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The asymptotic analysis of generalized eigenvectors of some Jacobi operators. Jordan box case*

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This paper presents two methods of finding asymptotic formulae for a basis of solutions of the second order difference equations in the Jordan box case. An application to spectral analysis of Jacobi operators is also sketched.

Keywords: Difference equation; Jacobi operator; Asymptotic behavior; Jordan box; Spectral analysis; Non-oscillatory solution

AMS Subject Classification (2000): 39A10; 47B25

1. Introduction

Interesting works on the asymptotic formulae of solutions of difference equations (in particular of the second order equations) have appeared in the last two decades. Let us mention only two comprehensive papers presenting the current state of art [1,4]. Some results on the asymptotic form of solutions, in the case of double characteristic root, have been recently found in references [6,7]. In the above works, we were mainly motivated by the spectral analysis of self-adjoint Jacobi operators. This work also describes asymptotic behavior of solutions in the Jordan box case (for a class of power like coefficients) and its application to the spectral theory of Jacobi operators.

In what follows we consider the system of difference equations

$$\lambda_{n-1}x(n-1)+(q_n-\lambda)\,x(n)+\lambda_n x(n+1)=0,\quad \text{where}\quad \lambda_n>0,\quad \lim\lambda_n=\infty,$$
$$q_n,\ \ \lambda\ \text{ are real, and } n\geq 2. \tag{1.1}$$

Let

$$\vec{x}_n:=\begin{pmatrix} x(n-1)\\ x(n)\end{pmatrix},\quad B_n:=\begin{pmatrix} 0 & 1\\ -\lambda_{n-1}\lambda_n^{-1} & (\lambda-q_n)\lambda_n^{-1}\end{pmatrix}$$

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Then equation (1.1) reads as

$$\vec{x}_{n+1} = B_n \vec{x}_n. \tag{1.2}$$

Assume that $\lim_n \lambda_{n-1}\lambda_n^{-1} = 1$ and $\lim_n q_n \lambda_n^{-1} = \delta$. Let $A := \begin{pmatrix} 0 & 1 \\ -1 & -\delta \end{pmatrix}$, then $B_n = A + R_n$, where $\lim_n R_n = 0$.

Sometimes in the spectral analysis of Jacobi operators it is helpful to consider the product

$$B_{2n} \cdot B_{2n-1} = A^2 + AR_{2n-1} + R_{2n}A + R_{2n} \cdot R_{2n-1}.$$

If discr $A^2 = (trA^2)^2 - 4\det A^2 \neq 0$, then A^2 has distinct characteristic roots and asymptotic behavior of solutions of equation (1.2) can be found by applying, for example, the results contained in the above mentioned works [1,4,6]. In particular, discrete analogues of Levinson's theorem were used in our earlier works on Jacobi operators [9,11].

Since discr $A^2 = 0$ if and only if ($\delta = 0$ or $\delta = \pm 2$, then there are two cases.

1. If $\delta = 0$, then $A^2 = -\mathrm{I}$ and the asymptotic analysis of solutions of equation (1.2) is more subtle. However, some progress was made in this case in our recent works [6,7].
2. If $\delta = \pm 2$, then A^2 is similar to the Jordan box $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and the analysis of asymptotic behavior of solutions of equation (1.2) becomes very delicate.

The special case: $\lambda_n = n + a$, $q_n = \pm 2n$ was already considered by us in [10]. We shall say more about the method used in [10] in section 4.

In this work, we study asymptotic behavior of solutions of equation (1.1) for λ_n and q_n given by

$$\lambda_n = n^{\alpha}(1 + r(n)), \quad q_n = -2n^{\alpha}(1 + s(n)), \tag{1.3}$$

where $\alpha \in (0, 1)$ and

$$r(n) = \frac{a}{n} + \frac{D}{n^{2\alpha}} + \frac{V(n)}{n}, \quad s(n) = \frac{b}{n} + \frac{E}{n^{2\alpha}} + \frac{W(n)}{n} \tag{1.4}$$

with $V(n)$, $W(n)$ satisfying some additional assumptions (sections 3 and 4).

Note that equation (1.1) with $q_n = 2n^{\alpha}(1 + s(n))$ reduces to equation (1.3) via the change of variables: $x(n) = (-1)^n u(n)$, $\lambda = -\mu$, $\mu > 0$.

We make a comment on the above form of $r(n)$ and $s(n)$ in section 5.

The methods we use depend essentially on the sign of λ.

For $\lambda > 0$ (non-oscillatory case) the method proposed by W. Kelley in [12] is employed. Kelley found asymptotic of solutions of equation (1.1) in the case of rational coefficients. This method also works for power like case given by equation (1.3) and presumably can be extended to other classes of coefficients in equation (1.1).

In turn for $\lambda < 0$ we apply the *ansatz* approach (compare [10]).

In particular, the asymptotic formulae for the solutions of equation (1.1) depend in nontrivial way on all parameters a, b, E, D and α. Complete proofs will be given only for $\alpha \in (1/3, 2/3)$.

Although the main motivation for this paper comes from the spectral theory of Jacobi operators, we hope that the asymptotic results obtained in the work may be of independent interest. Finally, observe that the asymptotic formulae of solutions found in this work are not covered by recent general results of S. Elaydi in [4] or R.J. Kooman in [14]. In the last paper an interesting result on the asymptotic behavior of second order difference operators is proved (Theorem 10.1 in [14]). However, this result does not apply to the coefficients λ_n and q_n of this work, due to the assumptions on $V(n)$ and $W(n)$ (see equation (3.4)).

The form of $r(n)$ and $s(n)$ also shows that $\alpha = 1/2$ is essential value of the parameter α. In this work we shall restrict our attention to the interval (1/3, 2/3). The methods apply to $\alpha \leq 1/3$ and $\alpha \geq 2/3$ but asymptotic formulae become more and more complicated, as α tends to 1 or 0. This is clear due to the presence of both $1/n$ and $1/n^{1-\alpha}$ in the Taylor expansion of $\sqrt{-\beta(n)}$, see equation (2.2).

I thank the referee for useful remarks and for bringing to my attention the work in [14].

The paper is divided into five sections. Section 2 contains some preparatory facts which will be used in section 3. In section 3 asymptotic formula for the solutions of equation (1.1) is proved, in the case $\lambda > 0$. In turn section 4 presents asymptotic result for $\lambda < 0$. Finally, in section 5 some comments on assumptions imposed on $r(n)$ and $s(n)$ are given and an application to spectral analysis of Jacobi operators is briefly described. The same section contains two open questions.

Throughout the paper the notation($a(\cdot) \ll b(\cdot)$ means that $\lim_{n\to\infty} a(n)/b(n) = 0$.

2. Preliminaries

Divide equation (1.1) by λ_n then equation (1.1) is equivalent to

$$x(n+1) + \frac{q_n - \lambda}{\lambda_n} x(n) + \frac{\lambda_{n-1}}{\lambda_n} x(n-1) = 0.$$

Denoting $\lambda_{n-1}/\lambda_n = 1 + q(n)$ and $(q_n - \lambda)/\lambda_n = -2 + p(n)$ the above equation says

$$x(n+1) + (-2 + p(n))x(n) + (1 + q(n))x(n-1) = 0. \tag{2.1}$$

We make the change of variable in equation (2.1) ([3] or [12])

$$w(n) := x(n) \prod_{i=1}^{n-1} \frac{2}{2 - p(i)}$$

and obtain

$$w(n+1) - 2w(n) + \frac{4(1 + q(n))}{(p(n) - 2)(p(n-1) - 2)} w(n-1) = 0.$$

Write

$$1 + \beta(n) := \frac{4(1 + q(n))}{(p(n) - 2)(p(n-1) - 2)} \tag{2.2}$$

then the above equation can be written as

$$w(n+1) - 2w(n) + (1 + \beta(n))w(n-1) = 0. \tag{2.3}$$

Using the formulae for λ_n and q_n it is obvious that $\beta(n) \to 0$, as $n \to \infty$. Note that $\beta(n)$ depends on λ. We claim that all real ($\neq 0$) solutions $w(n)$ of equation (2.2) satisfy

$$\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = 1, \tag{2.4}$$

provided $\lambda > 0$.

Let $\Delta a_n := a_{n+1} - a_n$.

LEMMA 1 *Fix $\lambda > 0$. Then any nontrivial real solution $w(n)$ of equation* (2.2) *satisfies equation* (2.4).

Proof. Obviously equation (2.3) is non-oscillatory if and only if equation (1.1) is non-oscillatory. It is known that every nontrivial solution of equation (2.3) satisfies equation (2.4) if and only if equation (2.3) is non-oscillatory ([5]).

Write equation (1.1) in the form

$$\Delta(\lambda_{n-1}\Delta x(n-1)) + p_{n-1} = 0, \tag{2.5}$$

where $p_{n-1} = \lambda_n + \lambda_{n-1} - \lambda + q_n$.

Lemma 1.2 in [2] says that equation (2.5) is non-oscillatory if and only if there exists a sequence u_n with $u_n > -\lambda_n$, $n \geq N$ for some $N > 0$, satisfying

$$\Delta u_n + \frac{p_n u_n + u_n u_{n+1}}{\lambda_n} + p_n \leq 0. \tag{2.6}$$

Put $u_n \equiv 0$. Then (2.6) is equivalent to

$$p_n = \lambda_{n+1} + \lambda_n + q_n - \lambda \leq 0, \quad n \geq N.$$

Since $\lambda > 0$ and $\alpha \in (0, 1)$

$$\lambda_{n+1} + \lambda_n + q_n = n^{\alpha}\left[\left(1+\frac{1}{n}\right)^{\alpha} + 1 - 2 + \left(1+\frac{1}{n}\right)^{\alpha} r(n+1) + r(n) - 2s(n)\right]$$

tends to zero as $n \to \infty$ (by the form of $r(n)$ and $s(n)$) and so it is clear that (2.6) holds for $n \geq N = N(\lambda)$. This completes the proof.

Due to equation (2.4), we have

$$\frac{w(n+1)}{w(n)} = 1 + X(n), \tag{2.7}$$

where $\lim_{n\to\infty} X(n) = 0$.

Below we repeat for the reader convenience the reasoning given by Kelley ([12], p. 168). Dividing equation (2.3) by $w(n)$ and using equation (2.7) we obtain

$$X(n) = (1+\beta(n))\frac{X(n-1)}{1+X(n-1)} - \beta(n). \tag{2.8}$$

Now we write equation (2.8) for large n as follows

$$X(n) - (1 + \beta(n))X(n-1) = -\beta(n) - (1 + \beta(n))\sum_{k=2}^{\infty}(-X(n-1))^k. \qquad (2.9)$$

Let

$$f(n) = \prod_{j=j_0}^{n-1}(1 + \beta(j)),$$

where j_0 is chosen such that $1 + \beta(j) \neq 0$, for $j \geq j_0$.

Using the formula for the solution of the first order linear equation, we have

$$X(n) = f(n+1)\left[C - \sum_{s=1}^{n}\frac{\beta(s) + (1 + \beta(s))\sum_{k=2}^{\infty}(-X(s-1))^k}{f(s+1)}\right], \qquad (2.10)$$

for some constant C.

In the next section, we shall find formal approximations of solutions of equation (2.3) by using equations (2.9) and (2.10).

In the section we shall need the following formulae.

PROPOSITION 2.2 *If λ_n and q_n are as above then*

$$\left(\prod_{i=1}^{n-1}\frac{2}{2 - p(i)}\right)^{-1}$$

$$= \begin{cases} A(n)\exp\left[\frac{\lambda}{2}(1-\alpha)^{-1}n^{1-\alpha} + (b-a)\ln n\right], & \text{when } \alpha \in \left(\frac{1}{2}, \frac{2}{3}\right), \\ A_1(n)\exp\sum_{i=1}^{n-1}\left[\frac{\lambda}{2i^{-\alpha}} + \left(D - E - \frac{\lambda^2}{8}\right)i^{-2\alpha} + (b-a)i^{-1}\right], & \text{when } \alpha \in \left(\frac{1}{3}, \frac{1}{2}\right], \end{cases}$$

where $A(n)$ and $A_1(n)$ are some sequences convergent to positive constants.

Proof. Straightforward computation.

It may happen that $p(i) = 2$ for a certain λ and i. Then in the above product the corresponding i-factor equals 1.

For the reader convenience we recall some results in [12].The first one is a lemma from [12].

LEMMA 2.3 *Assume that $f(n)$ is given by*

$$f(n) = \prod_{s=1}^{n-1}(1 + \beta(s)),$$

where $\beta(s) \to 0$, as $s \to \infty$ is a sequence of real numbers. If $\sum_s \beta(s)$ or $\sum_s \beta^2(s)$ diverges, then let $\mathscr{H}$ be the largest integer such that $\sum_s \beta^{\mathscr{H}}(s)$ diverges. Define

$$h(n) = \sum_{i=1}^{\mathscr{H}}\frac{(-1)^{i-1}}{i}\beta^i(n),$$

then

$$f(n) = F(n) \exp\left[\sum_{s=1}^{n-1} h(s)\right].$$

where $F(n) \to F > 0$ *as* $n \to \infty$.

The following appear as Theorems 1 and 2 in [12].

THEOREM 2.4 *Assume that* $1 + \beta(n) \geq 0$ *for* $n > n_0$ *and that* $v(n)$ *and* $w(n)$ *satisfy the inequalities*

$$v(n) \leq \frac{(1+\beta(n))v(n-1)}{1+v(n-1)} - \beta(n), \quad n \geq n_0 + 1 \tag{2.11}$$

$$w(n) \geq \frac{(1+\beta(n))w(n-1)}{1+w(n-1)} - \beta(n), \quad n \geq n_0 + 1 \tag{2.12}$$

$$w(n_0) \geq v(n_0)$$

$$v(n) > -1, \quad n \geq n_0.$$

If $X(n_0) \in [v(n_0), w(n_0)]$ *and* $X(n)$ *satisfies equation* (2.8) *for* $n \geq n_0 + 1$, *then* $v(n) \leq X(n) \leq w(n)$, $n \geq n_0$.

THEOREM 2.5 *Assume that* $1 + \beta(n) \geq 0$, $v(n)$ *satisfies* (2.11), $w(n)$ *satisfies* (2.12), $v(n) \geq w(n)$, $|v(n)| < 1$, *and* $|w(n)| < 1$ *for* $n \geq n_0$. *Then* (2.8) *has a solution* $X(n)$ *so that* $v(n) \geq X(n) \geq w(n)$ *for* $n \geq n_0$.

3. The case of positive half line $(0, +\infty)$

Fix $\lambda > 0$. As we already mentioned in the Introduction asymptotic behavior of solutions $x(n)$ of equation (1.1) depends in essential way on α. This will become clear below. We shall consider separately only two intervals of α:(1/2, 2/3) and (1/3, 1/2). However, the method also works for the remaining $\alpha's$.

3.1 Formal calculations

(a)
$$\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right).$$

Then we have inequalities:

$$2\alpha < 1 + \frac{\alpha}{2} < \frac{5\alpha}{2} < 1 + \alpha < 2 - \frac{\alpha}{2},$$

which will be frequently used below. First the asymptotic of solutions of equation (2.8) is found formally, and next using results in [12] precise proof of the obtained formal asymptotic formulae will be given.

Our task is to find formal solutions $X_{\pm}(n)$ of equation (2.8) modulo some terms of the order $O(n^{-3\alpha}) + O(n^{(\alpha/2)-2})$.

By definition of $\beta(n)$ we have

$$\beta(n) = -\frac{\lambda}{n^{\alpha}} - \frac{\alpha}{n} + 2r(n-1) - s(n-1) - s(n) + \frac{3\lambda^2}{4n^{2\alpha}} + \frac{\lambda c_1}{n^{1+\alpha}} + \frac{\lambda c_2}{n^{3\alpha}} + \frac{\lambda(3W(n) - 2V(n))}{n^{1+\alpha}} + O\left(\frac{1}{n^2}\right). \tag{3.1}$$

The growth conditions $r(n) = O(1/n) = s(n)$ also has been used in computing equation (3.1).

It follows that

$$\sqrt{-\beta(n)} = \sqrt{\frac{\lambda}{n^{\alpha}}}(1 + X_n)^{1/2}, \tag{3.2}$$

where

$$X_n = \frac{\alpha}{\lambda n^{1-\alpha}} - 2(r(n) - s(n))\frac{n^{\alpha}}{\lambda} - \frac{3\lambda}{4n^{\alpha}} + \frac{c_1}{n} + \frac{c_2}{n^{2\alpha}} + \frac{3W(n) - 2V(n)}{n} + O\left(\frac{1}{n^{2-\alpha}}\right). \tag{3.3}$$

In order to estimate $X_{n+1} - X_n$ we impose on $V(n)$ and $W(n)$ the condition

$$V(n) = O\left(\frac{1}{n^{x}}\right) = W(n), \quad \Delta V(n) = O\left(\frac{1}{n}\right) = \Delta W(n), \tag{3.4}$$

where $x > \alpha/2$.

How to make a reasonable guess on formal solutions $X_{\pm}(n)$? We follow the idea of Kelley [12]. First, using the Euler summation formula one can check that $f(n)\sum_{s=1}^{n-1}\beta(s)f(s+1)^{-1}$ tends to $c(\alpha - 1)$, for a nonzero constant c, as $n \to +\infty$.

Since we know that $X(n) \to 0$, as $n \to +\infty$, looking at equation (2.10) we see that the next largest term $X^2(k-1)$ in the numerator of the sum in equation (2.10) must influence the asymptotic behavior of $X(n)$. Therefore, we assume

$$\beta(s) + X^2(s-1) \ll \beta(s), \quad \text{as} \quad s \to +\infty.$$

It follows that

$$X(s-1) = \pm\sqrt{-\beta(s)} + \gamma(s), \tag{3.5}$$

where $\gamma(s) \ll 1/\sqrt{s^{\alpha}}$, as $s \to +\infty$, see equation (3.1).

On substituting $X(n-1)$ given by equation (3.5) into equation (2.8) we have

$$\begin{aligned} &X(n) + \beta(n) - (1 + \beta(n))\,[X(n-1) - X^2(n-1) + X^3(n-1) \\ &- X^4(n-1) + X^5(n-1)] + O(X^6(n-1)) = 0. \end{aligned} \tag{3.6}$$

By equations (3.2) and (3.3) the order of $O(X^6(n-1)) = O(1/n^{3\alpha})$. We want to solve equation (3.6) modulo $O(1/n^{2-\alpha/2}) + O(1/n^{3\alpha})$. It is convenient to write equation (3.6) in the form

$$X(n) - X(n-1) + [\beta(n) + X^2(n-1)] - [X(n-1)\beta(n) + X^3(n-1)]$$
$$+ [X^2(n-1)\beta(n) + X^4(n-1)] - [X^3(n-1)\beta(n) + X^5(n-1)] + O\left(\frac{1}{n^{3\alpha}}\right) = 0. \tag{3.7}$$

In equation (3.7) appears the difference $X(n) - X(n-1)$. Therefore, we must to analyze

$$\sqrt{-\beta(n+1)} - \sqrt{-\beta(n)},$$

(see equation (3.5)).

Write

$$(+)\sqrt{-\beta(n)} = \sqrt{\frac{\lambda}{n^{\alpha}}}\left(1 + \frac{X_n}{2} - \frac{X_n^2}{8} + \frac{X_n^3}{16} + O\left(X_n^4\right)\right),$$

Note that

$$\sqrt{\frac{\lambda}{n^{\alpha}}}X_n^4 = O\left(\frac{1}{n^{4(1-\alpha)+\alpha/2}}\right).$$

Since equation (3.4) implies that

$$\Delta r(n) = O\left(\frac{1}{n^2}\right) = \Delta s(n),$$

combining the above equalities and definition of X_n we have

$$X_{n+1} - X_n = O\left(\frac{1}{n^{2-\alpha}}\right).$$

Therefore, using (+) we check

$$\sqrt{-\beta(n+1)} - \sqrt{-\beta(n)} = \frac{-\alpha\sqrt{\lambda}}{2n^{1+\alpha/2}} + O\left(\frac{1}{n^{2-\alpha/2}}\right)$$

(because $4(1-\alpha) + \alpha/2 \geq 2 - \alpha/2$).

By equation (3.5) and the last formula we obtain

$$X(n) - X(n-1) = \pm\left[\frac{-\alpha\sqrt{\lambda}}{2n^{1+\alpha/2}}\right] + \gamma(n+1) - \gamma(n) + O\left(\frac{1}{n^{2-\alpha/2}}\right) \tag{3.8}$$

$$\beta(n) + X^2(n-1) = \pm 2\sqrt{-\beta(n)}\gamma(n) + \gamma^2(n) \tag{3.9}$$

$$\beta(n)X(n-1) + X^3(n-1) = -2\beta(n)\gamma(n) \pm 3\sqrt{-\beta(n)}\gamma^2(n) + \gamma^3(n). \tag{3.10}$$

As we shall see below it is not necessary to compute the two remaining terms

$$X^2(n-1)\beta(n) + X^4(n-1) \quad \text{and} \quad X^3(n-1)\beta(n) + X^5(n-1).$$

Inserting formulae (3.8)–(3.10) into (3.7) we have

$$\pm\left[\frac{-\alpha\sqrt{\lambda}}{2n^{1+\alpha/2}}+2\sqrt{-\beta(n)}\gamma(n)\right]+\gamma(n+1)-\gamma(n)+\gamma^2(n)+2\beta(n)\gamma(n)$$

$$\mp 3\sqrt{-\beta(n)}\gamma^2(n)-\gamma^3(n)+X^2(n-1)[\beta(n)+X^2(n-1)]-X^3(n-1) \tag{3.11}$$

$$\times[\beta(n)+X^2(n-1)]+O\left(\frac{1}{n^{2-\alpha/2}}\right)+O\left(\frac{1}{n^{3\alpha}}\right)=0.$$

We want $-\alpha\sqrt{\lambda}/2n^{1+\alpha/2}+2\sqrt{-\beta(n)}\gamma(n)$ to cancel out up to the order $1/n^{1+\alpha/2}$. Therefore,

$$\gamma(n)=\frac{\alpha}{4n}+\delta(n), \quad \text{with} \quad \delta(n)\ll\frac{1}{n}. \tag{3.12}$$

Hence (3.11) after substitution of (3.12) turns out to be equivalent to

$$\delta(n+1)-\delta(n)+2\beta(n)\left[\frac{\alpha}{4n}+\delta(n)\right]\pm 2\sqrt{-\beta(n)}\delta(n)+O\left(\frac{1}{n^{2-\alpha/2}}\right)$$

$$+O\left(\frac{1}{n^{3\alpha}}\right)=0, \tag{3.13}$$

because all the remaining terms of equation (3.11) are of a order higher than $O(1/n^{2-\alpha/2})$. Next we want the terms $\pm 2\sqrt{-\beta(n)}\delta(n)$ and $\beta(n)\alpha/2n$ to cancel out up to the order $1/n^{1+\alpha}$.

Thus

$$\delta_{\pm}(n)=\pm\frac{\alpha\sqrt{\lambda}}{4n^{1+\alpha/2}}+\varepsilon(n), \quad \text{with} \quad \varepsilon(n)\ll\frac{1}{n^{1+\alpha/2}}. \tag{3.14}$$

This way we have formally computed that

$$X_{\pm}(n-1)=\pm\sqrt{-\beta(n)}+\frac{\alpha}{4n}\pm\frac{\alpha\sqrt{\lambda}}{4n^{1+\alpha/2}}+\varepsilon(n), \tag{3.15}$$

where $\varepsilon(n)\ll 1/n^{1+\alpha/2}$.

(b)
$$\alpha\in\left(\frac{1}{3},\frac{1}{2}\right].$$

The reasoning in this case is similar to the one given in (a). Therefore, we shall be brief. First note the scale of inequalities is different from the case (a): $3\alpha/2\le(1-\alpha/2)<5\alpha/2\le(1+\alpha/2)\le(2-3\alpha/2)$.

The assumptions on $V(\cdot)$ and $W(\cdot)$ are given again by equation (3.4).

Using definition of $\beta(n)$ after lengthy computations we have

$$\beta(n)=-\frac{\lambda}{n^{\alpha}}(1+y_n), \tag{3.16}$$

where

$$y_n := \frac{c_1}{\lambda n^{\alpha}} + \frac{c_2}{\lambda n^{1-\alpha}} + \frac{c_3}{\lambda n^{2\alpha}} + \frac{c_4}{n} + \frac{c_5}{n^{3\alpha}} + \frac{c_6}{n^{1+\alpha}} + 2(W(n) - V(n))\lambda n^{\alpha-1}$$
$$+ (5W(n) - 2V(n))\lambda n^{-1} + O(n^{\alpha-2}) + O(n^{-4\alpha})$$

and

$$c_1 = 2(E - D) - 3\left(\frac{\lambda}{2}\right)^2, \quad c_2 = 2(a - b) + \alpha, \quad c_3 = 3(D - E) + \frac{\lambda^2}{2}.$$

The form of remaining constants c_s, $s = 4, 5, 6$ are not essential for formal computations. Now

$$\sqrt{-\beta(n)} = \sqrt{\tfrac{\lambda}{n^{\alpha}}}\left[\sum_{s=0}^{3} \binom{\frac{1}{2}}{s} y_n^s + O\left(y_n^4\right)\right]. \tag{3.17}$$

Rewrite equation (2.8) for large n as follows

$$X(n) - X(n-1) + \sum_{s=0}^{6} [\beta(n)X^s(n-1) + X^{2+s}(n-1)](-1)^s$$
$$- (1 - X(n-1))\beta(n)X^7(n-1) + O(X^9(n-1)) = 0. \tag{3.18}$$

We want to solve equation (3.18) modulo $O(n^{-(1+3\alpha/2)})$, for large n. As in the above case (a) we look for $X(n-1)$ in the form

$$X(n) = \pm\sqrt{-\beta(n+1)} + \gamma(n+1),$$

with $\gamma(n) \ll n^{-\alpha/2}$.

Using equations (3.16) and (3.17) we find

$$X(n) - X(n-1) = \pm\left[\frac{-\alpha\sqrt{\lambda}}{2n^{1+\alpha/2}} + cn^{-(1+3\alpha/2)}\right] + \gamma(n+1) - \gamma(n) + \gamma^2(n)$$
$$+ O(n^{-9\alpha/2}) + O(n^{\alpha/2-1}),$$

for some constant c.

In turn

$$\beta(n) + X^2(n-1) = \pm 2\gamma(n)\left[\sqrt{\lambda}n^{-\alpha/2} + e_1 n^{-3\alpha/2}\right] + O(n^{\alpha/2-1}) + \gamma^2(n),$$

for some constant e_1.

Denote by $L(n) := X(n) - X(n-1) + \beta(n) + X^2(n-1)$.

We demand that the terms $\pm[-\alpha\sqrt{\lambda}/2n^{1+\alpha/2}]$ and $\pm\gamma(n)\sqrt{\lambda}n^{-\alpha/2}$ of $L(n)$ cancel up to the order $n^{-1-\alpha/2}$

Hence

$$\gamma(n) = \alpha(4n)^{-1} + \delta(n), \tag{3.19}$$

where $\delta(n) \ll 1/n$.

Using equations (3.19) and (3.10) the sum $L(n) - [X(n-1)\beta(n) + X^3(n-1)]$ can be written as

$$\delta(n+1) - \delta(n) \pm \left[e_2 n^{-1-3\alpha/2} + 2\delta(n)\sqrt{\lambda}n^{-\alpha/2} + e_3\delta(n)n^{-3\alpha/2}\right]$$
$$+ 2\beta(n)[\alpha(4n)^{-1} + \delta(n)] + O(n^{-9\alpha/2}) + O(n^{\alpha/2-1}),$$

for some constants e_2 and e_3

Now the terms $\pm 2\delta(n)\sqrt{\lambda}n^{-\alpha/2}$ and $\alpha/2\beta(n)n^{-1}$ cancel up to the order $n^{-1-\alpha}$ provided that

$$\delta(n) = \pm\frac{\sqrt{\lambda}\alpha}{4n^{1+\alpha/2}} + \varepsilon(n), \tag{3.20}$$

where $\varepsilon(n) \ll 1/n^{1+\alpha/2}$.

Since $X^2(n-1)[\beta(n) + X^2(n-1)]$ is of the order $n^{-1-3\alpha/2}$ and the terms $X^s(n-1)$ $[\beta(n) + X^2(n-1)]$ and $O(X^9(n-1))$ are of the order higher than $O(n^{-1-2\alpha})$

This reasoning shows that $X(n)$ is also given by the formula (3.15).

3.2 Proof of formal approximations

(a) $$\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right).$$

Following the ideas of Kelly we will prove below that the formal expression given by equation (3.15) is indeed right approximation of genuine solutions of equation (2.8). We restrict ourselves to the proof in the + sign case of equation (3.15). The proof for the − sign in equation (3.15) is similar.

Define

$$\nu(n) := \sqrt{-\beta(n+1)} + \frac{\alpha}{4n} + \frac{d}{4n^{1+\alpha/2}}, \quad \text{here} \quad d \in \mathbb{R} \tag{3.21}$$

is a free constant.

We claim that choosing suitable d we have

$$\nu(n) + \beta(n) - (1 + \beta(n))\frac{\nu(n-1)}{1 + \nu(n-1)} < 0, \tag{3.22}$$

for all n sufficiently large.

Indeed, using equation (3.21) and repeating the computation made above (see equations (3.8) and (3.9), etc.) we find that the left hand side of (3.22) reduces to

$$\frac{1}{2}\left(d - \alpha\sqrt{\lambda}\right)\frac{\sqrt{\lambda}}{n^{1+\alpha}} + O\left(\frac{1}{n^{2-\alpha/2}}\right) + O\left(\frac{1}{n^{3\alpha}}\right).$$

This proves the claim provided that $d < \alpha\sqrt{\lambda}$ and n is sufficiently large. The same computations prove that

$$w(n) := \sqrt{-\beta(n+1)} + \frac{\alpha}{4n} + \frac{d_1}{4n^{1+\alpha/2}}$$

satisfies

$$w(n)+\beta(n)-(1+\beta(n))\frac{w(n-1)}{1+w(n-1)}>0, \tag{3.23}$$

if $d_1>\alpha\sqrt{\lambda}$ and n is large enough. Inequalities (3.22) and (3.23) allow to apply Theorem 2.4 and get a solution $X_1(n)$ of equation (2.8) such that

$$\left|X_1(n)-\sqrt{-\beta(n+1)}-\frac{\alpha}{4n}\right|\le\frac{M_1}{n^{1+\alpha/2}}, \tag{3.24}$$

for large n.

Similarly, by Theorem 2.5 we can find a second solution $X_2(n)$ of equation (2.8) for which we have

$$\left|X_2(n)+\sqrt{-\beta(n+1)}-\frac{\alpha}{4n}\right|\le\frac{M_2}{n^{1+\alpha/2}}, \tag{3.25}$$

for n sufficiently large.

Now the first asymptotic formulae for a basis of solutions of equation (1.1) are described by

THEOREM 3.1 *Let $\alpha\in(1/2,\ 2/3)$. Suppose that λ_n and q_n are given by equation* (1.3), *where V_n and W_n satisfy conditions* (3.4). *Then equation* (1.1) *has a basis of solutions $y_\pm(n)$ with the asymptotic behavior given by*

$$y_\pm(n)=C_\pm(n)n^{-\alpha/4}\exp\ \pm\left[\frac{\sqrt{\lambda}}{1-\alpha/2}n^{1-(\alpha/2)}+\psi n^{\alpha/2}+\rho n^{1-(3\alpha/2)}\right], \tag{3.26}$$

where $C_\pm(n)\to C_\pm\neq 0$, as $n\to\infty$ and

$$\rho=-\left(\frac{\lambda^{3/2}}{24}+\frac{D-E}{\sqrt{\lambda}}\right)\left(1-\frac{3\alpha}{2}\right)^{-1},\quad \psi=\frac{\alpha+2(b-a)}{\alpha\sqrt{\lambda}}.$$

Proof. Let $X_1(n)$, $X_2(n)$ be solutions of (2.8) satisfying estimates (3.24) and (3.25). Since

$$1+X_i(n)=\frac{w_i(n+1)}{w_i(n)},\quad i=1,2,$$

applying Lemma 2.3 we have

$$w_i(n)=F_i(n)\exp\sum_{k=1}^{n-1}\left[X_i(k)-\frac{1}{2}X_i^2(k)+\frac{1}{3}X_i^3(k)\right]. \tag{3.27}$$

Note that

$$\left\{X_i^4(n)\right\}\in l^1\quad(i=1,2)$$

by estimates equations (3.24) and (3.25), and so 3 is the largest integer $\mathscr{H}$ for which $\sum_k X_i^{\mathscr{H}}(k)$ is divergent.

Again due to estimates (3.24) and (3.25), the assumption (3.4) and the form of $\sqrt{-\beta(k+1)}$ one can write for large k

$$X_i(k) = \pm\left[\sqrt{\frac{\lambda}{k^{\alpha}}} + \frac{\alpha - 2(a-b)}{2\sqrt{\lambda}k^{1-\alpha/2}} - \frac{3}{8}\left(\sqrt{\frac{\lambda}{k^{\alpha}}}\right)^3 - \frac{D-E}{\sqrt{\lambda}k^{3\alpha/2}}\right]$$

$$+\frac{\alpha}{4k} + O\left(k^{(\alpha/2)-1-x}\right) + O(k^{(3\alpha/2)-2})$$

$$X_i^2(k) = \frac{\lambda}{k^{\alpha}} + \frac{\alpha - 2(a-b)}{k} + O(k^{\alpha-2})$$

$$X_i^3(k) = \pm\left(\frac{\lambda}{k^{\alpha}}\right)^{3/2} + O(k^{-(1+\alpha/2)}),$$

where in the above equalities $+$ sign corresponds to $i = 1$ and $-$ sign for $i = 2$.

Using the above formulae and the Euler summation formula equation (3.27) can be written as

$$w_i(n) = \widetilde{F}_i(n)n^{\alpha/4}\exp\, \pm\left[An^{1-\alpha/2} + Bn^{\alpha/2} + Cn^{1-3\alpha/2}\right], \tag{3.28}$$

with $A = \sqrt{\lambda}(1-\alpha/2)^{-1}$, $B = (\alpha - 2a + 2b)(\alpha\sqrt{\lambda})^{-1}$, $C = \rho$, for some sequences $\widetilde{F}_i(n)$ converging to positive constants $\widetilde{F}_i$.

Combining Proposition 2.2 and equation (3.28) we obtain equation (3.26).

The proof is complete.

(b) $$\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right].$$

Again the reasoning is the same as in the case (a).

Define

$$v_1(n) := \sqrt{-\beta(n+1)} + \frac{\alpha}{4n} + \frac{d_1}{4n^{1+\alpha/2}}, \tag{3.29}$$

where d_1 is a real constant.

Then one can choose d_1 such that

$$v_1(n) + \beta(n) - (1+\beta(n))\frac{v_1(n-1)}{1+v_1(n-1)} < 0 \tag{3.30}$$

for all n large enough.

In fact, direct computation of the left hand side of (3.30) reveals (by using the expansion of $(1+v_1(n-1))^{-1} = \sum_{s=0}^{9}(-1)^s v_1^s(n-1) + O(v_1^{10}(n-1))$ that it is equal (for large n) to

$$\left[(d\sqrt{\lambda} - \alpha\lambda)\right]n^{-1-\alpha} + f_1 n^{-(1+3\alpha/2)} + O(n^{-1-2\alpha}) + O(n^{\alpha/2-2}),$$

for a certain constant f_1. This expression can be made negative for a suitable d_1 and n large enough. The same computations show that

$$v_2(n) := \sqrt{-\beta(n+1)} + \frac{\alpha}{4n} + \frac{d_2}{n^{1+\alpha/2}}$$

satisfies for another constant d_2 (namely such that $d_2\sqrt{\lambda} - \alpha\lambda > 0$) and large n

$$\upsilon_2(n) + \beta(n) - (1 + \beta(n))\frac{\upsilon_2(n-1)}{1 + \upsilon_2(n-1)} > 0. \tag{3.31}$$

Applying Theorem 2.4 we obtain a solution $\tilde{X}_1(n)$ of equation (2.8) such that

$$\left|\tilde{X}_1(n) - \sqrt{-\beta(n+1)} - \frac{\alpha}{4n}\right| \leq \frac{M_1}{n^{1+\alpha/2}}, \tag{3.32}$$

for n sufficiently large and some M_1. Similarly, by Theorem 2.5 we can find another solution $\tilde{X}_2(n)$ of equation (2.8) such that

$$\left|\tilde{X}_2(n) - \sqrt{-\beta(n+1)} - \frac{\alpha}{4n}\right| \leq \frac{M_2}{n^{1-\alpha/2}}, \tag{3.33}$$

for some M_2 and large values of n.

Consequently, we obtain

THEOREM 3.2 *Let $\alpha \in (1/3, 1/2]$. Assume that $V(n)$ and $W(n)$ satisfy equation* (3.4). *Then equation* (1.1) *has a basis of solutions $X_\pm(n)$ with the asymptotic behavior of the form*

$$X_\pm(n) = A_\pm(n)n^{-\alpha/4}\exp \pm\left[\sqrt{\lambda}\left(1 - \frac{\alpha}{2}\right)^{-1} n^{1-\alpha/2} + \psi n^{\alpha/2} + \rho n^{1-3\alpha/2} + \frac{\eta}{1 - 5\alpha/2} n^{1-5\alpha/2}\right], \tag{3.34}$$

where $A_\pm(n) \to A_\pm \neq 0$, as $n \to \infty$, ρ and ψ are the same as in equation (3.26), $\eta = \sqrt{\lambda}[(c_3 - c_1^2/4 + c_1)/4 + \lambda^2/5]$ *with c_s given in equation* (3.16).

Proof. Let $\tilde{X}_1(n), \tilde{X}_2(n)$ be solutions of (2.8) satisfying estimates equations (3.32) and (3.33). As we know

$$1 + \tilde{X}_i(n) = \frac{\tilde{w}_i(n+1)}{\tilde{w}_i(n)}, \quad i = 1, 2$$

and applying Lemma 2.3 we obtain

$$\tilde{w}_i(n) = F_i(n)\exp\sum_{k=1}^{n-1}\left[\tilde{X}_i(k) - \frac{1}{2}\tilde{X}_i^2(k) + \frac{1}{3}\tilde{X}_i^3(k) - \frac{1}{4}\tilde{X}_i^4(k) + \frac{1}{5}\tilde{X}_i^5(k)\right], \tag{3.35}$$

with $F_i(n) \to F_i > 0$, as $n \to \infty$.

Note that $\{\tilde{X}_i^6(n)\} \in l^1$ $(i = 1, 2)$ due to (3.32) and (3.33), and so $\mathscr{H} = 5$. Using equation (3.17) and the assumption $x > \alpha/2$ we check that for some sequences $r_s(k) \in l^1$ and large k

the following equalities hold

$$\tilde{X}_i(k) = \pm\left[\sqrt{\frac{\lambda}{k^{\alpha}}} + \sqrt{\lambda}\frac{g_1}{2k^{3\alpha/2}} + \frac{g_2}{2\sqrt{\lambda}k^{1-\alpha/2}} + \frac{g_3\sqrt{\lambda}}{k^{5\alpha/2}}\right] + \frac{\alpha}{4k} + r_1(k),$$

$$\text{with}\quad g_1 = \frac{c_1}{\lambda},\quad g_2 = c_2,\quad g_3 = \frac{c_3}{2} - \frac{c_1^2}{8},$$

$$\tilde{X}_i^2(k) = \frac{\lambda}{k^{\alpha}} + \frac{g_1\lambda}{k^{2\alpha}} + \frac{g_2}{k} + r_2(k),$$

$$\tilde{X}_i^3(k) = \pm\left(\left(\frac{\lambda}{k^{\alpha}}\right)^{3/2} + \frac{3g_1\lambda^{3/2}}{2n^{5\alpha/2}}\right) + r_3(n),$$

$$\tilde{X}_i^4(k) = \frac{\lambda^2}{k^{2\alpha}} + r_4(k),$$

$$\tilde{X}_i^5(k) = \pm\left(\frac{\lambda}{k^{\alpha}}\right)^{5/2} + r_5(k).$$

Here + sign corresponds to $i = 1$ and − sign to $i = 2$.

Using equation (3.35) the above equalities and Proposition 2.2 we obtain

$$X_{\pm}(n) = A_1(n)\exp\left[\sum_{k=1}^{n-1}\left(\frac{\lambda}{2k^{-\alpha}} + \left(D - E - \frac{\lambda^2}{8}\right)k^{-2\alpha} + (b-a)k^{-1}\right)\right]$$

$$B(n)\exp\ \pm\sum_{k=1}^{n-1}\left[\sqrt{\frac{\lambda}{k^{\alpha}}} + \rho k^{-3\alpha/2} + \frac{g_2}{2\sqrt{\lambda}k^{1-\alpha/2}} + \eta k^{-5\alpha/2}\right]$$

$$\exp\left[-\sum_{k=1}^{n-1}\left(\frac{\lambda}{2k^{-\alpha}} + \left(D - E - \frac{\lambda^2}{8}\right)k^{-2\alpha} + (b-a)k^{-1} + \frac{\alpha}{4k}\right)\right]$$

$$= A_1(n)B_1(n)n^{-\alpha/4}\exp\ \pm\sum_{k=1}^{n-1}\left[\sqrt{\frac{\lambda}{k^{\alpha}}} + \rho k^{-3\alpha/2} + \frac{\psi}{k^{1-\alpha/2}} + \eta k^{-5\alpha/2}\right]$$

where $\eta = \sqrt{\lambda}(c_3 - c_1^2/4) + c_1)/4 + 1/5(\lambda)^{5/2}$, ρ and ψ are the same as in equation (3.21), and $B(n)$, $B_1(n)$ are some sequences convergent to positive constants.

The Euler formula completes the proof.

4. The case of negative half-line $(-\infty, 0)$

Asymptotic behavior of solutions of equation (1.1) for $\lambda < 0$ is much simpler to find. Assumptions on $r(n)$ and $s(n)$ are also less restrictive. Namely, $r(n)$ and $s(n)$ do not change their form (equation (1.4)) but in this section $V(n)$ and $W(n)$ only decay faster than $1/n^{\alpha/2}$, i.e.

$$V(n) = O\left(\frac{1}{n^x}\right) = W(n)\quad \text{for large}\quad n, \tag{4.1}$$

where $x > \alpha/2$.

The idea of proof is based on right *ansatz* about the form of asymptotic of solutions of equation (1.1). This approach was already successfully used in [10] (for negative λ and $\alpha = 1$). The form of the *ansatz* we make below is inspired by Theorem 3.1.

The case $\alpha \in (1/2, 2/3)$

THEOREM 4.1 *Let $\alpha \in (1/2, 2/3)$. Suppose that λ_n and q_n are the same as above, i.e. $\lambda_n = n^{\alpha}(1 + r(n))$, $q_n - 2n^{\alpha}(1 + s(n))$ and $r(n) = a/n + D/n^{2\alpha} + V(n)/n$, $s(n) = b/n + E/n^{2\alpha} + W(n)/n$. If $V(\cdot)$ and $W(\cdot)$ satisfy (4.1) and B_n is given in (1.1), then for any $\lambda < 0$ there are two linearly independent solutions $\vec{z}_{\pm}(n)$ of*

$$\vec{z}(n+1) = B_n \vec{z}(n) \tag{4.2}$$

with the asymptotics given by

$$z_{\pm}(n) = n^{-\alpha/4} \exp\left[\pm i \left(F n^{1-\alpha/2} - G n^{\alpha/2} + H n^{1-3\alpha/2}\right)\right](1 + o(1)),$$

as $n \to \infty$ and

$$i = \sqrt{-1}; \quad \vec{z}(n) := \begin{pmatrix} z(n-1) \\ z(n) \end{pmatrix}$$

$$F := \sqrt{-\lambda}\left(1 - \frac{\alpha}{2}\right)^{-1}, \quad G := \frac{\alpha + 2(b-a)}{\alpha\sqrt{-\lambda}},$$

$$H := \left[\frac{\left(\sqrt{-\lambda}\right)^3}{24} + \frac{D-E}{\sqrt{-\lambda}}\right]\left(1 - \frac{3\alpha}{2}\right)^{-1}.$$

Proof. We make the ansatz

$$z_n := n^{\gamma} \exp\left[\sum_1^n (Ak^{\delta} + Bk^{\varepsilon} + Ck^{\rho})\right],$$

where $-1 \le \rho < \varepsilon < \delta < 0$, and $A, B, C, \gamma \in \mathbb{R}$ are some numbers.

Define the matrix

$$S_n := \begin{pmatrix} \bar{z}_{n-1} & z_{n-1} \\ \bar{z}_n & z_n \end{pmatrix},$$

here $\bar{z}_n$ denotes the complex conjugate of z_n.

We want to choose $A, B, C, \gamma, \varepsilon, \delta$ and ρ such that

$$S_{n+1}^{-1} B_n S_n = I + R_n, \tag{4.3}$$

for some matrices R_n with $\{\|R_n\|\} \varepsilon l^1$. It follows that an arbitrary solution of equation (4.2) has the form

$$\vec{z}_{n+1} = S_{n+1} \vec{w}_n,$$

where $\vec{w}_n$ is a sequence of vectors which tends to a non-zero vector.

Therefore, the form of the asymptotic of $\vec{z}_{n+1}$ will be determined by the matrix S_{n+1}, i.e. by the parameters A, B, C, γ, δ, ε and ρ.

Let $A = \pm\sqrt{\lambda}$, $B = \pm(\alpha + 2(b-a))/(2\sqrt{\lambda})$, $C = ((-A^3/24) + (E-D)/A)$, $\delta = -\alpha/2$, $\varepsilon = \alpha/2 - 1$, $\gamma = -\alpha/4$ and $\rho = -3\alpha/2$.

Direct computation shows that

$$\det S_{n+1} = [n(n+1)]^{\gamma} 2A(n+1)^{-\alpha/2}\left[1 + BA^{-1}(n+1)^{\varepsilon-\delta} + \cdots\right],$$

where the above dots mean the higher order terms.

It follows that

$$(\det S_{n+1})^{-1} = \pm i\left(2\sqrt{-\lambda}\right)^{-1} n^{\alpha}\left[1 + O\left(\frac{1}{n^{1-\alpha}}\right)\right], \tag{4.4}$$

for large n.

Denote

$$\varphi(n) := 1 - \lambda_{n-1}\lambda_n^{-1}, \quad \psi(n) := \lambda\lambda_n^{-1} + 2(1+s(n))(1+r(n))^{-1} - 2.$$

Then

$$B_n = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \varphi(n) & \psi(n) \end{pmatrix}.$$

We have

$$S_{n+1}^{-1} B_n S_n = (\det S_{n+1})^{-1}\left[\begin{pmatrix} \rho_n & \eta_n \\ -\bar{\eta}_n & -\bar{\rho}_n \end{pmatrix} + \begin{pmatrix} s_n & t_n \\ -\bar{t}_n & -\bar{s}_n \end{pmatrix}\right], \tag{4.5}$$

where

$$\rho_n = |z_n|^2\left(\bar{z}_{n-1}\bar{z}_n^{-1} + z_{n+1}z_n^{-1} - 2\right),$$

$$\eta_n = z_n^2\left(z_{n-1}z_n^{-1} + z_{n+1}z_n^{-1} - 2\right),$$

$$s_n = |z_n|^2\left(-\psi(n) - \varphi(n)\bar{z}_{n-1}\bar{z}_n^{-1}\bar{z}_{n-1}\bar{z}_n^{-1}\right),$$

$$t_n = z_n^2\left(-\psi(n) - \varphi(n)z_{n-1}z_n^{-1}\right)$$

Below we shall estimate the off diagonal element: (det $S_{n+1})^{-1}(\eta_n + t_n)$ of $S_{n+1}^{-1}B_nS_n$.

We compute

$$z_{n-1}z_n^{-1} = \left(1 - \frac{\gamma}{n}\right)\Big[1 - (An^{\delta} + Bn^{\varepsilon} + Cn^{\rho}) + \frac{1}{2}(An^{\delta} + Bn^{\varepsilon} + Cn^{\rho})^2$$

$$- \frac{1}{3!}(An^{\delta} + Bn^{\varepsilon} + Cn^{\rho})^3 + \frac{1}{4!}(An^{\delta} + Bn^{\varepsilon} + Cn^{\rho})^4 + O(n^{5\delta})\Big] \tag{4.6}$$

$$z_{n+1}z_n^{-1} = \left(1 + \frac{\gamma}{n}\right)\Big\{1 + A(n+1)^{\delta} + B(n+1)^{\varepsilon} + C(n+1)^{\rho}$$

$$+ \frac{1}{2}[A(n+1)^{\delta} + B(n+1)^{\varepsilon} + C(n+1)^{\rho}]^2 + \frac{1}{3!}[\ldots]^3 + \frac{1}{4!}[\ldots]^4 + O(n^{5\delta})\Big\}. \tag{4.7}$$

Hence using equations (4.6) and (4.7) and the form of λ_n we have for large n

$$\eta_n + t_n = z_n^2\Big\{A(n+1)^{\delta} + B(n+1)^{\varepsilon} + C(n+1)^{\rho} - (An^{\delta} + Bn^{\varepsilon} + Cn^{\rho})$$

$$+ \frac{1}{2}[(A(n+1)^{\delta} + \ldots)^2 + (An^{\delta} + \ldots)^2]$$

$$+ \frac{1}{3!}[(A(n+1)^{\delta} + \ldots)^3 + (An^{\delta} + \ldots)^3]$$

$$+ \frac{1}{4!}[(A(n+1)^{\delta} + \ldots)^4 + (An^{\delta} + \ldots)^4] + O(n^{5\delta})$$

$$+ 2\gamma[An^{\delta-1} + Bn^{\varepsilon-1} + Cn^{\rho-1}] - \frac{\lambda}{n^{\alpha}} + \frac{\lambda r(n)}{n\alpha}$$

$$- 2[s(n) - r(n) - s(n)r(n)] + O(r^2(n)) + \frac{\alpha}{n}(An^{\delta} + Bn^{\varepsilon} + Cn^{\rho})$$

$$- \frac{\alpha}{n} - \Delta r(n-1)[1 + O(n^{\delta})]\Big\}$$

$$= z_n^2\Big\{\left(A^2 n^{2\delta} - \frac{\lambda}{n^{\alpha}}\right) + \left(\frac{2AB}{n^{-(\varepsilon+\delta)}} - \frac{\alpha}{n} - 2\frac{b-a}{n}\right)$$

$$+ \frac{1}{12}(A^4 n^{4\delta} + 24ACn^{4\delta} - 24[E-D]n^{-2\alpha}) + (A\delta + 2A\lambda + \alpha A)n^{\delta-1}$$

$$-B(\alpha-1)n^{\varepsilon-1} - C\alpha n^{\rho-1} + B^2 n^{2\varepsilon} + O(n^{5\delta}) + O\left(\frac{1}{n^{1+x}}\right)\Big\} \tag{4.8}$$

we also have used assumptions on α, $s(n)$ and $r(n)$. Due to definitions of A, B, C, δ, ε and ρ the first four terms in the above brackets $\{\ldots\}$ vanish and the remaining terms reduce, for large n, to the sum $O(n^{\alpha-2}) + O(n^{-5\alpha/2}) + O(1/n^{1+x})$.

Hence

$$|\eta_n + t_n| = \left| z_n^2 \left[O\left(n^{\alpha-2}\right) + O\left(n^{-5\alpha/2}\right) + O\left(\frac{1}{n^{1+x}}\right) \right] \right|$$
$$= O\left(n^{(\alpha/2)-2}\right) + O\left(n^{-3\alpha}\right) + O\left(\frac{1}{n^{1+x+\alpha/2}}\right),$$

so for large n

$$\left|(\det S_{n+1})^{-1}(\eta_n + t_n)\right| = O\left(n^{(3\alpha/2)-2}\right) + O\left(n^{-2\alpha}\right) + O\left(\frac{1}{n^{1+x-\alpha/2}}\right), \tag{4.9}$$

see equation (4.4). Since $\alpha \in (1/2, 2/3)$ and $x > \alpha/2$ it is clear that the off diagonal elements of $S_{n+1}^{-1} B_n S_n$ are summable.

Concerning the diagonal element $(\det S_{n+1})^{-1}(\rho_n + s_n)$ note that

$$\begin{aligned} |\rho_n + s_n - \det S_{n+1}| &= |z_n|^2 |\bar{z}_{n-1}\bar{z}_n^{-1} + z_{n+1}z_n^{-1} - 2 - \psi(n) - \varphi(n)\bar{z}_{n-1}\bar{z}_n^{-1} \\ &\quad - \left(z_{n+1}z_n^{-1} - \bar{z}_{n+1}\bar{z}_n^{-1}\right)| \\ &= |z_n|^2 |\bar{z}_{n-1}\bar{z}_n^{-1} + \bar{z}_{n+1}\bar{z}_n^{-1} - 2 - \psi(n) - \varphi(n)\bar{z}_{n-1}\bar{z}_n^{-1}| \\ &= |z_{n-1}z_n^{-1} + z_{n+1}z_n^{-1} - 2 - \psi(n) - \varphi(n)z_{n-1}z_n^{-1}| \\ &= |\eta_n + t_n|. \end{aligned} \tag{4.10}$$

Combining equations (4.10) and (4.9) we conclude the proof of equation (4.3) and the statement of theorem.

One could also formulate and prove analogous formula for $\alpha \in (1/3, 1/2]$. However, details of computations are long and not too interesting.

Instead we have the following asymptotic formula for the case $\alpha \in [2/3, 3/4)$.

THEOREM 4.2 *Let $\alpha \in (2/3, 3/4)$. Suppose that λ_n and q_n are the same as in Theorem* 4.1 *and satisfy the same assumptions as above. Then for any $\lambda < 0$ there are two linearly independent solutions of equation* (4.2) *with the asymptotic given by*

$$z_{\pm}(n) = n^{-\alpha/4} \exp\left[\pm \sum_1^n i\left(\sqrt{-\lambda}k^{-\alpha/2} - \frac{\alpha + 2(b-a)}{2\sqrt{-\lambda}} k^{\alpha/2-1} \pm \frac{\alpha + 2(b-a)}{8(-\lambda)^{3/2}} k^{3\alpha/2-2} \right) \right]$$
$$\times (1 + o(1)),$$

as $n \to \infty$.

Proof. (The idea).

We proceed in a similar way as above. First we make the same type of *ansatz*, i.e.

$$z_n := n^{\gamma_1} \exp\left[\sum_1^n (Mk^{\delta_1} + Nk^{\varepsilon_1} + Pk^{\rho_1}) \right],$$

where $-1 \le \rho_1 < \varepsilon_1 < \delta_1 < 0$, and $M, N, P, \gamma_1 \in \mathbb{R}$ are some numbers.

Then by repeating the calculations presented above we find that

$$S_{n+1}^{-1} B_n S_n = I + T_n,$$

with $(\|T_n\|) \in l^1$ provided we take $M = \pm\sqrt{\lambda}$, $N = \pm[\alpha + 2(b-a)](4\lambda)^{-1/2}$, $P = -\pm\frac{\alpha+2(b-a)}{8(\lambda)^{3/2}}$, $\gamma_1 = \frac{-\alpha}{4}$, $\delta_1 = \frac{-\alpha}{2}$, $\varepsilon_1 = \frac{\alpha}{2} - 1$, $\rho_1 = \frac{3\alpha}{2} - 2$. We omit the details.

5. Comments and applications to Jacobi operators

5.1 Comments

First note that the perturbation of n^{α} by $n^{\alpha-1} + Dn^{-\alpha}$ is quite natural because even when $a = b = D = E = 0$ in the asymptotic formula (3.26) appears $\exp \pm [(1/\sqrt{\lambda})n^{\alpha/2}) + \rho n^{1-(3\alpha/2)}]$, with ρ defined in Theorem 3.1. In turn the third term $V(n)n^{-1}$ has the order higher than $Dn^{-2\alpha}$ for $\alpha < 2/3$, $(1 + x > 1 + \alpha/2 > 2\alpha)$ or the order higher than an^{-1} for $\alpha < 1/2$. Concerning the assumptions (3.4) it is clear that they have been essentially used in the above estimates.

Under what type of perturbation of λ_n and q_n the asymptotic formulae are stable?

First note that in the asymptotic formulae for $X_s(n)$, $s = 1, 2$ appears $\sqrt{-\beta(n+1)}$. Therefore any additive perturbation of $-\beta(n)$, i.e. $-\beta(n) + r(n)$ leads to the same asymptotic form of the solutions $X_{\pm}(n)$ (modulo a convergent sequence) provided that $\sqrt{-\beta(n) + r(n)} = \sqrt{-\beta(n)} + r_1(n)$, with $r_1(\cdot) \in l^1$.

However, the following formulation in terms of additive perturbation of λ_n and q_n seems more natural.

Proposition 5.1 *Let λ_n and q_n be as in Theorem* 3.1 *and* $\alpha \in (1/3, 2/3)$.

Consider $\tilde{\lambda}_n = \lambda_n + \varphi_n$, $\tilde{q}_n = q_n + \psi_n$ with

$\varphi_n = O(n^{-y}) = \psi_n$ and $y + 3\alpha/2 > 1$.

Then the asymptotic formulae given in Theorem 3.1 (*resp. Theorem* 3.2) *for solutions of equation* (1.1) (*with $\tilde{\lambda}_n$ and $\tilde{q}_n$*) *remain unchanged.*

Proof. Computing $\widetilde{\beta(n)}$ corresponding to $\tilde{\lambda}_n$ and $\tilde{q}_n$ we find:

$$\widetilde{\beta(n)} = \beta(n)(1 + O(n^{-(y+\alpha)})), \quad \alpha \in \left(\frac{1}{3}, \frac{2}{3}\right),$$

for large n. It follows that $\sqrt{\widetilde{\beta(n)}} = \sqrt{\beta(n)}(1 + O(n^{-(y+\alpha)}))$ and the above note completes the proof.

Remark 5.2 However, the method used for $\lambda > 0$ cannot be applied for general l^1 perturbation, i.e. when $\tilde{\lambda}_n = \lambda_n(1 + r_n)$ and $\tilde{q}_n = q_n(1 + s_n)$, where r_n and s_n are only summable sequences. We need to know how fast decay these perturbations in the power scale (see the arguments used in the formal proofs in section 3). Nevertheless, summable perturbations are allowed by adapting the method used for $\lambda > 0$ in section 4, and this will be shown in a separate work (joint with E. Chernova and S. Naboko).

5.2 Application to Jacobi operators

As is well known asymptotic behavior of generalized eigenvectors of Jacobi operators is strongly related to the spectral analysis of them (via Khan-Pearson subordination theory [13]). This idea has been used by us in several recent papers [6–9,11].

Below we present only one standard application of Theorems 3.1 and 4.1. The application is rather straightforward and more interesting ones will be given in the future work.

For λ_n and q_n given by equation (3) the Jacobi operator J acts in $l^2(\mathbb{N})$ by the formula

$$(Ju)_n = \lambda_{n-1}u_{n-1} + q_n u_n + \lambda_n u_{n+1},$$

with the maximal domain.

Since $\sum_k \lambda_k^{-1} = +\infty$, J defines a self-adjoint operator in $l^2(\mathbb{N})$.

As an easy consequence of Theorems 3.1 and 4.1 we have

THEOREM 5.3 *Let $\alpha \in (1/2, 2/3)$ and let λ_n, q_n be as in equation* (1.3). *Then J is purely absolutely continuous on* $(-\infty, 0)$ *and purely point in* $(0, +\infty)$.

Proof. (Sketch) Fix $\lambda < 0$. By repeating standard reasoning and applying Theorem 4.1 one can check that equation (1.1) has no subordinated solutions (see the proof of Theorem 2.2 in [6]). This fact combined with Khan-Pearson theory proves that J is purely absolutely continuous in $(-\infty, 0)$

In turn Theorem 3.1 implies that equation (1.1) has (for positive λ) exponentially decreasing solution and this, again due to Khan–Pearson theory, implies that J may have only point spectrum on the positive half line. We omit the details.

5.3 Open questions

(a) Extend the results (for arbitrary real λ) for the weights ($\lambda_n = n^{\alpha}(1 + r(n)) + c$, where $c \neq 0$.
(b) Do the methods apply to other sequences λ_n and q_n?

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