1}, \iota_{\sigma_2}\}_\omega$$
is called the contact bracket (with respect to the contact structure).
- For the canonical contact form $\alpha = \mathrm{d}z - p_a\,\mathrm{d}x^a$ on the first jet bundle of $M \times \mathbb{R}$, the contact bracket on $C^\infty(M)$ reads
$$\{f,g\} = \left(f - p_a\frac{\partial f}{\partial p_a}\right)\frac{\partial g}{\partial z} - \left(g - p_a\frac{\partial g}{\partial p_a}\right)\frac{\partial f}{\partial z} + \frac{\partial f}{\partial x^a}\frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial x^a}\frac{\partial f}{\partial p_a}\,.$$

The Lie algebroid of a contact structure

- The contact bracket is a first order differential operator with respect to each argument, so that
$$\{\sigma,\cdot\}\in \mathsf{DO}^1(L^*,L^*)$$
whose principal part is a vector field X_σ on M called the contact vector field associated with σ. The section σ is the contact Hamiltonian of X_σ, $[X_\sigma,X_{\sigma'}]=X_{\{\sigma,\sigma'\}}$.
- Contact vector fields preserve the contact distribution $L\subset \mathsf{T}^*M$.
- If C is a contact structure, then we can use the tangent lift to lift the symplectic and $\mathbb{R}^\times$-principal structure to $\mathsf{T}(C^\times)$. The homogeneity structure on the vector bundle $\mathsf{T}(C^\times)$ (multiplication by reals) is compatible with the $\mathbb{R}^\times$-principal bundle structure (multiplication by reals commutes with the $\mathbb{R}^\times$-principal action), so the bi-degree and a vector bundle structure on $\mathsf{T}(C^\times)/\mathbb{R}^\times$ are well defined
- Functions of bi-degree $(1,1)$ on $\mathsf{T}(C^\times)$ are closed with respect to the Poisson bracket on $\mathsf{T}(C^\times)$ that defines a Lie algebroid structure on $C^*\otimes_M(\mathsf{T}(C^\times)/\mathbb{R}^\times)^*$ (the Lie algebroid of the contact structure).

Linear contact structures

- We can consider linear contact structures as vector bundles equipped with a compatible contact structure: the multiplication by reals in the vector bundle commutes with the $\mathbb{R}^{\times}$-action and makes the symplectic form linear (of degree 1).
- One can also characterize linear contact structure as locally generated by contact forms which are linear.

Theorem

Any linear contact structure is equivalent to the canonical contact structure on the first jets $\mathrm{J}^1 L$ *of a line bundle* $L \to M$*.*

- In the Darboux coordinates the generating contact forms can be written as

$$\alpha = \mathrm{d}z - p_a \,\mathrm{d}x^a \,.$$

(Note that any contact one-form can locally be written in this form).

- The corresponding contact Lie algebroid is in this case isomorphic with the Lie algebroid $\mathrm{DO}^1(L, L)$ of first order differential operators from L to L.

Kirillov manifolds

- All this can be generalized to the case in which the homogeneous symplectic structure is replaced with a homogeneous Poisson structure (of degree -1): Poisson $\mathbb{R}^{\times}$-principal bundles.
- They are sometimes called also Kirillov manifolds, as the dual objects are local Lie algebras in the sense of Kirillov: homogeneous Poisson structure on an $\mathbb{R}^{\times}$-principal bundle $L^{\times}$ defines a local (in fact differential of the first order) Lie algebra bracket on the sections of the line bundle L^*. The line bundle L^* with such a bracket is called also a Jacobi bundle; for trivial line bundles $L = M \times \mathbb{R}$ this yields Jacobi structures on M.
- One can obtain a Lie algebroid bracket on $L^* \otimes_M (\mathsf{T}(L^{\times}))^*$ considering functions of bi-degree $(1,1)$ on $\mathsf{T}(L^{\times})$.
- In the case of a Jacobi structure on M, we get a Lie algebroid on $\mathsf{T}^*M \times \mathbb{R}$, first found by Kerbrat and Souici-Benhammadi. If the Jacobi is simply Poisson, we get the well-known Lie algebroid on T^*M.

Philosophy

- It is well known that the choice of one of equivalent definitions influences strongly our way of thinking and makes the formulations of some concepts and generalisations easier or harder, depending on the choice made.
- We insist in this talk in understanding Jacobi/contact geometry in terms of its homogeneous symplectization/poissonisation.
- However, we understand the homogeneity not as associated with a vector field but an $\mathbb{R}^\times$-principal bundle structure.
- The appearance of this principal bundle structure is absolutely fundamental for the whole picture, as the proper framework for Jacobi geometry are rank 1 modules (generally nontrivial), so line bundles.
- All derived concepts like 'Jacobi algebroid', 'Jacobi groupoid', 'Jacobi structure with background', etc., should be understood as the corresponding objects in Poisson geometry, equipped additionally with a principal $\mathbb{R}^\times$-action which is compatible with the other structures. The only thing to be decided is a reasonable notion of compatibility.

Contact groupoids

- The natural compatibility between a groupoid structure $\mathcal{G} \rightrightarrows M$ and an $\mathbb{R}^\times$-principal bundle structure on $\mathcal{G}$ is expressed as the fact that $\mathbb{R}^\times$ acts by groupoid automorphisms. Then, we get $\mathbb{R}^\times$-groupoids.
- According to our general philosophy we propose the following.

Definition

A Kirillov groupoid is a $\mathbb{R}^\times$-groupoid equipped with a homogeneous multiplicative Poisson structure of degree -1, i.e. an $\mathbb{R}^\times$-groupoid which has a Poisson groupoid structure of degree -1. Kirillov groupoids with trivial $\mathbb{R}^\times$-bundle will be called Jacobi groupoids. If the Poisson structure is non-degenerate, i.e. a symplectic structure, then we will speak of a contact groupoid.

- In other words, a contact groupoid is a homogeneous symplectic groupoid, i.e. a symplectic groupoid $(\mathcal{G}, \omega)$ equipped additionally with a compatible principal $\mathbb{R}^\times$-principal bundle structure such that $\mathbb{R}^\times$ acts by groupoid isomorphisms and ω is homogeneous of degree 1.

Contact groupoids

- Symplectic groupoids have been defined by Weinstein and, under different names, independently by Karasev and Zakrzewski. They can be understood as groupoids $\mathcal{G} \rightrightarrows M$ equipped with a multiplicative symplectic form ω.
- Note that the original definition of a contact groupoid $(\mathcal{G}, \alpha, f)$ says that it is a Lie groupoid $\mathcal{G} \rightrightarrows M$ equipped with a contact form α on $\mathcal{G}$ and a multiplicative function $f : \mathcal{G} \to \mathbb{R}$ such that

$$\alpha_{gh}(X_g \oplus_{\mathsf{T}\mathcal{G}} Y_h) = \alpha_g(X_g) + e^{f(g)}\alpha_h(Y_h)\,,$$

  where $\oplus_{\mathsf{T}\mathcal{G}}$ is the partial multiplication in the tangent Lie groupoid $\mathsf{T}\mathcal{G} \rightrightarrows \mathsf{T}M$.
- The whole complication (e.g. the multiplicative function) comes from assuring that the $\mathbb{R}$-principal bundle is trivial.

Dazord's definition

- Our understanding of a contact groupoid is equivalent to the Dazord's:
  A contact groupoid is a Lie groupoid $\mathcal{G}_0 \rightrightarrows M$ equipped with a contact distribution $C^0 \subset \mathsf{T}\mathcal{G}_0$ which is closed with respect to the multiplication in the tangent groupoid $\mathsf{T}\mathcal{G}_0 \rightrightarrows \mathsf{T}M$.
- In other words a contact groupoid is a contact distribution on a Lie groupoid $\mathcal{G}_0$ which is simultaneously a Lie subgroupoid of the tangent groupoid.
- The relation with our definition is given by $\mathcal{G} = C^\times$ and $\mathcal{G}_0 = \mathcal{G}/\mathbb{R}^\times$.
- Starting with a Lie groupoid $\mathcal{G} \rightrightarrows M$ we can construct a canonical symplectic groupoid structure on $\mathsf{T}^*\mathcal{G} \rightrightarrows \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebroid of $\mathcal{G}$.
- We can also construct a canonical contact groupoid $\mathcal{C}(\mathcal{G}_0)$ as follows.

Canonical construction

- Let $\mathcal{G}$ be a Lie groupoid and consider the symplectic groupoid $\mathsf{T}^*\mathcal{G}$. The manifold of units is the dual $\mathfrak{g}^*$ of the Lie algebroid $\mathfrak{g}$ of $\mathcal{G}$, embedded into $\mathsf{T}^*\mathcal{G}$ as the conormal bundle ν^*M.
- $\mathsf{T}^*\mathcal{G}$ has a vector bundle structure compatible with the groupoid structure in the sense that homotheties $h_t(\theta_y) = t.\theta_y$ in the vector bundle $\mathsf{T}^*\mathcal{G} \to \mathcal{G}$ act as groupoid morphisms (it is a $\mathcal{VB}$-*groupoid*).
- The source and the target maps $s, t : \mathsf{T}^*\mathcal{G} \to \mathfrak{g}^*$ intertwine the homotheties in $\mathsf{T}^*\mathcal{G} \to \mathcal{G}$ with those in $\mathfrak{g}^* \to M$. It is now clear that removing the level sets $Z_s = s^{-1}(\{0\})$ and $Z_t = t^{-1}(\{0\})$ gives us an open-dense subgroupoid of $\mathsf{T}^*\mathcal{G}$:

$$\mathcal{C}(\mathcal{G}) = \mathsf{T}^*\mathcal{G} \setminus \{Z_s \cup Z_t\} \rightrightarrows (\mathfrak{g}^*)^{\times}$$

- In other words, $\mathcal{C}(\mathcal{G})$ consists of covectors from $\mathsf{T}^*\mathcal{G}$ which vanish on vectors tangent to source or target fibres.

Canonical construction

- Of course, being an open subgroupoid of $\mathsf{T}^*\mathcal{G}$ it is still a symplectic groupoid, but as the zero section of $\mathsf{T}^*\mathcal{G}$ has been removed and as $\mathcal{C}(\mathcal{G})$ remains $\mathbb{R}^\times$-invariant, the group $\mathbb{R}^\times$ acts on $\mathcal{C}(\mathcal{G})$ by non-zero homotheties in a free and proper way.
- The symplectic form remains homogeneous of degree 1 with respect to this action, so we are dealing with a contact groupoid. The contact groupoid $\mathcal{C}(\mathcal{G})$ is canonically associated with the groupoid $\mathcal{G}$ and will be called the canonical contact groupoid of $\mathcal{G}$.
- In the traditional picture, it should be viewed as the reduced groupoid $\mathcal{C}(\mathcal{G})/\mathbb{R}^\times$ which is an open-dense submanifold of the projectivization bundle $\mathcal{P}(\mathsf{T}^*\mathcal{G})$.

Theorem

Any contact groupoid $\mathcal{G}$ with $\mathcal{G}/\mathbb{R}^\times = \mathcal{G}_0$ has a realization as a contact subgroupoid of the canonical contact groupoid $\mathcal{C}(\mathcal{G})$.

Contact supergeometry

- Since the objects we have considered, like $\mathbb{R}^\times$-principal bundles, symplectic forms, one-forms, distributions, etc., make sense also in supergeometry we can consider contact supergeometry in a fully analogous way.
- The difference is that we can consider even contact super-structures as well as odd contact super-structures (e.g. on first jets of an odd line bundle).
- The even contact super-structures ale locally generated by even contact forms, the odd contact super-structures ale locally generated by odd contact forms.
- It seems that the concept of a 'mixed' contact structure does not make much sense.
- Note that a given $\mathbb{R}^\times$-principal bundle P over a supermanifold gives rise to an even line bundle and an odd line bundle (the transition functions are formally the same).

Contact super-forms

Definition

So formally, an even/odd contact super-structure on a supermanifold $\mathcal{M}$ is an $\mathbb{R}^\times$-principal bundle over $\mathcal{M}$ equipped with a homogeneous even/odd symplectic form.

Theorem

(a) Every even contact form α on a supermanifold $\mathcal{M}$ can be locally written as

$$\alpha = \mathrm{d}z - p_a\,\mathrm{d}x^a + \epsilon_j\theta^j\,\mathrm{d}\theta^j, \quad \epsilon_j = \pm 1$$

for certain local coordinates (z, x^a, p_b, θ^j) on $\mathcal{M}$ among which (z, x^a, p_b) are even and (θ^j) are odd.
(b) Every odd contact form α on a supermanifold $\mathcal{M}$ can be locally written as

$$\alpha = \mathrm{d}\xi - \theta^a\,\mathrm{d}x^a$$

for certain local coordinates (x^a, θ^b, ξ) on $\mathcal{M}$ among which (x^a) are even and (θ^b, ξ) are odd.

Symplectic $\mathbb{R}^\times$-principal 2-manifolds

- It is well known that with an $\mathbb{N}$-graded manifold, in particular a 2-manifold $\mathcal{M}$, we can associate a tower of fibrations

$$\mathcal{M} = \mathcal{M}_{(2)} \to \mathcal{M}_{(1)} \to \mathcal{M}_{(0)} \,,$$

  corresponding to the filtration $\mathcal{A}_{(0)}(\mathcal{M}) \subset \mathcal{A}_{(1)}(\mathcal{M}) \subset \mathcal{A}_{(2)} = \mathcal{A}(\mathcal{M})$ of the polynomial algebra on $\mathcal{M}$, where $\mathcal{A}_{(i)}$ is the subalgebra in $\mathcal{A}$ generated by polynomial functions of weight $\leq i$, $i = 0, 1, 2$.
- Hence, $\mathcal{M}_{(i)}$ is an i-manifold. In particular, $\mathcal{M}_{(0)}$ is an even manifold and $\mathcal{M}_{(1)} = F[1]$ is a vector bundle over $\mathcal{M}_{(0)}$ with odd fibers.
- In the case of an $\mathbb{R}^\times$-principal 2-manifold, the fibrations intertwine the $\mathbb{R}^\times$-action, so $\mathcal{M}_{(0)}$ and $\mathcal{M}_{(1)}$ are also principal $\mathbb{R}^\times$-bundles and $\mathcal{M} \to \mathcal{M}_{(1)} \to \mathcal{M}_{(0)}$ is a morphism of principal $\mathbb{R}^\times$-bundles.
- On symplectic $\mathbb{R}^\times$-principal 2-manifolds the symplectic form is homogeneous of degree two with respect to the $\mathbb{N}$-gradation and of degree one with respect to the principal structure.

Pseudo-Euclidean principal $\mathbb{R}^\times$-bundles

- If $(\mathcal{M},\omega)$ is a symplectic 2-manifold, i.e. the symplectic form of degree 2, the bundle $F[1]$ is a pseudo-Euclidean with the pseudo-scalar product $\langle\cdot,\cdot\rangle$ obtained as the Poisson bracket on functions of degree one.
- The pseudo-Euclidean product induces a Poisson structure of on the principal $\mathbb{R}^\times$-bundle $\mathcal{M}_{(1)} = F[1]$ which makes it into a linear odd principal Poisson $\mathbb{R}^\times$-bundle, as the pseudo-Euclidean product is an $\mathbb{R}^\times$-homogeneous linear odd Poisson bracket. Such linear odd principal Poisson $\mathbb{R}^\times$-bundles we will call pseudo-Euclidean principal $\mathbb{R}^\times$-bundles.
- As an example, note that if F is a purely even linear principal $\mathbb{R}^\times$-bundle over $\mathcal{M}$ with the linear $\mathbb{R}^\times$ action h^0, then $F\oplus_{\mathcal{M}} F^*$ is canonically a pseudo-Euclidean principal $\mathbb{R}^\times$-bundle with respect to action $\widehat{h}^0_t = h^0_t \oplus t\cdot\left(h^0_{t^{-1}}\right)^*$ and the pseudo-Euclidean product induced from the canonical pairing between F and F^*,

$$\langle X+\mu, Y+\nu\rangle = \frac{1}{2}\left(i_X\nu + i_Y\mu\right).$$

Constructing contact 2-manifolds

- This structure can be viewed as a reductions of the symplectic principal $\mathbb{R}^\times$-bundle $\mathsf{T}^*[2]F[1]$ of degree 2. If F is a pseudo-Euclidean principal $R^\times$-bundle, then the isometric bundle embedding

$$F[1] \to (F \oplus_{\mathcal{M}} F^*)[1]\,, \quad X \mapsto X + \langle X, \cdot \rangle\,,$$

  is simultaneously a morphism of linear principal $\mathbb{R}^\times$-bundles. Now we can take P to be the symplectic principal bundle of degree 2 being the pull-back of $\mathsf{T}^*[2]F[1]$, i.e. completing the commutative diagram of morphisms of linear principal Poisson $\mathbb{R}^\times$-bundles

$$\begin{array}{ccc} P & \longrightarrow & \mathsf{T}^*[2]F[1] \\ \downarrow & & \downarrow \\ F[1] & \longrightarrow & (F \oplus_{\mathcal{M}} F^*)[1] \end{array} \qquad (2)$$

- We get the following ‘contact variant’ of a Roytenberg’s result.

Courant algebroids

Theorem

Contact 2-manifolds are in one-to-one correspondence with pseudo-Euclidean principal $\mathbb{R}^\times$-bundles. The correspondence is given by the above construction.

- Courant algebroids, defined originally by Lie, Weinstein and Xu, have been recognized by Roytenberg as symplectic 2-manifolds equipped with a qubic Hamiltonian.
- To be more precise, let us start with a symplectic 2-manifold $(\mathcal{M}, \omega, h^1)$ with $\mathcal{M}_{(1)} = F[1]$, so that functions from $\mathcal{A}^1(\mathcal{M})$ (of weight 1) represent linear functions on F and any section e of F represents a linear function ι_e on F^*.
  Let $\{\!\{\cdot,\cdot\}\!\}_\mathcal{M}$ be the Poisson bracket associated with the symplectic form ω of weight 2. The symplectic Poisson bracket induces a nondegenerate pairing

$$\{\!\{\cdot,\cdot\}\!\}_\mathcal{M} : \mathcal{A}^1(\mathcal{M}) \otimes_{\mathcal{A}^0(\mathcal{M})} \mathcal{A}^1(\mathcal{M}) \to \mathcal{A}^0(\mathcal{M})$$

Courant algebroids

- Being of degree -1, the derived bracket is closed on $\mathcal{A}^1(\mathcal{M})$, so that it can be viewed as a Loday bracket, the *Courant-Dorfman bracket*, on sections of the vector bundle F which will be denoted, with some abuse of notation, also $\{\cdot,\cdot\}$.
- The bracket $\{\iota(e),f\}$ between functions $\iota(e)\in\mathcal{A}^1(\mathcal{M})$ and basic functions $f\in\mathcal{A}^0(\mathcal{M})=C^\infty(\mathcal{M}_{(0)})$ is a derivative with respect to f and corresponds to a vector bundle morphism $\rho:F\to\mathsf{T}\mathcal{M}_{(0)}$ (the anchor map),

$$\{e,fe'\}=f\{e,e'\}+\rho(e)(f)e'\,,$$

for all sections e,e' of F and all $f\in C^\infty(\mathcal{M}_{(0)})$.
- The structures $\{\cdot,\cdot\}$ and ρ, enriched with the pseudo-Euclidean product, form a so called Courant algebroid structure on F. Moreover, maps $\{\cdot,\cdot\}$, ρ, and $\langle\cdot,\cdot\rangle$ arise in this way if and only if the following identities hold true (version simplifying the original one):

$$\langle\{e,e'\},e'\rangle=\langle e,\{e',e'\}\rangle\,,\tag{3}$$

$$\rho(e)\langle e',e'\rangle=2\langle\{e,e'\},e'\rangle\,.\tag{4}$$

Contact Courant algebroids

- Our general philosophy forces the following definition *á la* Roytenberg:

Definition

A contact Courant algebroid is a contact 2-manifold P equipped with a an $\mathbb{R}^\times$-homogeneous cubic homological Hamiltonian $H \in \mathcal{A}^{(1,3)}(P)$, $\{\!\{H, H\}\!\} = 0$.

In this sense a contact Courant algebroid is an $\mathbb{R}^\times$-homogeneous Courant algebroid.

- To find a more 'classical' description of the contact Courant algebroid, note that the space $\mathcal{A}^{(1,1)}(P)$, as well as the space $\mathcal{A}^{(1,0)}(P)$ of homogeneous functions on $P_{(0)}$, is an $\mathcal{A}^{(0,0)}(P)$-module, where elements of $\mathcal{A}^{(0,0)}(P)$ represent functions on the base $M = P_{(0)}/\mathbb{R}^\times$ of the principal $\mathbb{R}^\times$-bundle $P_{(0)} \to M$. Homogeneous functions on $P_{(0)}$ represent, in turn, sections of the even line bundle $L = P^e_{(0)}$ induced by the $\mathbb{R}^\times$-principal bundle $P_{(0)}$.

Contact Courant algebroids

('Classical' definition) A contact Courant algebroid is a structure $(\mathcal{E}, L, \{\cdot,\cdot\}, \langle\cdot,\cdot\rangle, \rho)$ such that

(a) $\mathcal{E}$ is a vector bundle over M and L is a line bundle over M,

(b) $\{\cdot,\cdot\}$ is a Loday bracket on sections of $\mathcal{E}$,

(c) $\langle\cdot,\cdot\rangle$ is a pseudo-Euclidean product with values in L,

(d) $\rho : \mathcal{E} \to \mathsf{DO}^1(L, L)$ is a vector bundle morphism associating to any section e of $\mathcal{E}$ a first-order differential operator $\rho(e)$ from sections of L into sections of L,

(e) the identities

$$\langle\{e, e'\}, e'\rangle = \langle e, \{e', e'\}\rangle\,,$$
$$\rho(e)\langle e', e'\rangle = 2\langle\{e, e'\}, e'\rangle\,,$$

are satisfied for all sections e, e' of $\mathcal{E}$.

Back to Roytenberg

- Consider a contact Courant algebroid $(\mathcal{E}, L, \{\cdot,\cdot\}, \langle\cdot,\cdot\rangle, \rho)$ defined as above. We can always find a local trivializations of L and $\mathcal{E}$ with associated local affine coordinates (x^a, z) and (x^a, θ^i), respectively.
- The pseudo-Euclidean product with values in L is in the local trivialization of L a standard product with values in $\mathbb{R}$. Moreover, a basis (ε_i) of local sections of $\mathcal{E}$ can always be chose such that $\langle\varepsilon_i, \varepsilon_j\rangle = g_{ij}$ and the basic functions g_{ij} are constants. Since by construction $\theta^i(\varepsilon_j) = \delta^i_j$, our identification of sections with linear functions *via* the pseudo-Euclidean product yields $\varepsilon_i = g_{ij}\theta^j$.
- Let us write in local coordinates

$$\rho(e_i) = r_i^a(x)\partial_{x^a} + r_i(x)$$

and

$$\Theta(\varepsilon_i, \varepsilon_j, \varepsilon_k) = \langle\{\varepsilon_i, \varepsilon_j\}, \varepsilon_k\rangle = A_{ijk}(x)\,,$$

Since the product $\langle\cdot,\cdot\rangle$ is nondegenerate, the above formula defines the brackets $\{\varepsilon_i, \varepsilon_j\}$ uniquely. Note that $A_{ijk}(x)$ is totally skew-symmetric in (i, j, k).

Back to Roytenberg

- Out of this data we can construct a symplectic principal $\mathbb{R}^\times$ bundle P of degree 2. In local Darboux coordinates $(t, x^a, \theta^i, z, p_a)$ the symplectic form reads as

$$\omega = \mathrm{d}z\,\mathrm{d}t + t \cdot \left(\mathrm{d}p_a\,\mathrm{d}x^a + \frac{1}{2} g_{ij}\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j\right) .$$

- Note that, among local coordinates, t, x^a are of weight 0 thus even, θ^i are of weight 1 thus odd, and z, p_a are of weight 2 thus even, so that ω is of weight 2 and even. Moreover, with respect to the $\mathbb{R}^\times$-action, $x^a, \theta^i, , p_a, z$ are invariant (weight 0), and t, $t \neq 0$, is homogeneous of degree 1, so that ω is homogeneous of degree 1.
- The corresponding homogeneous cubic Hamiltonian is then of the form

$$H = t\theta^i\left(r_i^a(x)p_a + r_i(x)z\right) - \frac{t}{6} A_{ijk}(x)\theta^i\theta^j\theta^k .$$

- These are the data of the contact Courant algebroid in the Roytenberg's form.

Canonical example

Consider the contact 2-manifold $\mathsf{T}^*[2]\mathsf{T}[1](\mathbb{R}^\times \times M)$ for a purely even manifold M. As the cubic Hamiltonian H associated with the canonical vector field on $\mathsf{T}[1](\mathbb{R} \times M)$ (being the de Rham derivative) is 1-homogeneous, we obtain a homogeneous Courant bracket on the linear principal $\mathbb{R}^\times$-bundle $F = \mathsf{T}(\mathbb{R}^\times \times M) \oplus_{(\mathbb{R}^\times \times M)} \mathsf{T}^*(\mathbb{R}^\times \times M)$, so that $\mathcal{E} = (\mathbb{R} \times \mathsf{T}M) \oplus_M (\mathbb{R}^* \times \mathsf{T}^*M)$. Sections of the latter are of the form $(X, f) + (\alpha, g)$. The contact Courant algebroid structure on $\mathcal{E}$ consists of

(a) the Loday bracket on sections of $\mathcal{E}$ of the form

$$\begin{aligned}\{(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)\} &= ([X_1, X_2], X_1(f_2) - X_2(f_1)) \\ &+ (\mathcal{L}_{X_1}\alpha_2 - i_{X_2}\,\mathrm{d}\alpha_1 + f_1\alpha_2 - f_2\alpha_1 + f_2\,\mathrm{d}g_1 + g_2\,\mathrm{d}f_1\ , \\ &X_1(g_2) - X_2(g_1) + i_{X_2}\alpha_1 + f_1 g_2)\ ,\end{aligned}$$

(b) the pseudo-Euclidean product of the form

$$\langle (X, f) + (\alpha, g), (X, f) + (\alpha, g)\rangle = \alpha(X) + fg\ ,$$

(c) the vector bundle morphism $\rho : \mathcal{E} \to \mathsf{DO}^1(M)$ of the form

$$\rho\left((X, f) + (\alpha, g)\right) = X + f\ .$$

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THANK YOU FOR YOUR ATTENTION!