Alternative approaches to asymptotic behaviour of eigenvalues of some unbounded Jacobi matrices

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Abstract

In this article we calculate the asymptotic behaviour of the point spectrum for some special self-adjoint unbounded Jacobi operators J acting in the Hilbert space $l^2 = l^2(\mathbb{N})$. For given sequences of positive numbers λ_n and real q_n the Jacobi operator is given by $J = SW + WS^* + Q$, where $Q = \mathrm{diag}(q_n)$ and $W = \mathrm{diag}(\lambda_n)$ are diagonal operators, S is the shift operator and the operator J acts on the maximal domain. We consider a few types of the sequences $\{q_n\}$ and $\{\lambda_n\}$ and present three different approaches to the problem of the asymptotics of eigenvalues of various classes of J's. In the first approach to asymptotic behaviour of eigenvalues we use a method called successive diagonalization, the second approach is based on analytical models that can be found for some special J's and the third method is based on an abstract theorem of Rozenbljum.
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1. Introduction

In this work we calculate the point spectrum or the asymptotic behaviour of the point spectrum for some special unbounded Jacobi operators J acting in the Hilbert space $l^2 = l^2(\mathbb{N})$. We shall present three different approaches to the problem of the asymptotics of eigenvalues of various classes of J's. Let $\{e_n\}_{n=1}^\infty$ be the standard orthonormal basis in l^2. S stands for the shift operator on l^2 given on the basis vectors as follows $Se_n = e_{n+1}$, $n \geqslant 1$. For given sequences of positive numbers λ_n and real q_n one defines the Jacobi operator by $J = SW + WS^* + Q$, where Q and W are the associated diagonal operators, $Qe_n = q_n e_n$ ($Q = \mathrm{diag}(q_n)$) and $We_n = \lambda_n e_n$ ($W = \mathrm{diag}(\lambda_n)$) for $n \geqslant 1$ and the operator J acts on the maximal domain. We call λ_n's the weights of J. It turns that in the case $\lim \inf_{n \to \infty} q_n^2 (\lambda_n^2 + \lambda_{n-1}^2)^{-1} > 2$ and $\lim_{n \to \infty} |q_n| = +\infty$ the operator J is self-adjoint and its spectrum is discrete [9].

As far as we know the problem of finding the asymptotics of eigenvalues of Jacobi matrices has not been systematically studied in a general situation. However, the density of eigenvalues of rational Jacobi matrices was analysed by Dehesa in [2,3]. Let us also mention that there are special Jacobi matrices which appear in quantum optics with known

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1   asymptotics of their eigenvalues [16]. In [11] the asymptotics of eigenvalues for a special class of Jacobi matrices was
    found. Moreover, the method proposed in [11] (so called successive diagonalization method) can be applied for a quite
3   general set of *J*'s (Theorem 2.2, Section 2 of this paper). The method presented in Theorem 2.2 resembles the idea
    used in [14] in his studies of the asymptotic of eigenvalues of some pseudo-differential operators on the unit circle. By
5   using his abstract result [14, Theorem 1] we can find asymptotics of eigenvalues of a certain *J* when Theorem 2.2 does
    not apply (see Section 4). Our application of Rozenbljum result is rather simple, and we hope to find more interesting
7   one in the future.

    The third approach we propose here concerns special classes of Jacobi matrices for which analytical model can be
9   found. Then we deal with differential operator $\tilde{J}$ (unitarily equivalent to *J*) of the first order acting in a Hilbert space of
    holomorphic functions. This method is rather limited to special *J*'s but the results give very sharp asymptotical formulae
11 (exponential decays of errors). Let us mention that the method of successive diagonalization has been recently applied
    in [1] by Boutet de Monvel et al. to find the asymptotics of eigenvalues of a modified Jaynes–Cummings model.
13 This paper only deals with Jacobi matrices with discrete spectra. There is already a large literature devoted to infinite
    self-adjoint Jacobi matrices and their relation to the theory of orthogonal polynomials. In recent years several methods
15 were found for studies of spectral properties of general Jacobi matrices in $l^2(\mathbb{N})$ and $l^2(\mathbb{Z})$, see [5,4,8–11]. Finally, let
    us also mention the recent monograph of Teschl [15].
17 The paper is organized as follows. In Section 2 the method of successive diagonalization is briefly described. Section
    3 contains several natural examples of Jacobi matrices for which analytical models can be found and uses them to
19 compute precise asymptotic behaviour of their eigenvalues. Finally, in Section 4 simple applications of the Rozenbljum
    abstract theorem are given.

21 **2. Successive diagonalization method**

    The idea of successive diagonalization was introduced for a special class of Jacobi matrices in [11]. It is based on
23 the following general lemma:

    Lemma 2.1 (*Lemma 2.1, Janas and Naboko [11]*). *Let D be a self-adjoint diagonal operator in a Hilbert space H*
25 *given by $De_n = \mu_n e_n$, where $\{e_n\}$ is an orthonormal basis in H and simple eigenvalues $\mu_n \to \infty$ are ordered by*
    $|\mu_k| \leqslant |\mu_{k+1}|$. *Assume that $|\mu_i - \mu_k| \geqslant \varepsilon_0$ if $i \neq k$. If R is compact (not necessary self-adjoint) operator in H then the*
27 *operator $T = D + R$ has discrete spectrum which consists of complex eigenvalues $\lambda_n(T)$ and $\lambda_n(T) = \mu_n + \mathrm{O}(\|R^* e_n\|)$*
    for large n.

29 Successive diagonalization method was applied in [11] to a special class of Jacobi operators. It was used there in
    three steps. Then every next step allows to obtain the asymptotics of the eigenvalues of Jacobi operator more precisely
31 then the previous one. In this article we apply this method using only one step, but it gives (we hope) satisfactory
    explanation of the main idea of successive diagonalization.
33 Let $\Sigma_{1/p}$ $(p > 0)$ denote the set of compact operators such that its singular numbers s_k satisfy $s_k = \mathrm{O}(1/k^p)$. By
    $\Sigma_{1/p}^b$ we mean the subset of $\Sigma_{1/p}$ of operators in l^2 which possess a band type matrix form in the basis $\{e_n\}_{n=1}^\infty$.

35 **Theorem 2.2.** *Let*

$$J_1 = Q + SW + WS^*, \tag{2.1}$$

    where $Q = \mathrm{diag}(q_n)$, $W = \mathrm{diag}(\lambda_n)$ and assume that

37

    (i) $q_n = \rho n^\alpha (1 + \Delta_n)$, $\Delta_n \to 0$, $\rho \in \mathbb{R}\backslash\{0\}$;
39      $\lambda_n = n^\beta (1 + w_n)$, $w_n \to 0$;
    (ii) $\beta \geqslant 0$, $\alpha > 2\beta + 1$;
41 (iii) $\Delta_{n+1} - \Delta_n = \mathrm{o}(1/n)$.

    Then the eigenvalues $\lambda_n(J_1)$ of J_1 satisfy $\lambda_n(J_1) = q_n + \mathrm{O}(1/n^{\alpha-2\beta-1})$ for large n.

Proof. Following the method introduced in [11] we look for the diagonal operators Λ_1, $\Lambda_2 \in \Sigma^b_{1/(\alpha-\beta-1)}$ such that

$$X := (I + S\Lambda_1 + \Lambda_2 S^*)J_1 - Q(I + S\Lambda_1 + \Lambda_2 S^*) \in \Sigma^b_{1/p} \tag{2.2}$$

for some p. If we use the notation $[A, B] = AB - BA$ for any operators A and B then we have

$$X = [S, Q]\Lambda_1 + \Lambda_2[S^*, Q] + SW + WS^* + S\Lambda_1 SW + \Lambda_2 W + S\Lambda_1 WS^* + \Lambda_2 S^* WS^*.$$

Note that $S\Lambda_1 SW$, $\Lambda_2 W$, $S\Lambda_1 WS^*$, $\Lambda_2 S^* WS^* \in \Sigma^b_{1/(\alpha-2\beta-1)}$ provided Λ_1, Λ_2 are chosen as above.

Denote $(\Delta q)(n) := q_{n+1} - q_n$ then $[S, Q]e_n = -(\Delta q)(n)e_{n+1} = SDe_n$, where $D = \mathrm{diag}(-(\Delta q)(n))$ and similarly we have that $[S^*, Q] = -DS^*$. Next, note that

$$(\Delta q)(n) = \rho(n+1)^\alpha(1 + \Delta_{n+1}) - \rho n^\alpha(1 + \Delta_n)$$
$$= \rho(n+1)^\alpha - \rho n^\alpha + \rho[(n+1)^\alpha - n^\alpha]\Delta_{n+1} + \rho n^\alpha(\Delta_{n+1} - \Delta_n)$$

and $(n + 1)^\alpha - n^\alpha = \alpha n^{\alpha-1} + \mathrm{O}(n^{\alpha-2})$. But $n^\alpha(\Delta_{n+1} - \Delta_n) = \mathrm{o}(n^{\alpha-1})$ by (iii), so we have

$$(\Delta q)(n) = \rho\alpha n^{\alpha-1} + \mathrm{o}(n^{\alpha-1}).$$

There exists a finite dimensional diagonal operator Q_1 such that $D + Q_1$ is invertible and we can define operators $\Lambda_1 := -(D + Q_1)^{-1}W$ and $\Lambda_2 := W(D + Q_1)^{-1}$. The operators Λ_1 and Λ_2 are diagonal and belong to $\Sigma^b_{1/(\alpha-1-\beta)}$ by their definitions. Then the operators

$$[S, Q]\Lambda_1 + SW = S[(D + Q_1)\Lambda_1 + W] - SQ_1\Lambda_1 = -SQ_1\Lambda_1$$

and

$$\Lambda_2[S^*, Q] + WS^* = \Lambda_2 Q_1 S^*$$

are finite dimensional and belong to $\Sigma^b_{1/p}$ for every p and consequently X belongs to $\Sigma^b_{1/(\alpha-2\beta-1)}$. Denote $K = I + S\Lambda_1 + \Lambda_2 S^*$. We can choose Q_1 such that K is invertible, then $KJ_1 - QK = X$, i.e.

$$J_1 = K^{-1}(Q + XK^{-1})K.$$

Therefore, $\sigma(J_1) = \sigma(Q + XK^{-1})$ and by Lemma 2.1

$$\lambda_n(J_1) = \lambda_n(Q + XK^{-1}) = q_n + \mathrm{O}(\|(K^{-1})^* X^* e_n\|)$$
$$= q_n + \mathrm{O}(1/(n^{\alpha-2\beta-1})). \qquad \square$$

3. Special classes of Jacobi matrices with analytic models

In this section we shall study a few examples of unbounded Jacobi matrices in l^2, which do not fall into the class considered in Theorem 2.2. It happens that all examples are defined by sequences of power like behaviour i.e. $q_n = \delta n^\alpha$, $\lambda_n = \gamma n^\beta(1 + \Delta_n)$, $\Delta_n \to 0$, $n \to 0$, with $\alpha - \beta$ equals, respectively, to 1, $\frac{1}{2}$, 0. Nevertheless, by using specific forms of their weights and diagonals, we shall find the asymptotic of the eigenvalues. This approach was used for example in [7]. In all three examples, unitarily equivalent operator models of J's in some Hilbert spaces of holomorphic functions. These model operators turn out to be differential operators. The original equation for the eigenvalues $Jf = \lambda f$ is transformed into the corresponding differential equation. However, analysis of these differential equations is not so trivial because they do not fall in general into any (known to us) classical classes describing special functions. In this section we investigate the spectra of Jacobi matrices given by (3.1), (3.12), (3.28).

Recall that the Hardy spaces $H^2 = H^2(\mathbb{D})$ in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ consists of all holomorphic functions $f(z) = \sum_{n=0}^\infty a_n z^n$ in $\mathbb{D}$ such that $\sup_{r<1} \int_0^{2\pi} |f(re^{it})|^2 \, \mathrm{d}t = \sum_{n=0}^\infty |a_n|^2 < +\infty$.

Let $\Lambda_0 = \mathrm{diag}(n)_{n=1}^\infty$ be the diagonal operator determined by the sequence $\{n\}_{n=1}^\infty$.

1 *3.1. Analytic model for an example of a Jacobi matrix with $\alpha - \beta = 1$.*

We start with a simple operator

3
$$J_2 = \Lambda_0 + cS + cS^*. \tag{3.1}$$

Note that J_2 is formally in the class of Jacobi matrices defined in Theorem 2.2 with $\alpha = 1$ and $\beta = 0$. Therefore, it is clear
5 that the assumptions of Theorem 2.2 are not satisfied. The spectrum of J_2 is discrete and consists of the eigenvalues
only. Assume that $c \neq 0$ (the case $c = 0$ is trivial and $\sigma(J_2) = \{1, 2, \ldots\}$).

7   Recall the first type Bessel functions for $k \in \mathbb{N}$ and $x \in \mathbb{C}$ given by the power series $\mathbf{J}_k(x) = \sum_{n=0}^{\infty} (-1)^n 1/n!(n + k)!(\frac{x}{2})^{k+2n}$ [17]. Using the Gamma function $\Gamma(z)$ and taking $1/\Gamma(z) = 0$ for $z \in \{-1, -2, \ldots\}$ the Bessel functions
9 can be defined for complex indices by

$$\mathbf{J}_v(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\Gamma(n + v + 1)} \left(\frac{x}{2}\right)^{v+2n}$$

11 for $v \in \mathbb{C}$, $x \in \mathbb{R}$ ($\mathbf{J}_v(0) = \infty$ if $\mathscr{R}ev < 0$). Moreover; $\mathbf{J}_{-k}(x) = (-1)^k \mathbf{J}_k(x)$ for $k \in \mathbb{N}$ [17].

Theorem 3.1. *Let J_2 be an operator on l^2 given by* (3.1) *then*

$$\sigma(J_2) = \left\{\lambda \in \mathbb{R} : \lambda + c^2 - 1 = \sum_{k=2}^{\infty} \frac{c^k \mathbf{J}_k(2c)}{(k-2)!} \frac{1}{k - \lambda}\right\} \cup \{k \in \mathbb{N} : \mathbf{J}_k(2c) = 0\}$$

13
$$= \{\lambda_n(J_2) : n \geqslant k_0\},$$

where

15
$$\lambda_n(J_2) = n - r_n, \quad r_n = \mathrm{O}\left(\frac{c^{2n}}{n!(n-1)!}\right), \quad n \geqslant k_0$$

for some k_0.

17 **Proof.** Let us consider the equation

$$J_2 f = \lambda f, \tag{3.2}$$

19 where $f = \{f_n\}_{n=1}^{\infty}$. In (3.2) is the second order difference equation, we will change it to a differential equation by
the method suggested in [7]. Take the coherent state for the shift operator S in l^2 [13] $f_z = \sum_{n=1}^{\infty} z^{n-1} e_n$ and let
21 $f = \sum_{n=1}^{\infty} f_n e_n$. Then (3.2) is equivalent to the relation $(f_z, J_2 f) = \lambda(f_z, f)$ for all $z \in \mathbb{D}$.
In the other words $(f_z, \Lambda_0 f) + c(f_z, Sf) + c(f_z, S^* f) = \lambda(f_z, f)$ for $z \in \mathbb{D}$.
23 Denote by

$$\Phi(z) := (f_z, f). \tag{3.3}$$

25 Straightforward calculations prove that

$$(f_z, \Lambda_0 f) = \Phi(z) + z\Phi'(z);$$

27
$$(f_z, Sf) = z\Phi(z);$$

$$(f_z, S^* f) = \frac{1}{z}(\Phi(z) - \Phi(0)) \tag{3.4}$$

29 and we get the equation

$$z\Phi'(z) + (1 + cz)\Phi(z) + c\frac{1}{z}(\Phi(z) - \Phi(0)) = \lambda\Phi(z). \tag{3.5}$$

31 Note that (3.2) has a solution $f \in l^2\backslash\{0\}$ if and only if (3.5) has solution $\Phi \in H^2\backslash\{0\}$. Assume that we have $\lambda \in \mathbb{R}$
such that (3.5) has a solution $\Phi \in H^2$. If $\Phi(0) = 0$ then by taking the limit at 0 in (3.5) we get $c\Phi'(0) = 0$. Because

$c \neq 0$ we have $\Phi(0) = f_1 = 0$ and $\Phi'(0) = f_2 = 0$. By the difference (3.2) we have $f = 0$, so $\lambda \notin \sigma_p(J)$. Therefore, instead of (3.5) we can consider the equation

$$z^2 \Phi'(z) + [cz^2 + (1 - \lambda)z + c]\Phi(z) = c, \quad \Phi(0) = 1. \tag{3.6}$$

We will look for all parameters $\lambda \in \mathbb{R}$ for which (3.6) has non trivial solution in H^2. It is clear that the point spectrum of J_2 consists of all such numbers λ. Let Φ be in the form $\Phi(z) = e^{-cz}\varphi(z)$. Of course, $\Phi \in H^2 \Leftrightarrow \varphi \in H^2$. Then (3.5) is equivalent to

$$z^2 \varphi'(z) + [(1 - \lambda)z + c]\varphi(z) = ce^{cz}, \quad \varphi(0) = 1. \tag{3.7}$$

Let φ have the power series

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then (3.7) is satisfied when

$$
\begin{aligned}
&a_0 = 1, \\
&ca_1 + (1 - \lambda)a_0 = c^2, \\
&(n - \lambda)a_{n-1} + ca_n = \frac{c^{n+1}}{n!}, \quad n \geqslant 2.
\end{aligned}
\tag{3.8}
$$

Denote $d_k = (\lambda - k) \ldots (\lambda - 2)(-1)^{k-1}, \ k \geqslant 2$, then we calculate the solution of (3.8)

$$a_1 = c^{-1}(\lambda + c^2 - 1) \tag{}$$

and

$$a_n = c^{-n} \sum_{k=0}^{n} \frac{c^{2k}}{k!}(\lambda - n) \ldots (\lambda - (k+1)), \quad n \geqslant 2. \tag{3.9}$$

For $\lambda \notin \{2, 3, \ldots\}$ we can rewrite (3.9) as

$$a_n = \frac{(-1)^{n-1}d_n}{c^{n-1}} \left(a_1 + c^{-1} \sum_{k=2}^{n} \frac{(-1)^{k-1}c^{2k}}{d_k k!} \right), \quad n \geqslant 2. \tag{3.10}$$

We should answer the question: for which $\lambda \in \mathbb{R}$ the sequence $\{a_n\} \in l^2$.

Consider the case $\lambda \in \{0, 1, 2, 3, \ldots\}$. Using (3.9)

$$a_n = c^{-n} \sum_{k=\lambda}^{n} \frac{c^{2k}}{k!}(\lambda - n) \ldots (\lambda - (k+1)) = \frac{(-1)^{n-\lambda}(n - \lambda)!}{c^{n-2\lambda}} \sum_{k=0}^{n-\lambda} \frac{(-1)^k c^{2k}}{(k+\lambda)! k!}, \quad n > \lambda.$$

Now, notice that in this case

$$\{a_n\} \in l^2 \Leftrightarrow \sum_{k=0}^{\infty} \frac{(-1)^k c^{2k}}{(k+\lambda)! k!} = 0 \Leftrightarrow \mathbf{J}_\lambda(2c) = 0 \Leftrightarrow \mathbf{J}_{-\lambda}(2c) = 0.$$

1   For $\lambda \notin \{0, 1, 2, \ldots\}$, using (3.10), the following relations hold:

$$\{a_n\} \in l^2 \Leftrightarrow a_1 = c^{-1} \sum_{k=2}^{\infty} \frac{(-1)^k c^{2k}}{d_k k!}$$

$$\Leftrightarrow \lambda - 1 + c^2 = \sum_{k=2}^{\infty} \frac{(-1)^k c^{2k}}{d_k k!}$$

$$\Leftrightarrow \sum_{k=0}^{\infty} \frac{(-1)^k c^{2k}}{\Gamma(1 + k - \lambda) k!} = 0$$

$$\Leftrightarrow \mathbf{J}_{-\lambda}(2c) = 0.$$

3   Because

$$\frac{1}{d_k} = \sum_{j=2}^{k} \frac{(-1)^j}{(k-j)!(j-2)!} \frac{1}{j - \lambda}$$

5   then we have

$$\sum_{k=2}^{\infty} \frac{(-1)^k c^{2k}}{d_k k!} = \sum_{k=2}^{\infty} \frac{(-1)^k c^{2k}}{k!} \sum_{j=2}^{k} \frac{(-1)^j}{(k-j)!(j-2)!} \frac{1}{j - \lambda}$$

$$= \sum_{k=2}^{\infty} \frac{(-1)^k}{(k-2)!} \left[\sum_{j=0}^{\infty} \frac{(-1)^{j+k} c^{2(j+k)}}{(j+k)! j!} \right] \frac{1}{k - \lambda}$$

$$= \sum_{k=2}^{\infty} \frac{c^{2k}}{(k-2)!} \left[\sum_{j=0}^{\infty} \frac{(-1)^j c^{2j}}{(j+k)! j!} \right] \frac{1}{k - \lambda}$$

$$= \sum_{k=2}^{\infty} \frac{c^k \mathbf{J}_k(2c)}{(k-2)!} \frac{1}{k - \lambda}.$$

7   So, by the facts given above and the properties of the Bessel functions, we obtain

$$\sigma_p(J_2) = \{\lambda \in \mathbb{R} : \mathbf{J}_{-\lambda}(2c) = 0\}$$

$$= \left\{ \lambda \in \mathbb{R} : \lambda + c^2 - 1 = \sum_{k=2}^{\infty} \frac{c^k \mathbf{J}_k(2c)}{(k-2)!} \frac{1}{k - \lambda} \right\} \cup \{k \in \mathbb{N} : \mathbf{J}_k(2c) = 0\}. \tag{3.11}$$

9   Because $\mathbf{J}_n(2c) > 0$ for $n \geqslant c^2$, there exists a number $\lambda \in \sigma_p(J_2) \cap (n - 1, n)$ if $n > c^2$. Let $\lambda = n - r_n \in \sigma_p(J_2)$, $r_n \in (0, 1)$, $n > c^2$ and denote $\Delta_k = c^k \mathbf{J}_k(2c)/(k - 2)!$ then

11

$$n - r_n + c^2 - 1 = \sum_{k=2}^{n-1} \frac{\Delta_k}{k - n + r_n} + \frac{\Delta_n}{r_n} + \sum_{k=n+1}^{\infty} \frac{\Delta_k}{k - n + r_n} \leqslant \sum_{k=2}^{\infty} |\Delta_k| + \frac{\Delta_n}{r_n},$$

so $n + c^2 - 2 - \sum_{k=2}^{\infty} |\Delta_k| \leqslant \Delta_n / r_n$.

13   There exists $n_1 > c^2$ such that $\frac{1}{2} n \leqslant \Delta_n / r_n$ for $n \geqslant n_1$. So, we get $r_n \leqslant 2\Delta_n / n$, $n \geqslant n_1$, where $\Delta_n \leqslant c^{2n}/(n - 2)!$ $\sum_{j=0}^{\infty} c^{2j}/(j+n)! j! \leqslant e^{c^2} c^{2n}/(n-2)! n!$. Finally, using the above estimation it can be proved that for large $n \geqslant n_0 > n_1$

15   there exists exactly one number $\lambda = n - r_n \in (n - 1, n) \cap \sigma_p(J_2)$. □

Let us mention that it is not difficult to prove the following fact:

Remark 3.2. If $\Lambda = \mathrm{diag}(n)_{n=-\infty}^{+\infty}$ is a diagonal operator and S is a shift operator in $l^2(\mathbb{Z})$ then $J_2 = \Lambda + cS + cS^*$ considered in $l^2(\mathbb{Z})$ has the spectrum equal to $\mathbb{Z}$. $\square$

3.2. Analytic model for an example of a Jacobi matrix with $\alpha - \beta = 0$.

The Jacobi matrix $J = Q + SW + WS^*$ in $l^2(\mathbb{Z})$ with diagonals given by $Q = \mathrm{diag}(\delta n)_{n=-\infty}^{+\infty}$, $W = \mathrm{diag}$ $(\sqrt{(n+c)^2 + b^2})_{n=-\infty}^{+\infty}$ has been considered in [12] and in [6]. They have found the formulae for eigenvalues of this operator. Here and in Section 4.1 we investigate Jacobi matrix J_3 with the same entries as J but acting on $l^2(\mathbb{N})$. It turns out that spectral analysis of J_3 looks differently than that of J. Our approach is again based on analytical model of J_3 in H^2.

This section is devoted to the operator

$$J_3 = \delta\Lambda_0 + S(\Lambda_0 + cI) + (\Lambda_0 + cI)S^*, \tag{3.12}$$

in $l^2 = l^2(\mathbb{N})$. Assume that $\delta > 2$, $c \geqslant 0$, for such parameters $\sigma(J_3) = \sigma_p(J_3)$ and the spectrum is discrete.

Let us consider the relation $J_3 f = \lambda f$. Again take the coherent state $f_z = \sum_{n=1}^{\infty} z^{n-1}e_n$ and $f = \sum_{n=1}^{\infty} f_n e_n$ then using $(f_z, J_3 f) = \lambda(f_z, f)$ we get

$$(f_z, S\Lambda_0 f) + c(f_z, Sf) + (f_z, \Lambda_0 S^* f) + c(f_z, S^* f) + \delta(f_z, \Lambda_0 f) = \lambda(f_z, f),$$

for $|z| < 1$. Let $\Phi(z) = (f_z, f)$ then we have

$$\begin{aligned}
(f_z, S\Lambda_0 f) &= z^2 \Phi'(z) + z\Phi(z), \\
(f_z, \Lambda_0 S^* f) &= \Phi'(z), \\
(f_z, S^* f) &= \frac{1}{z}(\Phi(z) - \Phi(0))
\end{aligned} \tag{3.13}$$

and by (3.4)

$$z^2 \Phi'(z) + z\Phi(z) + cz\Phi(z) + \Phi'(z) + c\frac{1}{z}(\Phi(z) - \Phi(0)) + \delta(\Phi(z) + z\Phi'(z)) = \lambda\Phi(z) \tag{3.14}$$

and so

$$z(z^2 + \delta z + 1)\Phi'(z) = -((1+c)z^2 + (\delta - \lambda)z + c)\Phi(z) + c\Phi(0),$$

for $z \in \mathbb{D}$. As in the previous situation, this differential equation has a solution in $H^2 \backslash \{0\}$ exactly when $J_3 f = \lambda f$ has a solution $f \in l^2 \backslash \{0\}$.

Notice that $z^2 + \delta z + 1 = (z - z_1)(z - z_2)$, where $z_1 = -(\delta + \sqrt{\delta^2 - 4})/2$, $z_2 = -(\delta - \sqrt{\delta^2 - 4})/2$ and $z_1 < -1 < z_2 < 0$. We consider the following two cases of the parameter c.

(I) $c = 0$.

In this case, (3.14) is equivalent to

$$\frac{\Phi'(z)}{\Phi(z)} = -\frac{z + (\delta - \lambda)}{z^2 + \delta z + 1},$$

because

$$\frac{z + (\delta - \lambda)}{z^2 + \delta z + 1} = \frac{A}{z - z_1} + \frac{B}{z - z_2},$$

where $A = (z_1 - \lambda + \delta)/(z_1 - z_2)$ and $B = (\lambda - \delta - z_2)/(z_1 - z_2)$, so $\ln \Phi(z) = -A \ln(z - z_1) - B \ln(z - z_2) + C$, $\ln \Phi(z) = C + \ln(z-z_1)^{-A}(z-z_2)^{-B}$, $\Phi(z) = C_1(z-z_1)^{-A}(z-z_2)^{-B}$. Therefore; $\Phi \in H^2$ if and only if $(z-z_2)^{-B} \in H^2$, but this condition is equivalent to the fact that $-B = k$, where $k = 0, 1, 2, \ldots$. It follows that the spectrum of J_3 consists

1   of the numbers

$$\lambda_k(J_3) = k\sqrt{\delta^2 - 4} + \frac{1}{2}\left(\sqrt{\delta^2 - 4} + \delta\right) \tag{3.15}$$

3   for $k = 0, 1, 2, \ldots$.
      (II) $c \in (0, 1]$.

5   We can assume that $\Phi(0) = 1$. Indeed, suppose that Φ is an analytic function in some open set that contains 0, $\Phi(0) = 0$ and satisfies (3.14) in this set. Then taking the limit at 0 of both sides of (3.14) we get $\Phi'(0)(c+1) = 0$ and so $\Phi'(0) = 0$

7   (because $c + 1 \neq 0$). Thus $f_1 = \Phi(0) = 0$, $f_2 = \Phi'(0) = 0$ and using the relation $J_3 f = \lambda f$, we have $f_n = 0$, $n \geqslant 1$ and λ is not the eigenvalue of J_3.

9   Now, we will look for the function Φ of the form $\Phi(z) = (z - z_1)^A \varphi(z)$ for some real number A. Notice that $\Phi \in H^2$ if and only if $\varphi \in H^2$, because $|z_1| > 1$.

11   We have the new equation:

$$z(z - z_1)^{A+1}(z - z_2)\varphi'(z) = -[(1 + c + A)z^2 + (\delta - \lambda - z_2 A)z + c](z - z_1)^A \varphi(z) + c.$$

13   One can choose A such that $(1 + c + A)z^2 + (\delta - \lambda - z_2 A)z + c = (z - z_1)(pz + q)$, for some p, q. It follows that $pz + q = (A + 1 + c)z + \delta - \lambda + z_1(1 + c) + A(z_1 - z_2)$ and $A = (\delta(c - 1) - z_1 + \lambda)/(z_1 - z_2)$. Hence

$$(z - z_2)[z\varphi'(z)] + [(A + 1 + c)z - cz_2]\varphi(z) = \frac{c}{(z - z_1)^{A+1}}. \tag{3.16}$$

The function of the right hand side of (3.16) is analytic in $\mathbb{D}$,

$$\frac{c}{(z - z_1)^{A+1}} = \sum_{n=0}^{\infty} b_n z^n,$$

(the power series converge in $|z| < |z_1|$) and

19    $$b_0 = c(-z_2)^{A+1},$$

$$b_n = c(-1)^n(-z_2)^{A+1+n}\frac{(A + 1)(A + 2)\ldots(A + n)}{n!}, \quad n \geqslant 1. \tag{3.17}$$

21   We look for an analytic in $\mathbb{D}$ function $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfying (3.16). Thus the coefficients of φ must satisfy

$$a_0 = \frac{b_0}{-cz_2} = (-z_2)^A;$$

23    $$a_1 = \frac{b_1 - (A + 1 + c)a_0}{-(c + 1)z_2} = \frac{(-z_2)^A}{(c + 1)z_2}[(A + 1)(cz_2^2 + 1) + c]; \tag{3.18}$$

and the recurrence relation

25    $$(n + A + c)a_{n-1} - (c + n)z_2 a_n = b_n, \quad n \geqslant 2.$$

This is the first order linear difference equation. We can solve it:

27    $$a_n = g_n a_1 - \sum_{k=1}^{n} \frac{b_k}{(c + k)z_2} \prod_{l=k+1}^{n} \frac{l + A + c}{(c + l)z_2} \tag{3.19}$$

or

29    $$a_n = g_n\left[a_1 - \sum_{k=2}^{n} \frac{b_k}{(c + k)z_2 g_k}\right], \quad n \geqslant 2 \tag{3.20}$$

1   if $l + A + c \neq 0$ for all $l \geqslant 1$, where

$$g_n = \prod_{k=2}^{n} \frac{k + A + c}{(c + k)z_2}.$$

3   Notice that if $c \notin \{-1, -2, \ldots\}$ then $(c + k)z_2 \neq 0$ for every $k \in \mathbb{N}$.

Proposition 3.3.

5   (1) *If $k + A + c \neq 0$ for all $k \in \{2, 3, \ldots\}$ then*

$$\{a_n\} \in l^2 \Leftrightarrow a_1 = \sum_{k=2}^{\infty} \frac{b_k}{z_2(c + k)g_k}. \tag{3.21}$$

7   (2) *If $c \in (0, 1]$ and $k_0 + A + c = 0$ for some $k_0 \in \{2, 3, \ldots\}$ then $\{a_n\} \notin l^2$.*

Proof. Ad.1$\Rightarrow$ Because $|g_n| \to \infty$ it must be that the second term of the product in (3.20) must be convergent to 0.
9    $\Leftarrow$ In this situation we have (see the definition of g_k)

$$a_n = g_n \sum_{k=n+1}^{\infty} \frac{b_k}{z_2(c + k)g_k} = \frac{b_{n+1}}{n + 1 + A + c} + \sum_{k=n+2}^{\infty} \frac{b_k}{k + A + c} \prod_{l=n+1}^{k-1} \left(z_2 - \frac{z_2 A}{l + A + c} \right).$$

11   By (3.17) $|b_n| \leqslant Ch^n$ and $|z_2 - z_2 A/(n + A + c)| \leqslant h$ for large n, where $h = 1/|z_1| + \varepsilon = |z_2| + \varepsilon < 1$, and we have
$|a_n| \leqslant Ch^{n+1} + C\sum_{k=n+2}^{\infty} h^k h^{k-n-2} \leqslant C(h^{n+1} + h^{n+2}1/(1 - h)) < C'h^{n+1}$ for large n, this implies $\{a_n\} \in l^2$.
13   **Ad.2.** Let $A = -k_0 - c$, then $A + l + c \neq 0$ for all $l > k_0$ and using (3.19) and (3.17) for $n > k_0$, we have

$$
\begin{aligned}
a_n &= \sum_{k=k_0}^{n} \frac{b_k}{z_2(k + c)} \prod_{l=k+1}^{n} \frac{A + l + c}{(c + l)z_2} \\
&= c(-z_2)^{A+1} \sum_{k=k_0}^{n} z_2^k \frac{(-k_0 - c + 1)\ldots(-k_0 - c + k)}{k!}(z_2^{-1})^{n-k} \prod_{l=k+1}^{n} \frac{l - k_0}{c + l}.
\end{aligned}
$$

15   Hence we can write (for some constant $M(c, z_2, k)$)

$$
\begin{aligned}
a_n &= M(c, z_2, k_0)z_1^{n+1} \sum_{k=k_0}^{n} z_2^{2k} \frac{(1 - k_0 - c)\ldots(k - k_0 - c)}{k!} \prod_{l=k+1}^{n} \frac{l - k_0}{l + c} \\
&= M(c, z_2, k)\frac{z_2^{2k_0}}{k_0!}z_1^{n+1} \prod_{l=k_0+1}^{n} \frac{l - k_0}{l + c} \left[1 + \sum_{k=1}^{n-k_0} z_2^{2k} \frac{(1 - c)\ldots(k - c)}{(k_0 + 1)\ldots(k_0 + k)} \prod_{l=k_0+1}^{k_0+k} \frac{l + c}{l - k_0} \right]
\end{aligned}
$$

17   and so there is a constant $M_1(c, z_2, k) > 0$ such that

$$|a_n| \geqslant M_1(c, z_2, k_0)|z_1|^{n+1} \prod_{l=k_0+1}^{n} \frac{l - k_0}{l + c} \to +\infty, \quad n \to \infty. \qquad \square$$

19   Let us come back to (3.21). Using (3.17), after some calculations, we obtain

$$\sum_{k=2}^{\infty} \frac{b_k}{z_2(c + k)g_k} = c(-z_2)^{A+1} \sum_{k=2}^{\infty} z_2^{2k-2}\gamma_k \frac{A + k}{k} \prod_{l=0}^{k-2} \left(\frac{A + 1 + l}{A + 2 + l + c} \right),$$

21   where

$$\gamma_2 = 1, \quad \gamma_k = \prod_{l=2}^{k-1} \left(1 + \frac{c}{l} \right), \quad k > 2. \tag{3.22}$$

1 Then condition (3.21) from Proposition 3.3 can be transformed to

$$-\frac{1}{c(c+1)}[(A+1)(cz_2^2+1)+c] = \sum_{k=2}^{\infty} z_2^{2k}\gamma_k\frac{A+k}{k}\prod_{l=1}^{k-1}\left(1-\frac{c+1}{A+1+l+c}\right).$$

3 Denote by $x = A + 1 + c$, then we have the condition for λ to belong to the spectrum of J_3:

$$\lambda \in \sigma_p(J_3) \Leftrightarrow \lambda = (x-1-c)(z_1-z_2) + z_1 - \delta(c-1),$$

5 where x is a real number such that

$$\underbrace{-\frac{1}{c(c+1)}[(x-c)(cz_2^2+1)+c]}_{L(x)} = \underbrace{\sum_{k=2}^{\infty} z_2^{2k}\gamma_k\frac{x-1-c+k}{k}\prod_{l=1}^{k-1}\left(1-\frac{c+1}{x+l}\right)}_{P(x)}. \tag{3.23}$$

7 The computations given below will allow to localize the point λ very precisely. Notice that

$$\prod_{l=1}^{k-1}\left(1-\frac{c+1}{x+l}\right) = 1 + \sum_{p=1}^{k-1}\frac{A_{kp}}{x+p},$$

9 where

$$A_{21} = -(c+1),$$

11
$$A_{k\,1} = -(c+1)\prod_{l=1}^{k-2}\left(1-\frac{c+1}{l}\right), \quad A_{k,k-1} = -(c+1)\gamma_k(k-1); \quad k>2,$$

$$A_{k\,p} = -(c+1)\gamma_{p+1}p\prod_{l=1}^{k-p-1}\left(1-\frac{c+1}{l}\right); \quad k>2, \quad 2\leqslant p\leqslant k-2.$$

13 Denote $d_k = z_2^{2k}\gamma_k(x-1-c+k)/k$, then

$$P(x) = \sum_{k=2}^{\infty} d_k + \underbrace{\sum_{k=2}^{\infty} d_k\sum_{j=1}^{k-1}\frac{A_{kj}}{x+j}}_{P_1},$$

$$P_1 = \sum_{k=2}^{\infty}\left(\sum_{l=k}^{\infty} d_l A_{l(k-1)}\right)\frac{1}{x+(k-1)}$$

$$= \sum_{k=1}^{\infty}\left(\sum_{l=k+1}^{\infty} z_2^{2l}\gamma_l\frac{x+l-1-c}{l}A_{lk}\right)\frac{1}{x+k}$$

$$= \sum_{k=1}^{\infty}\left(\sum_{l=k+1}^{\infty} z_2^{2l}\gamma_l\frac{A_{lk}}{l}\right)\frac{x+k-k}{x+k} + \sum_{k=1}^{\infty}\left(\sum_{l=k+1}^{\infty} z_2^{2l}\gamma_l\frac{l-1-c}{l}A_{lk}\right)\frac{1}{x+k}$$

$$= \sum_{k=1}^{\infty}\left(\sum_{l=k+1}^{\infty} z_2^{2l}\gamma_l\frac{A_{lk}}{l}\right) + \sum_{k=1}^{\infty}\left[-k\left(\sum_{l=k+1}^{\infty} z_2^{2l}\gamma_l\frac{A_{lk}}{l}\right) + \sum_{l=k+1}^{\infty} z_2^{2l}\gamma_l\frac{l-1-c}{l}A_{lk}\right]\frac{1}{x+k}$$

15
$$= \sum_{k=1}^{\infty}\left(\sum_{l=k+1}^{\infty} z_2^{2l}\gamma_l\frac{A_{lk}}{l}\right) + \sum_{k=1}^{\infty}\left[\sum_{l=k+1}^{\infty} z_2^{2l}\gamma_l\frac{l-k-1-c}{l}A_{lk}\right]\frac{1}{x+k}.$$

1   Denote

$$a = -\left(\frac{cz_2^2 + 1}{c(c+1)} + \sum_{k=2}^{\infty} \frac{z_2^{2k} \gamma_k}{k} \right)$$

3   and

$$b = \frac{cz_2^2}{1+c} - \sum_{k=2}^{\infty} z_2^{2k} \gamma_k \frac{k-1-c}{k} - \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} z_2^{2l} \gamma_l \frac{A_{lk}}{l}.$$

5   Notice that $a < 0$ because $c > 0$ and $\gamma_k > 0$. By the above calculations we look for $x \in \mathbb{R}$ such that

$$ax + b = \sum_{k=1}^{\infty} z_2^{2k+2} \gamma_{k+1} \left[\frac{-cA_{k+1\,k}}{k+1} + \sum_{l=k+2}^{\infty} z_2^{2(l-k-1)} \frac{\gamma_l}{\gamma_{k+1}} \frac{A_{lk}}{l} (l - k - 1 - c) \right] \frac{1}{x+k}$$

$$= \sum_{k=1}^{\infty} z_2^{2k+2} \gamma_{k+1} \left[\frac{-cA_{k+1\,k}}{k+1} + \sum_{j=2}^{\infty} z_2^{2(j-1)} \frac{\gamma_{k+j}}{\gamma_{k+1}} a_{k+j,k} \frac{j-1+c}{k+j} \right] \frac{1}{x+k}.$$

7   Therefore,

$$ax + b = c(c+1) \sum_{k=1}^{\infty} z_2^{2k+2} \gamma_{k+1}^2 k \underbrace{\left[\frac{1}{k+1} + \sum_{j=2}^{\infty} z_2^{2(j-1)} \frac{\gamma_{j+k}}{\gamma_{k+1}} \prod_{s=2}^{j-1} \left(1 - \frac{c+1}{s} \right) \frac{j-1-c}{j+k} \right]}_{B_k} \frac{1}{x+k}. \qquad (3.24)$$

9   Denote by

$$A_k = c(c+1) z_2^{2(k+1)} \gamma_{k+1}^2 k B_k. \qquad (3.25)$$

11   Notice that

$$\frac{\gamma_{j+k}}{\gamma_{k+1}} = \prod_{s=k}^{j+k-1} \left(1 + \frac{c}{s} \right) \leqslant \gamma_j,$$

$$|B_k| \leqslant 1 + \sum_{j=2}^{\infty} z_2^{2(j-1)} \gamma_j \prod_{s=2}^{j-1} \left| 1 - \frac{c+1}{s} \right| (1+c) =: M_1$$

13

and let

$$M_2 := c(c+1) M_1 \sum_{k=1}^{\infty} z_2^{2(k+1)} \gamma_{k+1}^2 k. \qquad (3.26)$$

15

Denote $f(x) = \sum_{k=1}^{\infty} A_k / (x+k)$. Note two simple facts.

17   **Fact 1.** *Because $A_k > 0$, then f is continues and decreasing in every interval $(-n-1, -n)$, where $n = 1, 2, 3, \ldots$ and in $(-1, +\infty)$; moreover $\lim_{x \to -n^{\pm}} f(x) = \pm\infty$. This implies that in $(-n-1, -n)$ there exists exactly one x such that*
19   $ax + b = f(x)$ *(for $n \in \{1, 2, \ldots\}$)*

Remember that we denoted by $L(x)$ the left side of Eq. (3.23) and by $P(x)$ the right side of this equation. We can
21   check that $L(0) > 0$ and $P(0) \leqslant 0$ (because we have assumed that $c \in (0, 1]$), so it is trivial that $L(0) > P(0)$, but this implies that $b > f(0)$. Therefore there exist exactly two points in $(-1, +\infty)$ satisfying equation $ax + b = f(x)$. Finally
23   $\{x \in \mathbb{R} : ax + b = f(x)\} = \{x_n = -n + r_n : n = 0, 1, \ldots\}$ for some $r_n \in (0, 1)$, when $n \geqslant 2$.

Fact 2. *If $f(x) = ax + b$ and $x + n = r_n \in (0, 1)$, then there exists n_1 such that $|r_n| \leqslant 2A_n / |a| n$ for $n \geqslant n_1$.*

1　　Indeed, we have $|a|n + ar_n + b = A_n/r_n + \sum_{k \neq n} A_k/(k - n + r_n) \leqslant A_n/r_n + \sum_{k=n+1}^{\infty} A_k/(k - n + r_n) \leqslant |A_n|/|r_n| + \sum_{k=1}^{\infty} A_k \leqslant |A_n|/|r_n| + 2M_2$, so $|a|n - |a| + b - M_2 \leqslant |A_n|/|r_n|$. Then there exists n_1 such that $|a|n/2 \leqslant |A_n|/|r_n|$ for
3　　all $n \geqslant n_1$.

　　Finally, we have just proved the following theorem:

5　　**Theorem 3.4.** *Let* $J_3 = \delta \Lambda_0 + S(\Lambda_0 + c) + (\Lambda_0 + c)S^*$ *be a Jacobi operator in* l^2 *and* $\delta > 2$ *then*

$$\sigma(J_3) = \{\lambda_n(J_3) : n = 0, 1, 2, \ldots\},$$

where

7

　　1. *in the case* $c = 0$

9
$$\lambda_n(J_3) = n\sqrt{\delta^2 - 4} + \frac{1}{2}\left(\sqrt{\delta^2 - 4} + \delta\right);$$

　　2. *in the case* $c \in (0, 1]$

11
$$\lambda_n(J_3) = n\sqrt{\delta^2 - 4} + \frac{1}{2}\left(\sqrt{\delta^2 - 4} + \delta\right) + c\left(\sqrt{\delta^2 - 4} - \delta\right) - r_n\sqrt{\delta^2 - 4}, \quad (3.27)$$

　　where $|r_n| \leqslant C(c, \delta)z_2^{2(n+1)}\gamma_n^2$ *for* $n > n_1$ *for some large* n_1.
13　　(*Note that* $z_2 = -(\delta - \sqrt{\delta^2 - 4})/2 \in (-1, 0)$ *and* $\gamma_k = \prod_{l=2}^{k-1}(1 + \frac{c}{l})$.) □

　　The spectrum of J_3 for the parameter $c > 1$ will be considered in Section 4.1.

15　　*3.3. Analytic model for an example of a Jacobi matrix with* $\alpha - \beta = \frac{1}{2}$

　　In this section we calculate the point spectrum for

17　　$$J_4 = \Lambda_0 + \gamma S \Lambda_0^{1/2} + \gamma \Lambda_0^{1/2} S^*, \quad (3.28)$$

where $\Lambda_0 = \text{diag}(n)_{n=1}^{\infty}$ and $\gamma \in \mathbb{R}$. The reason for presenting this here is twofold. Firstly, the analytic model of J_4 uses
19　　the Bargmann space. Secondly, knowledge of $\sigma(J_4)$ can be used to compute asymptotic of the eigenvalues of some
　　perturbations of J_4. It is easy to check that J_4 is unitarily equivalent to the operator $T = \gamma T_z + \gamma T_z^* + T_z^* T_z$, where T_z
21　　is the operator of multiplication by the variable z on the Bargmann space B^2 of entire functions on $\mathbb{C}$ that belong to
　　$L^2(\mu)$, with $\mathrm{d}\mu(z) = \frac{1}{\pi}e^{-|z|^2}\,\mathrm{d}m(z)$, where m is the Lebesgue measure [13].
23　　Then

$$Tf = \lambda f$$

25　　is equivalent to

$$\gamma z f + \gamma f' + (zf)' = \lambda f$$

27　　$$f'/f = (\lambda - 1 - \gamma z)(z + \gamma)^{-1}.$$

Hence

29　　$$f'/f = -\gamma + (\lambda + \gamma^2 - 1)(z + \gamma)^{-1}$$

$$\ln(f) = \ln[C(z + \gamma)^{\lambda + \gamma^2 - 1}e^{-\gamma z}]$$

31　　for some constant $C \neq 0$.

　　It follows that

33　　$$f = C(z + \gamma)^{\lambda + \gamma^2 - 1}e^{-\gamma z}$$

1 and so $f \in B^2 \backslash \{0\} \Leftrightarrow \lambda + \gamma^2 - 1 \in \{0, 1, 2, \ldots\}$. Therefore;

$$\sigma_p(J_4) = \sigma_p(T) = \{k + 1 - \gamma^2 : \; k = 0, 1, 2, \ldots\}.$$

3 **4. Rozenbljum theorem**

In this section we are going to find an asymptotic behaviour of eigenvalues for a few Jacobi operators applying the
5 following abstract theorem of Rozenbljum.

Theorem 4.1 (*Rozenbljum [14]*). *Assume that A and B are self-adjoint bounded below operators having purely discrete*
7 *spectra and 0 is not an eigenvalue of any of them. Let $\alpha \in (-\infty, 1)$. Assume that there exist bounded and boundedly
invertible operators K_1, K_2 and bounded operators C_1, C_2 such that*

9 $$K_1^{-1} A K_1 - B = C_1 |B|^\alpha,$$

$$K_2^{-1} B K_2 - A = C_2 |A|^\alpha.$$

11 *If $\sigma(A) = \{\lambda_n(A) : n \geqslant 1\}$, $\sigma(B) = \{\lambda_n(B) : n \geqslant 1\}$ and $\lambda_n(A) \leqslant \lambda_{n+1}(A)$, $\lambda_n(B) \leqslant \lambda_{n+1}(B)$, $n \geqslant 1$ then*

$$|\lambda_n(A) - \lambda_n(B)| = \mathrm{O}(|\lambda_n(A)|^\alpha), \quad n \to \infty. \quad \square$$

13 Let us mention that in [14] one can find much general version of Theorem 4.1 that includes similar result also for
operators with discrete spectra but not bounded below. We apply Theorem 4.1 to the following Jacobi operators.
15 At the beginning of we will complete analysis of asymptotics of eigenvalues for the Jacobi matrices on $l^2 = l^2(\mathbb{N})$
with the entries considered in Section 3.2 with the parameter $c > 1$. We consider the operator $J_3 = \delta \Lambda_0 + SW + WS^*$
17 (like the one given by 3.12), where $W = \Lambda_0 + cI$ for $c > 1$. In this case $c = \tilde{n} + \tilde{c}$ for some $\tilde{n} \in \{1, 2, \ldots\}$ and $\tilde{c} \in [0, 1]$.
Take the Jacobi operator

$$\widetilde{J} - \tilde{n}\delta I = (\delta \Lambda_0 - \tilde{n}\delta I) + S(\Lambda_0 + \tilde{c}I) + (\Lambda_0 + \tilde{c}I)S^*$$

$$= \begin{pmatrix} (1 - \tilde{n})\delta & 1 + \tilde{c} & & & & \\ 1 + \tilde{c} & (2 - \tilde{n})\delta & \ddots & & & \\ & \ddots & \ddots & \tilde{n} - 1 + \tilde{c} & & \\ & & \tilde{n} - 1 + \tilde{c} & 0\delta & \tilde{n} + \tilde{c} & \\ & & & \tilde{n} + \tilde{c} & \delta & 1 + \tilde{n} + \tilde{c} \\ & & & & 1 + \tilde{n} + \tilde{c} & 2\delta & 2 + \tilde{n} + \tilde{c} \\ & & & & & & \cdots \end{pmatrix}.$$

19 Notice that

21 $$\widetilde{J} - \tilde{n}\delta I = \begin{pmatrix} M_{\tilde{n}} & S_c \\ S_c^* & J_3 \end{pmatrix},$$

and so

23 $$(\widetilde{J} - \tilde{n}\delta I) - \begin{pmatrix} M_{\tilde{n}} & 0 \\ 0 & J_3 \end{pmatrix} = \begin{pmatrix} 0 & S_c \\ S_c^* & 0 \end{pmatrix} = C(\alpha)\Lambda_0^{-\alpha},$$

for some finite dimensional operator $C(\alpha)$ and for any number $\alpha \geqslant 1$. Observe that

25 $$\sigma_p(\widetilde{J} - \tilde{n}\delta I) = \{\lambda_n(\widetilde{J}) - \tilde{n}\delta : n = 0, 1, 2, \ldots\}$$

and

27 $$\sigma_p \begin{pmatrix} M_{\tilde{n}} & 0 \\ 0 & J_3 \end{pmatrix} = \underbrace{\{\alpha_0, \ldots, \alpha_{\tilde{n}-1}\}}_{\sigma_p(M_{\tilde{n}})} \cup \underbrace{\{\lambda_0(J_3), \lambda_1(J_3), \ldots\}}_{\sigma_p(J_3)}.$$

1 We apply the Rozenbljum theorem (Theorem 4.1) to operators $A = (\widetilde{J} - \tilde{n}\delta I) + \gamma I$ and $B = \begin{pmatrix} M_{\tilde{n}} & 0 \\ 0 & J_3 \end{pmatrix} + \gamma I$, where the parameter $\gamma \in \mathbb{R}$ is such that $\sigma(A), \; \sigma(B) \subset (0, +\infty)$. Therefore, by the Rozenbljum theorem we obtain

3 $$\lambda_n(J_3) = \lambda_{n+\tilde{n}}(\widetilde{J}) - \tilde{n}\delta + \widetilde{r}_n,$$

where $\widetilde{r}_n = \mathrm{O}(n^{-\alpha})$ for large n and any natural number $\alpha \geqslant 1$. Finally, using Theorem 3.4, calculate that

$$\lambda_n(J_3) = (n + \tilde{n})\sqrt{\delta^2 - 4} + \frac{1}{2}\left(\delta + \sqrt{\delta^2 - 4}\right) + \tilde{c}\left(\sqrt{\delta^2 - 4} - \delta\right) - r_n\sqrt{\delta^2 - 4} - \tilde{n}\delta + \widetilde{r}_n$$

5
$$= n\sqrt{\delta^2 - 4} + \frac{1}{2}\left(\delta + \sqrt{\delta^2 - 4}\right) + c\left(\sqrt{\delta^2 - 4} - \delta\right) - \widetilde{\widetilde{r}}_n,$$

for $n = 0, 1, 2, \ldots$, where $\widetilde{\widetilde{r}}_n = \mathrm{O}(n^{-\alpha})$ for large n and for any $\alpha \geqslant 1$.

7 Now, let us come back to the operator with diagonals like these considered in [12,6]. Let $q_n = \delta n$ and $\lambda_n = \sqrt{(n + c)^2 + b}$. Take the diagonal operators $Q = \mathrm{diag}(q_n)$, $W = \mathrm{diag}(\lambda_n)$ and define the Jacobi operator

9 $$J_5 = SW + WS^* + Q.$$

Let $B = J_5 + \gamma I$, $A = J_3 + \gamma I$, where J_3 is given by (3.12) with the parameter $\delta > 2$ and $c \geqslant 0$ and $\gamma \in \mathbb{R}$ is such that
11 $\sigma(A), \; \sigma(B) \subset (0, +\infty)$. Applying Theorem 4.1, take $K_1 = K_2 = I$, $\alpha = -1$. Define

$$R := \mathrm{diag}\left(\sqrt{(n + c)^2 + b} - (n + c)\right), \; \sqrt{(n + c)^2 + b} - (n + c) = \frac{b}{\sqrt{(n + c)^2 + b} + (n + c)} = \mathrm{O}\left(\frac{1}{n}\right).$$

13 Then we have $B - A = J_5 - J_3 = SR - RS^* = C_1 A^{-1} = C_2 B^{-1}$ for some bounded operators C_1 and C_2. Again by the Rozenbljum theorem and (3.27),

15 $$\lambda_n(J_5) = \lambda_n(J_3) + \mathrm{O}\left(\frac{1}{n}\right) = n\sqrt{\delta^2 - 4} + \frac{1}{2}\left(\delta + \sqrt{\delta^2 - 4}\right) + c\left(\sqrt{\delta^2 - 4} - \delta\right) + \mathrm{O}\left(\frac{1}{n}\right)$$

for large n.

17 Recall that Masson and Repka [12] and Edward [6] proved that if J_5' with a diagonal and weights like J_5 is considered as an operator in $l^2(\mathbb{Z})$ then

19 $$\sigma_p(J_5') = \left\{n\sqrt{\delta^2 - 4} + \frac{1}{2}\left(\delta + \sqrt{\delta^2 - 4}\right) + c\left(\sqrt{\delta^2 - 4} - \delta\right) : \; n \in \mathbb{Z}\right\}.$$

Moreover, they found exact formulae for eigenvalues of J_5' and proved that the eigenvalues do not depend on the
21 parameter $b \geqslant 0$. We see that the asymptotic behaviour in $+\infty$ of the point spectra of J_5' and J_5 is obviously similar, but generally the eigenvalues of the operator J_5 acting in $l^2(\mathbb{N})$ do not equal to formulae that give their asymptotics,
23 what can be observed by looking at (3.23) and (3.24) in Section 3.2.

The final (easy) application of Theorem 4.1 is given to a Jacobi operator which also does not fall into the class
25 described in Theorem 2.1. Let us try to calculate asymptotic behaviour of the point spectrum of

$$J_6 = SW + WS^* + Q,$$

27 where

$$W = \mathrm{diag}\left(\frac{1}{\sqrt{n}}\right), \quad Q = \mathrm{diag}(\sqrt{n}).$$

29 It is trivial that $J_6 - Q = SW + WS^* = C_1|J_6|^{-1} = C_2|Q|^{-1}$, so by Theorem 4.1

$$\lambda_n(J_6) = \sqrt{n} + \mathrm{O}\left(\frac{1}{\sqrt{n}}\right),$$

31 $n \geqslant 1$.

However, using the Rozenbljum theorem we can obtain even better result. Notice that $J_6^2 = Q^2 + W^2 + SW^2 S^* + (S + QSW) + (S + QSW)^* + (SW)^2 + (WS^*)^2$, $Q^2 = \Lambda_0$ and $S + QSW = 2S + SD$, where $De_n = \frac{1}{\sqrt{n}(\sqrt{n}+\sqrt{n+1})} e_n$ for $n = 1, 2, \ldots$. Then $J_6^2 - (\Lambda_0 + 2S + 2S^*) \in \Sigma_{1/1}^b$, so using the result of Section 3 and the Rozenbljum theorem we have the asymptotic for the eigenvalues of J_6^2 given by $\lambda_n(J_6^2) = n + \mathrm{O}(\frac{1}{n})$ and the asymptotics for J_6 is

$$\lambda_n(J_6) = \sqrt{\lambda_n(J_6^2)} = \sqrt{n} + \mathrm{O}\left(\frac{1}{n\sqrt{n}}\right).$$

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