Secondary Calculus

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XXII IWDGTM
Będlewo, August 19–26, 2007
Secondary Calculus generalizes standard calculus on manifolds to the (functional) space of solutions of a given PDE by using only differential geometry and homological algebra in the environment of an infinite jet manifold. In a sense

Secondary = Functional, Variational

Thus, Secondary Calculus has relevant applications to Physics and, in particular, Field Theory. As an example the geometry of the Covariant Phase Space may be formalized within Secondary Calculus.
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Differential Calculus may be formalized over any associative, unitary, (graded) commutative algebra $A$ over a ring $R$. In the case $A = C^\infty(M)$ the theory reduces to standard calculus on manifolds.

**Example**

Let $P, Q \in \text{Mod}_A$. A $k$–th order, $Q$–valued, differential operator over $P$ is defined to be any $R$–linear operator $\Box : P \rightarrow Q$ such that

$$[a_0, [a_1, [\cdots [a_k, \Box] \cdots ]]]) = 0, \quad \forall a_0, a_1, \ldots, a_k \in A.$$ 

$$\text{Diff}_k(P, Q) = \{ \Box : P \rightarrow Q \mid \Box \text{ is a } k\text{–th order differential operator} \}$$

**Remark**

$\text{Diff}_k(P, Q)$ is an $A$–module.
Remark

\[ \text{Mod}_A \ni Q \mapsto \text{Diff}_k(P, Q) \in \text{Mod}_A \]

is a functor and a representable one. I.e., for any \( Q \) there is a canonical isomorphism of modules

\[ \text{Diff}_k(P, Q) \cong \text{Hom}_A(\mathcal{J}^k(P), Q) \]

with \( \mathcal{J}^k(P) = \{ k - \text{jets of elements in } P \} \).

A number of functors may be similarly introduced in \( \text{Mod}_A \). An object of differential calculus is any among these functors, their repr. objects, etc.

Example

The de Rham complex \( 0 \to A \xrightarrow{d} \Lambda^1(A) \xrightarrow{d} \cdots \to \Lambda^q(A) \xrightarrow{d} \cdots \) is an object of differential calculus.
Remark

The solution space \( M \) of a PDE may be understood as the space of maximal integral submanifold of a distribution.

\[
\begin{align*}
(\ldots, x^\mu, \ldots, u^j_\sigma, \ldots) & \in J^k_\pi \\
(\ldots, x^\mu, \ldots, u^j, \ldots) & \in E \\
(\ldots, x^\mu, \ldots) & \in M
\end{align*}
\]

The Cartan distribution over \( J^\infty_\pi \) restricts to \( \mathcal{E} \)

\[
\mathcal{C} = \langle \ldots, D^\mathcal{E}_\mu, \ldots \rangle, \quad D^\mathcal{E}_\mu = \left( \frac{\partial}{\partial x^\mu} + u^j_\sigma + \mu \frac{\partial}{\partial u^j_\sigma} \right)|_{\mathcal{E}}
\]

Remark

\( \{ \text{solutions to } \mathcal{E} \} = \{ \text{maximal integral submanifolds of } (\mathcal{E}, \mathcal{C}) \} !!! \)
The tangent bundle to a differential equation splits \( T\mathcal{E} \cong \mathcal{C} \oplus \mathcal{V}\mathcal{E} \).

**The Variational Bi–Complex and the \( \mathcal{C}–\)Spectral Sequence**

\[
\begin{array}{cccccc}
\ldots & \xrightarrow{d} & \Lambda^{q,p+1}(\mathcal{E}) & \xrightarrow{\bar{d}} & \Lambda^{q+1,p+1}(\mathcal{E}) & \xrightarrow{\bar{d}} & \ldots \\
\uparrow{dv} & & \uparrow{dv} & & \uparrow{dv} & & \uparrow{dv} \\
\ldots & \xrightarrow{d} & \Lambda^{q,p}(\mathcal{E}) & \xrightarrow{\bar{d}} & \Lambda^{q+1,p}(\mathcal{E}) & \xrightarrow{\bar{d}} & \ldots \\
\end{array}
\]

\[
\{(CE_r(\mathcal{E}), d_r)\}_r \quad (CE_0(\mathcal{E}), d_0) \quad (\Lambda(\mathcal{E}), \bar{d})
\]

\[
CE^p,q_0(\mathcal{E}) \cong \Lambda^{q,p}(\mathcal{E}) \cong C^p \Lambda^p(\mathcal{E}) \otimes \bar{\Lambda}^q(\mathcal{E})
\]

\[
C^p \Lambda^p(\mathcal{E}) = \langle \ldots, i_\mathcal{E}^* (\omega^j_{\sigma_1} \wedge \ldots \wedge \omega^j_{\sigma_p}), \ldots \rangle, \quad \omega^j_\sigma = du^j_\sigma - u^j_{\sigma+\mu} dx^\mu
\]

\[
\bar{\Lambda}^q(\mathcal{E}) = \langle \ldots, i_\mathcal{E}^* (\bar{d}x^{\mu_1} \wedge \ldots \wedge \bar{d}x^{\mu_q}), \ldots \rangle, \quad \bar{d}x^\mu = dx^\mu.
\]
\( \mathcal{C} \) determines a “horizontal” differential calculus on \( \mathcal{E} \).

**Definition**

Let \( P, Q \in \text{Mod}_{\mathcal{F}(J^\infty \pi)} \) be modules of sections of vector bundles over \( J^\infty \pi \). A linear \( \mathcal{C} \)–diff. operator \( \Box : P \to Q \) is one locally in the form

\[
\Box(p) = \Box_{aA}^\sigma (D_{\sigma} p^a) \xi^A, \quad p = p^a e_a, \quad D_{\sigma} = D^\sigma_1 \circ \cdots \circ D^\sigma_n, \quad \Box_{aA}^\sigma \in \mathcal{F}(\mathcal{E})
\]

\( \ldots, e_a, \ldots \) a local basis of \( P; \quad \ldots, \xi^A, \ldots \) a local basis of \( Q \).

Any \( \mathcal{C} \)–differential operator \( \Box \) restricts to \( \mathcal{E} \): \( \Box \mapsto \Box_\mathcal{E} \).

Horizontal jet–spaces may be also defined. \( P \) module of sections of a vector bundle over \( \mathcal{E} \Rightarrow \overline{J}^k(P) \) module of horizontal jets of elements in \( P \), \( k \leq \infty \).

\[
\Box : P \to Q \quad \Rightarrow \quad \overline{\Psi}_\Box : \overline{J}^k(P) \to Q \quad \Rightarrow \quad \overline{\Psi}_\Box^\infty : \overline{J}^\infty(P) \to \overline{J}^\infty(Q)
\]
Adjoint Operators

$P$ – module of sections of a vector bundle over $\mathcal{E}$.

$P^* = \text{Hom}(P, \mathcal{F}(\mathcal{E}))$ – dual module.

$\hat{P} = P^* \otimes \Lambda^n(\mathcal{E})$ ($n = \dim M$) – adjoint module.

$\Box : P \to Q$ a $\mathcal{C}$–diff. operator $\Rightarrow \widehat{\Box} : \hat{Q} \to \hat{P}$ the adjoint operator, i.e.,

$$
\widehat{\Box}(\hat{q}) = (-1)^{|\sigma|} D_{\sigma}(\square_{aA}^\sigma \hat{q}^A)(e^a \otimes d^n x)
$$

$$
\hat{q} = \hat{q}^A(\varepsilon_A \otimes d^n x), \quad d^n x = dx^1 \wedge \cdots \wedge dx^n
$$

$\ldots, e^a, \ldots$ and $\ldots, \varepsilon_A, \ldots$ dual bases of $\ldots, e_a, \ldots$ and $\ldots, \varepsilon^A, \ldots$

For $\Delta : P \to Q$ and $\nabla : Q \to R$ $\mathcal{C}$–differential operators,

$$
\widehat{\Delta} = \Delta \text{ and } \widehat{\nabla} \circ \widehat{\Delta} = \widehat{\Delta} \circ \widehat{\nabla}.
$$
What is Differential Calculus?

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A Geometric Setting for Functional Spaces
The \( C \)-Spectral Sequence
\( C \)-differential operators
\( C \)-connections and \( C \)-modules

\( P \) – module of sections of a vector bundle \( O \to E \) over \( E \)

Algebraic Definition

A \( C \)-connection in \( P \) is an \( F(E) \)-linear corresp. \( \nabla : C \supset X \mapsto \nabla_X \), such that \( \nabla_X : P \to P \) is a der-operator (covariant derivative) over \( X \), i.e.

\[
\nabla_X(fp) = f\nabla_X p + X(f)p, \quad f \in F(E), \ p \in P.
\]

A \( \nabla \) is flat if \([\nabla_X, \nabla_Y] = \nabla_{[X,Y]}\), \( X, Y \subset C \)

Geometric Definition

A \( C \)-connection in \( O \) is an \( n \)-dimensional hor. distribution \( C(O) \) over \( O \) which projects isomorphically onto \( C \). \( C(O) \) is flat if it is involutive.

Definition

The module \( P \) of sections of a vector bundle over \( E \) endowed with a flat \( C \)-connection is called a \( C \)-module.
Example 1
the module of vertical vector fields $VD(\mathcal{E})$:

$$\nabla_X Z = [X, Z]^V, \quad X \subset \mathcal{C}, \ Z \in VD(\mathcal{E}).$$

Example 2
the module of Cartan $p$–forms $C^p \Lambda^p(\mathcal{E})$

$$\nabla_X \omega = L_X \omega, \quad X \subset \mathcal{C}, \ \omega \in C^p \Lambda^p(\mathcal{E}).$$

Example 3
$\Delta : P \to P_1$ a $\mathcal{C}$–differential operator and $\Psi^\Delta_{\infty} : J^\infty(P) \to J^\infty(P_1)$ the associated horizontal jet prolongation. $R_\Delta = \ker \Psi^\Delta_{\infty} \subset J^\infty(P)$:

$$\nabla_X (f \cdot j^\infty(p)) = X(f) \cdot j^\infty(p), \quad X \subset \mathcal{C}, \ f \in \mathcal{F}(\mathcal{E}), \ p \in P.$$
There is a de Rham–like complex associated with a $C$–module $P$

$$\cdots \rightarrow P \otimes \Lambda^q(E) \xrightarrow{\overline{d}_P} P \otimes \Lambda^{q+1}(E) \xrightarrow{\overline{d}_P} \cdots \quad (\star)$$

**Definition**

The graded cohomology vector space of $(\star)$, $\overline{H}^\bullet(P)$, is called the **horizontal cohomology** space of $P$. $\overline{d}_P$ is a $C$–differential operator.

**Example**

$P = C^p \Lambda^p(E) \Rightarrow \overline{d}_P = \overline{d}$, $(\star)$ is the $p$–th row of the variational bi–complex and $\overline{H}^\bullet(C^p \Lambda^p(E)) = CE^{p,\bullet}_1(E)$.

A connection in a bundle $\pi$ over a manifold $M$, $P = \Gamma(\pi)$ may be used to **integrate** a (suitably supported) element $p \in P \otimes \Lambda^q(M)$ over a $q$–fold $\gamma \subset M$. Similarly, the flat $C$–connection in a $C$–module $P$ may be used to **integrate** an element $p \in P \otimes \Lambda^q(E)$ over an integral $q$–fold of $C$. 
Horizontal Calculus on a PDE: Summary

On the infinite prolongation $E$ of a differential equation $E_0$, the Cartan distribution $\mathcal{C}$ determines

- the space $M$ of solutions of $E_0$;
- the class of $\mathcal{C}$–modules, $P$,
- the class of associated de Rham–like complexes, $(P \otimes \Lambda^\bullet(E), \bar{d}_P)$,
- the class of horizontal cohomology spaces, $\bar{H}^\bullet(P)$.
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Universal Linearization
Secondary First Order Calculus
The Secondarization Scheme
The Secondary Exterior Differential

\[ \mathcal{E} \xrightarrow{i_{\mathcal{E}}} J^{\infty, \pi} \xrightarrow{F} J^{\infty, \nu} \]
\[ \mathcal{E}_0 \xrightarrow{F_0} V \]
\[ \pi_k \quad \nu \quad M \]
\[ \ell^E_F(\chi) = (\frac{\partial F_\alpha}{\partial u_j} D_\sigma \chi^j)[x,u] = 0 \]
\[ \chi = (\ldots, \chi^j[x,u], \ldots) \]

Proposition
\[ H^0(\mathcal{V}D(\mathcal{E})) \simeq \ker \ell^E_F. \]
$H^0(\text{VD}(\mathcal{E}))$ is a space of (local) vector fields on $M$.

**Definition**

$$D(M) = \overline{H}^\bullet(\text{VD}(\mathcal{E})) = \{\text{secondary vector fields on } M\}.$$

Another example: elements in $\overline{H}^n(C^p\Lambda^p(\mathcal{E})) \simeq CE_1^{p,n}(\mathcal{E})$ identify with variational $p$–forms on $\mathcal{E}$. For $p = 0$, $[\omega] \in \overline{H}^q(\mathcal{F}(\mathcal{E})) \simeq CE_1^{0,q}(\mathcal{E})$ identifies with the functional

$$M \ni x \mapsto \int_{\gamma} j^\infty_*(x)(\omega), \quad \gamma \text{ a suitable } q\text{–fold in } M.$$

$\overline{H}^q(C^p\Lambda^p(\mathcal{E}))$ is a space of (local) differential $p$–forms on $M$.

**Definition**

$$\Delta^p(M) = \overline{H}^\bullet(C^p\Lambda^p(\mathcal{E})) \simeq CE_1^{p,\bullet}(\mathcal{E}) = \{\text{secondary } p\text{–forms on } M\}.$$

All standard operations with vector fields and forms have secondary analogues, defined in purely algebraic (and homological) way!
So far I defined

- \( M = \{\text{secondary points}\} = \{\text{solutions } x \text{ of } E_0\}, \)
- \( D(M) = \{\text{secondary vector fields}\} = \overline{H}^\bullet(V D(E)), \)
- \( \Lambda^\bullet(M) = \{\text{secondary differential forms}\} = \overline{H}^\bullet(C \cdot \Lambda^\bullet(E)). \)

**Secondaryization Principle**

The secondary version \( \Phi(M) \) of an object \( \Phi \) of differential calculus is the horizontal cohomology of the \( C \)–"module" of vertical analogues of "elements" in \( \Phi \).

The following **Secondaryization Scheme** may be used to define \( \Phi(M) \):

1. Define a vertical version \( V \Phi(E) \) of \( \Phi \) over \( E \),
2. Note that \( V \Phi(E) \) has got a canonical \( C \)–"module" structure,
3. Put \( \Phi(M) = \overline{H}^\bullet(V \Phi(E)). \)
Apply the Secondarization Scheme to the de Rham Complex.

1. The vertical version of the de Rham complex over $E$ is the vertical de Rham complex $\cdots \to C^p \Lambda^p(E) \xrightarrow{d^V} C^{p+1} \Lambda^{p+1}(E) \xrightarrow{d^V} \cdots$

2. $C^\bullet \Lambda^\bullet(E)$ has a $C$–module structure and $d^V$ is compatible with it, i.e., it extends to the var. bi–complex $(C^\bullet \Lambda^\bullet(E) \otimes \overline{\Lambda}^\bullet(E), d, d^V)$.

3. Put $\Lambda^p(M) = \overline{H}^\bullet(C^p \Lambda^p(E)) \simeq CE^p_1(\bullet)(E)$. moreover $d^V$ passes in horizontal cohomology, giving a complex (secondary de Rham complex) $\cdots \to \Lambda^p(M) \xrightarrow{d} \Lambda^{p+1}(M) \xrightarrow{d} \cdots$.

$(\Lambda^\bullet(M), d) = (CE^\bullet_1(\bullet)(E), d^\bullet_1(\bullet))$.

Calculus of variations is an aspect of Secondary Calculus. Put $\mathcal{E} = J^\infty \pi$.

Then $M = \{\text{sections of } \pi\}$. $S = [L] \in \overline{H}^n(\mathcal{F}(\mathcal{E})) \subset C^\infty(M)$ is an action functional: $L = \mathcal{L}[x, u] d^n x$, $S \simeq \int \mathcal{L}[x, u] d^n x$ and

$$dS = [d^V L] \equiv (\ldots, (-1)^\sigma D_\sigma(\frac{\partial \mathcal{L}}{\partial u_\sigma}), \ldots) \in \Lambda^1(M).$$
Theorem [Goldschmidt]

Let $\Delta : P \to P_1$ be a $C$–differential operator. There exists a formal resolution of $\ker \Delta$, i.e. a formally exact complex (compatibility complex) of $C$–diff. operators $P \xrightarrow{\Delta} P_1 \xrightarrow{\Delta_1} \cdots \to P_q \xrightarrow{\Delta_q} \cdots$, i.e. such that the sequence $\overline{J}^\infty(P) \xrightarrow{\overline{\Psi}_\Delta^\infty} \overline{J}^\infty(P_1) \xrightarrow{\overline{\Psi}_{\Delta_1}^\infty} \cdots \to \overline{J}^\infty(P_q) \xrightarrow{\overline{\Psi}_{\Delta_q}^\infty} \cdots$ is exact.

Theorem [Spencer]

Horizontal cohomologies of $R_\Delta = \ker \overline{\Psi}_\Delta^\infty$ are isomorphic to cohomologies of any compatibility complex of $\Delta$.

Corollary

Horizontal cohomologies of $R^*_\Delta$ are isomorphic to homologies of any adj. complex $\hat{P} \xleftarrow{\hat{\Delta}} \hat{P}_1 \xleftarrow{\cdots} \hat{P}_{q-1} \xleftarrow{\hat{\Delta}_q} \hat{P}_q \xleftarrow{\cdots}$ of a compat. complex of $\Delta$. 
The length of a compatibility complex of $\Delta$ measures the “degree of overdeterminacy” of the equation $\Delta(p) = 0$.

**Proposition**

$VD(\mathcal{E}) \cong R_{\ell_F^\mathcal{E}}$ and therefore hor. cohom. of $VD(\mathcal{E})$ is isomorphic to cohom. of a compatibility complex $\mathcal{E} \xrightarrow{\ell_F^\mathcal{E}} P_1 \xrightarrow{\Delta_1} \cdots \xrightarrow{} P_q \xrightarrow{\Delta_q} \cdots$.

Then, if equation $\ell_F^\mathcal{E}(\chi) = 0$ is not overdetermined

$$
\overline{H}^q(VD) \cong \begin{cases}
\text{ker } \ell_F^\mathcal{E} & \text{if } q = 0 \\
\text{coker } \ell_F^\mathcal{E} & \text{if } q = 1 \\
0 & \text{if } q > 1
\end{cases},
\overline{H}^q(C^1 \Lambda^1) \cong \begin{cases}
\text{coker } \widehat{\ell}_F^\mathcal{E} & \text{if } q = n \\
\text{ker } \widehat{\ell}_F^\mathcal{E} & \text{if } q = n - 1 \\
0 & \text{if } q < n - 1
\end{cases}.
$$

If $\mathcal{E} = J^\infty \pi$, then $D(M) \cong \kappa$ and $\Lambda^1(M) \cong \widehat{\kappa}$. 
At the moment Secondary Calculus only deals with local functionals!

**Example**

Multilocal functionals are not represented in Secondary Calculus

\[ \int \cdots \int \mathcal{L} [x_1, \ldots, x_r, u_1, \ldots, u_r] d^n x_1 \cdots d^n x_r \]

**Example**

Feynman–like functionals are not represented in Secondary Calculus

\[ \exp i \int \mathcal{L} [x, u] d^n x \]

The space of secondary functions is still too small!
A Lagrangian field theory is a bundle $\pi : E \to M$ together with an action $S = [L] \in \overline{H}^n(\mathcal{F}(J^\infty \pi))$, $L \in \overline{\Lambda}^n(J^\infty \pi)$ is a lagrangian density and $\mathcal{E}_0 : dS = 0$ the associated Euler–Lagrange equations of motion.

**Definition**

The **Covariant Phase Space $\mathcal{P}$** of the lagrangian field theory $(\pi, S)$ is the space of solutions of $\mathcal{E}_0$, i.e., $\mathcal{P} = \{\text{max. int. subman. of } (\mathcal{E}, C)\}$.

**Proposition [Zuckerman]**

There exists a canonical, closed (secondary) 2–form on $\mathcal{P}$.

**Proof.** $dS = [d^V L] \in \hat{\mathcal{X}} \hookrightarrow C^1 \Lambda^1 \otimes \overline{\Lambda}^n$. Thus, $d^V L - dS = \overline{d}\theta$, for some $\theta \in C^1 \Lambda^1 \otimes \overline{\Lambda}^{n-1}$. Put $\omega = i^*_\mathcal{E}(d^V \theta) \in C^2 \Lambda^2(\mathcal{E}) \otimes \overline{\Lambda}^{n-1}(\mathcal{E})$, and note that

$$\overline{d}\omega = 0 \quad \Rightarrow \quad \omega = [\omega] \in \overline{H}^{n-1}(C^2 \Lambda^2(\mathcal{E})) \subset \Lambda^2(\mathcal{P})$$

does only depend on $S$. Moreover, $d\omega = [d^V \omega] = 0$. 
The first Noether theorem has a formulation in terms of \((P, \omega)\)!

Let \(\chi \in \mathfrak{g}\) be a Noether symmetry of \((\pi, S)\), i.e., \(L_\chi S = 0\). \(\chi\) is, in particular, an infinitesimal symmetry of \(E_0\), i.e.,

\[
X = \chi|_E \in \ker \ell^E_{dS} \simeq \overline{H}^0(VD(E)) \Rightarrow X \in D(P)
\]

According to first Noether theorem there exists an associated conservation law

\[
f \in \overline{H}^{n-1}(\mathcal{F}(E)) \Rightarrow f \in C^\infty(P)
\]

**Proposition [LV]**

\[
df = -i_\chi \omega.
\]

Similar to hamiltonian mechanics!
The second Noether theorem has a formulation in terms of \((P, \omega)\).

Define \(\Gamma : D(P) \ni X \mapsto i_X \omega \in \Lambda^1(P)\).

In hamilt. mech.: degeneracy distrib. of presympl. form = \langle \text{gauge symm.} \rangle

\(\ker \Gamma \) (= degeneracy distribution of \(\omega\)) = \langle \text{gauge symmetries of } (\pi, S) \rangle?

---

**Standard Definition**

A *local* (or gauge) *symmetry* of \((\pi, S)\) is a \(C\)–differential operator \(G : P \to \mathfrak{g}\), such that \(G(p)\) is a Noether symmetry for any \(p \in P\).

---

**Remark**

\(\text{im } G \subset \ker \Gamma\).

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**(Natural) Definition**

A *gauge symmetry* of \((\pi, S)\) is an element in \(\ker \Gamma\).
In hamilt. mechanics: gauge symmetries ⇔ first class constraints

\[ \ell^\mathcal{E} \text{d}S = \hat{\ell}^\mathcal{E} \text{d}S \Rightarrow \text{If the eq. } \ell^\mathcal{E} \text{d}S(\chi) = 0 \text{ is overdetermined then it is also underdetermined (i.e. constrained). Let } \chi \xrightarrow{\ell^\mathcal{E} \text{d}S} \hat{\chi} \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} \ldots \text{ be a non trivial compat. complex and } \hat{\chi} \xleftarrow{\ell^\mathcal{E} \text{d}S} \chi \xleftarrow{\hat{\Delta}_1} P_2 \xleftarrow{\ldots} \text{ the adjoint complex.} \]

\[ \ell^\mathcal{E} \text{d}S \circ \hat{\Delta}_1 = 0. \]

**Theorem [LV]**

\[ \ker \Gamma = \operatorname{im} \hat{\Delta}_1. \]

**Corollary [LV]**

\( \omega \) is non–degenerate iff \( \ell^\mathcal{E} \text{d}S(\chi) = 0 \) is a non–constrained eq.
Suppose that $\omega$ is non-degenerate $\implies$ (as in hamiltonian mechanics) there are brackets $\{\cdot,\cdot\}$ on the space of secondary functions on $P$.

Let $f \in C^\infty(P)$. Put $X_f = \Gamma^{-1}(df) \in D(P)$.

Let $f, g \in \overline{H}^{n-1}(\mathcal{F}(\mathcal{E})) \subset C^\infty(P)$ and $H \in \overline{H}^n(\mathcal{F}(\mathcal{E})) \subset C^\infty(P)$. Put

$$\{f, g\}_0 = -L_{X_f}g, \quad \{f, H\}_1 = -L_{X_f}H.$$

**Proposition [LV]**

$(\overline{H}^{n-1}(\mathcal{F}(\mathcal{E})), \{\cdot,\cdot\}_0)$ is a Lie algebra and $(\overline{H}^n(\mathcal{F}(\mathcal{E})), \{\cdot,\cdot\}_1)$ an its representation.

**Theorem [Barnich–Henneaux–Schomblond]**

$\{\cdot,\cdot\}_0$ coincide with the Peierls bracket between conservation laws.
In hamilt. mech. gauges are quotiented out via symplectic reduction. How to define a secondary symplectic reduction?

Geometric Definition of Degeneracy Distribution

Let $X = [X] \in \overline{H}^0(VD(\mathcal{E})) \subset D(P)$ be a gauge symmetry. $X \in VD(\mathcal{E})$ is a standard vector field over $\mathcal{E}$. Put $\mathcal{G} = \langle X \mid [X] \in \ker \Gamma \rangle$ and $\tilde{\mathcal{C}} = \mathcal{C} + \mathcal{G}$.

Conjecture 1

$(\mathcal{E}, \tilde{\mathcal{C}})$ is (locally) isomorphic to the infinite prolongation of a PDE $\tilde{\mathcal{E}}_0$.

Put $\tilde{P} = \{\text{solutions of } \tilde{\mathcal{E}}_0\}$. There is a morphism $p^*: \Lambda^\bullet(\tilde{P}) \rightarrow \Lambda^\bullet(P)$.

Conjecture 2 on Secondary Symplectic Reduction

There exists a unique secondary 2–form $\tilde{\omega}$ on $\tilde{P}$ such that $p^*(\tilde{\omega}) = \omega$ and $\tilde{\omega}$ has zero degeneracy distribution.
### Secondary Calculus

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Geometry of the Covariant Phase Space


