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JOB NUMBER: MS 148972
JOURNAL: GDEA

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GDEA 148972—14/12/2005—THAHEER—193397

Journal of Difference Equations and Applications,
Vol. 00, No. 0, 2006, 1–31

New discrete Levinson type asymptotics of solutions of linear systems

JAN JANAS†¶ and MARCIN MOSZYŃSKI‡*

†Instytut Matematyczny Polskiej Akademii Nauk, ul. św., Tomasza 30, Kraków 31-027, Poland
‡Wydzial Matematyki Informatyki i Mechaniki, Uniwersytet Warszawski ul., Banacha 2, Warszawa 02-097, Poland

(Received 9 July 2005; revised 10 November 2005; in final form 17 November 2005)

We prove new discrete versions of Levinson type theorems describing asymptotic behavior of solutions of systems of linear difference equations. We show that for several cases of equations with coefficients possessing some "essential" oscillations the asymptotics should be also essentially corrected, comparing with the classical Levinson's cases studied, e.g. in [2,5,9]. The results obtained here allow to study the asymptotics for some systems with coefficients which are not necessary convergent. As an illustration, an application to spectral studies of some Jacobi matrices is presented, by using the asymptotics of generalized eigenvectors.

Keywords: Linear systems of difference equations; Assymptotics of solutions; The Levinson theorem; Stolz classes; Jacobi matrices; Spectral analysis

1. Introduction

The paper is devoted to the problem of asymptotic behavior of solutions of systems of linear equations having the form

$$x(n+1) = A(n+1)x(n), \quad n \geq n_0, \tag{1.1}$$

where $A = \{A(n)\}_{n \geq n_0}$ is a certain sequence of $d \times d$ invertible complex matrices (we shall be mainly concerned with $d = 2$). The asymptotics of such systems have been investigated in numerous papers. One of the most important is [2], where the classical Levinson's result (see [6]) on asymptotics for differential systems is adopted for the discrete case.

This paper is a continuation of our studies from [9], where some further discrete analogs of the Levinson type results were established. The main motivation of the previous, as well as of the present paper, comes from the spectral analysis of Jacobi operators. Note that the asymptotic methods have been used for spectral studies of Jacobi matrices in numerous papers—see [1,7,9,11–15].

In [9], the asymptotics was described for sequences A being a "small perturbation" of a sequence A' which can be diagonalized to a form satisfying some special conditions—*dichotomy condition*—see [2]) by a bounded variation sequence of diagonalizing matrices.

*Corresponding author. Email: mmoszyns@mimuw.edu.pl
¶Email: najanas@cyf-kr.edu.pl

Journal of Difference Equations and Applications
ISSN 1023-6198 print/ISSN 1563-5120 online © 2006 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/10236190500489897

In this case, the asymptotics for a base solution possesses the classical Levinson's form

$$\left(\prod_{s=n_0}^{n-1} \lambda(s)\right) v(n), \tag{1.2}$$

where $\lambda(n)$ is a suitably chosen eigenvalue of $A'(n)$ and $\{v(n)\}_{n\geq n_0}$ is a vector sequence, which is convergent to a non-zero limit.

Yet, such a simple situation does not occur, in general, for the classes of A' studied in this paper. Here, we study A' that belong to more general classes of matrices, resulting in a new type of asymptotics. We prove that now "the scalar term" in the asymptotics has the form

$$\prod_{s=n_0}^{n-1} \widetilde{\lambda(s)},$$

where $\widetilde{\lambda(n)} = \lambda(n) + r(n)$ is a perturbation of $\lambda(n)$ with some $r(n) \to 0$, and it can be computed by some explicit formulas. It is worth to note that contrary to the cases studied in [9], the term $\widetilde{\lambda(n)}$ depends in an essential way not only on the eigenvalues, but also on eigenvectors of $A'(n)$. Also, the vector term corresponding to $v(n)$ from equation (1.2) is more involved and contains the eigenvectors of $A'(n)$. In most cases, considered here the vector term "oscillates", i.e. it need not converge (but its norm can be estimated from below and above).

Note that asymptotic formulas with the scalar factor different than equation (1.2) can be found, for instance, in [2] (which is used in the proof of our Theorem 4.1).

The methods we use here are more or less standard (successive diagonalization procedure). However, it was necessary to find right classes of matrices A' which are regular enough to carry out the diagonalization method. We consider classes defined in terms of the Stolz D^k algebra (see [20]) and its generalization. In particular, we study matrix sequences given by

$$A'(n) = I + \frac{1}{\mu(n)} V(n), \tag{1.3}$$

with a suitable *weight sequence* $\mu(n) \to +\infty$ and V belonging to the $\mu-$*weighted* D^2 *class* (see Section 2.1). This class is especially interesting for us due to its application to the spectral analysis of Jacobi operators. It turns out that asymptotics of solutions can be found provided

$$\limsup_{n\to+\infty} \operatorname{discr} V(n) < 0 \quad \text{or} \quad \liminf_{n\to+\infty} \operatorname{discr} V(n) > 0.$$

Let us remind the usefulness of the D^k algebras in spectral analysis of difference operators (see [14,20]).

The importance of asymptotic studies of solutions of equation (1.1) for various kinds of applications can hardly be overestimated. We refer to [3,4,17] for interesting examples.

The paper is organized as follows. In Section 2, we introduce notation and some technical facts, which we use in the next sections. The most important is Lemma 2.2 on preserving of the weighted D^k classes by C^k transformations.

In Section 3, we study the case of $A' \in D^k$ with the *limes superior* of the discriminant being negative. We describe a general procedure for any $k \geq 2$ (Theorem 3.1), and we show the explicit formulas in the case $k = 2$ (Remarks 3.2, no. 3).

In Section 4, we prove a theorem (Theorem 4.1) which is formulated for any dimension d, with the assumption that $A(n)$ converge to a limit possessing d nonzero eigenvalues with pairwise different absolute values, and with some weak oscillation assumption. As a consequence, for $d = 2$, we obtain a positive discriminant analog (Corollary 4.3) of the "$k = 2$ result" of Section 3. Note that the oscillation assumption is here slightly different than in the previous section (see Remark 4.2).

The most difficult from the proof-technical point of view (but also the most interesting for the Jacobi matrix applications) are the results of the Section 5. We study there the class mentioned above, described by equation (1.3) with the positive or negative discriminant assumptions (Theorems 5.1 and 5.3).

The last section is devoted to some illustrations of the abstract results of the previous sections by examples related to the spectral analysis of Jacobi operators. We refer to our paper in preparation [10] where we shall present applications of the asymptotic results to spectral studies in more general situations.

Appendix contains some longer proofs of facts from Preliminaries.

2. Preliminaries

In this section, we introduce the notation used in the paper, and in several subsections we give definitions and prove some technical lemmas used in the main part of the paper. Some longer proofs are passed to Appendix. We start with the basic notation.

Let us fix $d \in \mathbb{N}$. By $M_d(K)$ we denote the set of $d \times d$ matrices with the entries in K for $K = \mathbb{C}$ or $\mathbb{R}$. We fix an arbitrary norm $\|\cdot\|$ in $\mathbb{C}^d$ and we use the same symbol also for the induced operator norm in $M_d(\mathbb{C})$. For $s \in \{1, \ldots, d\}$ the s-th standard base vector of $\mathbb{C}^d$ is denoted by e_s, if $v \in \mathbb{C}^d$, then the s-th coordinate of v is usually denoted by v_s or $(v)_s$.

For $A \in M_d(\mathbb{C})$ and $s, s' \in \{1, \ldots, d\}$ the entry from the s-th row and the s'-th column of A is denoted by $A_{ss'}$. By $\operatorname{Diag} A$ we denote the diagonal matrix with the diagonal of A, i.e. $\operatorname{Diag} A = \operatorname{diag}(A_{11}, \ldots, A_{dd})$, where $\operatorname{diag}(v_1, \ldots, v_d)$ is the diagonal matrix from $M_d(\mathbb{C})$ with $v_1, \ldots, v_d$ being the successive diagonal entries. If $d = 2$, the symbol $\operatorname{discr} A$ denotes the discriminant of the characteristic polynomial of A, i.e. $\operatorname{discr} A = (\operatorname{tr} A)^2 - 4 \det A$.

We shall use the following convention for products of matrices: $\prod_{j=k}^{l} A(j)$ equals $A(l) \cdot \ldots \cdot A(k)$ if $l > k$, if $l = k$ it equals A_k, and if $l < k$ it equals I.

Let X be a finite dimensional normed space (we mainly consider here $X = K, K^d, M_d(K)$, for $K = \mathbb{C}, \mathbb{R}$), and let $\mu := \{\mu(n)\}_{n \geq N}$ be a sequence of positive numbers ("the weight sequence"), with some $N \in \mathbb{Z}$. For $p \in (1, +\infty)$ and a sequence $x := \{x(n)\}_{n \geq n_0}$ of elements of X we say that x is a $l^p(\mu)$ sequence or simply $x \in l^p(\mu)$, iff $\sum_{n=\max(n_0,N)}^{+\infty} \|x(n)\|^p \mu(n) < +\infty$. In the case $\mu \equiv 1$, we write also l^p instead of $l^p(\mu)$. Note that we use the same notation $l^p(\mu)$ for any X and any starting index n_0 of the sequence. The set of bounded sequences is denoted by l^∞. We write $x(n) \to g$ to denote the convergence of x to the limit $g \in X$. The discrete derivative of x is denoted by Δx, i.e. $\Delta x = \{x(n+1) - x(n)\}_{n \geq n_0}$. For $k = 0, 1, 2, \ldots$ the symbol Δ^k denotes the k-th power of the operator Δ (and $\Delta^0 = I$).

The remaining notation is introduced in the subsections.

2.1 Stolz classes of matrices—generalizations and properties

In this subsection we study the classes D^k and $D^{k,r}$ introduced by Stolz in [20]. We present generalizations of some results and notions from [20].

We start from a generalization of the D^k classes. Let X, x and μ be as before, and let $k = 1, 2, \ldots$. We say that x is a $D^k(\mu)$ sequence or simply $x \in D^k(\mu)$, iff $x \in l^\infty$ and $\Delta^m x \in l^{(k/m)}(\mu)$ for $m = 1, \ldots k$. In the case $\mu \equiv 1$, we write also D^k instead of $D^k(\mu)$. We say that the weight sequence μ is shiftable iff $\{(\mu(n))/(\mu(n+1))\}_{n \geq N} \in l^\infty$. This condition guarantees that the $l^p(\mu)$ classes are left shift invariant. We say that μ is separated from zero iff $\liminf_{n \to 0} \mu(n) > 0$. This condition guarantees that $l^p(\mu) \subset l^q(\mu)$ when $p \leq q$.

Example 2.1 Consider $\mu(n) = n^\alpha$ with some $\alpha > 0$, and let x be a scalar sequence given by $x(n) = h(n^\gamma)$, where $\gamma > 0$ and h is a C^2 scalar function on $[1; +\infty)$. It can be easily proved that if

$$\int_1^{+\infty} (|h'(s)|^2 + |h''(s)|) s^{1 + \frac{\alpha - 1}{\gamma}} \, \mathrm{d}s < +\infty,$$

then $x \in D^2(\mu)$. Thus, if we assume that $\alpha < 1$, h' and h'' are bounded and

$$\frac{1 - \alpha}{2} > \gamma > 0 \tag{2.1}$$

then $x \in D^2(\mu)$. In particular, if equation 2.1 holds, then the sequences given by the formulas $\sin(n^\gamma)$ or $\cos(n^\gamma)$ are in $D^2(\mu)$.

The following lemma shows that the weighted D^k classes can be preserved under C^k transformations (i.e. transformations possessing continuous k-th differential). For $G \subset X$ containing all the elements $x(n)$ of x, a set X' and a function $f : G \to X'$ we define $f(x) = \{f(x(n))\}_{n \geq n_0}$.

Lemma 2.2 *Let μ be a shiftable weight sequence, separated from zero. Suppose that X, X' are finite dimensional real normed spaces and that $K \subset U \subset X$, where U is open and K is a compact and convex set. If $f : U \to X'$ is a C^k function and x is a $D^k(\mu)$ sequence of elements of K, then $f(x) := \{f(x(n))\}_{n \geq n_0}$ is a $D^k(\mu)$ sequence in X'.*

The proof is placed in Appendix.

Observe that choosing the space X and the function f properly, from the above lemma we can derive that acting by some operations on two (or more) $D^k(\mu)$ sequences we obtain a sequence being still in $D^k(\mu)$. It is true, for instance, for the product of scalar or matrix sequences (in the real and complex case). The problem of the quotient of two $D^k(\mu)$ scalar sequences x and y is slightly more delicate, even for y^{-1} being bounded, because of the convexity assumption in Lemma 2.2. Nevertheless, this problem can be solved easily, as it is shown below.

Lemma 2.3 *Let μ be a shiftable weight sequence, separated from zero. Suppose that $x := \{x(n)\}_{n \geq n_0}$ and $y := \{y(n)\}_{n \geq n_0}$ are two $D^k(\mu)$ scalar sequences and that*

$$\inf_{n \geq n_0} |y(n)| > 0. \tag{2.2}$$

Then $(x/y) \in D^k(\mu)$.

Proof. Observe first that using $x(n)(y(n))^{-1} = x(n)\overline{y(n)}|y(n)|^{-2}$ and Lemma 2.2 (for f being the product) we can reduce the problem to the case of y being real. We have $\Delta y \in l^k(\mu)$ and thus $y(n+1) - y(n) \to 0$, since μ is separated from zero. Hence, using equation (2.2), for real y we see that the sign of $y(n)$ is constant for n large enough. Thus, the assertion follows from Lemma 2.2. □

Let us recall the notion of Stolz's $D^{k,r}$ classes (we shall not need any "weighted generalization" of them). Let $r \in \{0,\ldots,k-1\}$, we say that x is a $D^{k,r}$ sequence or simply $x \in D^{k,r}$, iff $\Delta^j x \in l^{(k/(j+r))}$ for $j = 1,\ldots k-r$. Note that

$$D^{k,k-1} = D^1, \quad D^{k,0} \cap l^\infty = D^k \tag{2.3}$$

for any $k = 1,2,\ldots$.

By E we shall denote the matrix changing the order in the standard base of $\mathbb{C}^2$, i.e.

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Definition 2.4 *Consider sequences* $\{\Lambda(n)\}_{n\geq n_0}$, $\{S(n)\}_{n\geq n_0}$ *of complex* 2×2 *matrices, where* $\Lambda(n) = \mathrm{diag}\,(\lambda_+(n), \lambda_-(n))$ *for* $n \geq n_0$. *We say that the pair* $\{\Lambda(n)\}_{n\geq n_0}$, $\{S(n)\}_{n\geq n_0}$ *satisfies generalized Stolz conditions k, r (we shall also use the abbreviation* $GSC_{k,r}$*), iff the following conditions hold*

$$\inf_{n\geq n_0} \mathrm{Im}\,\lambda_+(n) > 0, \ \lambda_-(n) = \overline{\lambda_+(n)}, \ n \geq n_0, \tag{2.4}$$

$$\{\Lambda(n)\}_{n\geq n_0} \in D^k, \ \{S(n)\}_{n\geq n_0} \in D^{k,r} \cap l^\infty, \tag{2.5}$$

$$\det S(n) \neq 0 \quad for \quad n \geq n_0 \quad and \quad \{(S(n))^{-1}\}_{n\geq n_0} \in l^\infty, \tag{2.6}$$

$$\text{(a)} \quad \overline{S(n)} = ES(n)E, \quad or \ \ \text{(b)} \quad \overline{S(n)} = S(n)E. \tag{2.7}$$

If moreover equation 2.7 (*a*) *holds, we shall denote this case by* $GSC_{k,r}$ (*a*) *and similarly for* (*b*).

Note that the original Stolz conditions from [20, Theorem 4] contained one extra condition, namely

$$|\lambda_+(n)| \to 1. \tag{2.8}$$

We shall denote $\mathrm{GSC}_{k,r}$ with equation (2.8) by $\mathrm{OSC}_{k,r}$. The following lemma is a generalization of the Stolz result mentioned above.

Lemma 2.5 *If* $k \geq 2$, $r \in \{0,\ldots,k-2\}$ *and the pair* $\{\Lambda(n)\}_{n\geq n_0}$, $\{S(n)\}_{n\geq n_0}$ *satisfies* $GSC_{k,r}$, *then there exists a pair* $\{\Lambda'(n)\}_{n\geq n_0'}$, $\{S'(n)\}_{n\geq n_0'}$ *satisfying* $GSC_{k,r+1}$ (*a*), *where* $n_0' \geq n_0 + 1$ *and* $\Lambda'(n) = \mathrm{diag}\,(\lambda'_+(n), \lambda'_-(n))$, *such that the following conditions hold*

$$\Lambda(n)(S(n))^{-1}S(n-1) = S'(n)\Lambda'(n)(S'(n))^{-1}, \quad n \geq n_0', \tag{2.9}$$

$$S'(n) \to I, \tag{2.10}$$

$$\left|\frac{\lambda'_+(n)}{\lambda_+(n)}\right| \to 1. \tag{2.11}$$

Proof. In the original Stolz result GSC was replaced by OSC. Observe that

$$|\lambda_+(n)|^{-1} = \left((\mathrm{Re}\,\lambda_+(n))^2 + (\mathrm{Im}\,\lambda_+(n))^2\right)^{-\frac{1}{2}}, \quad n \geq n_0.$$

Thus, using the fact that $\{\lambda_+(n)\}_{n\geq n_0} \in D^k$ and that equation (2.4) holds, by Lemma 1.2 we see that $\{|\lambda_+(n)|^{-1}\}_{n\geq n_0} \in D^k$. Therefore, defining $\tilde{\Lambda}(n) = |\lambda_+(n)|^{-1}\Lambda(n), \quad n \geq n_0$, and using again Lemma 2.2, we obtain $\{\tilde{\Lambda}(n)\}_{n\geq n_0} \in D^k$ and moreover, the "rescalled" pair $\{\tilde{\Lambda}(n)\}_{n\geq n_0}$, $\{S(n)\}_{n\geq n_0}$ satisfies $\mathrm{OSC}_{k,r}$. Now we can use the Stolz result [20, Theorem 4.1], and we obtain $\{\tilde{\Lambda}'(n)\}_{n\geq n_0'}$, $\{S'(n)\}_{n\geq n_0'}$ satisfying $\mathrm{OSC}_{k,r+1}$ (a), equation (2.10), and the analog of equation (2.9) for the "rescalled" pair. Now it is enough to define $\Lambda(n)' = |\lambda_+(n)|\tilde{\Lambda}'(n), \quad n \geq n_0{}'$, and use the fact that $\{|\lambda_+(n)|\}_{n\geq n_0} \in D^k$ (by the arguments as above). □

2.2 Local diagonalization of matrices

DEFINITION 2.6 *Let $X_0 \in M_d(\mathbb{C})$. The triple $(U, \mathcal{D}, \mathcal{T})$, where U is an open neighborhood of X_0 in $M_d(\mathbb{C})$, and $\mathcal{D}, \mathcal{T} : U \to M_d(\mathbb{C})$ are C^1 functions (in the sense of $2d^2$ real variables functions) such that for any $X \in U$ $\mathcal{D}(X)$ is diagonal, $\mathcal{T}(X)$ is invertible and*

$$X = \mathcal{T}(X)\mathcal{D}(X)(\mathcal{T}(X))^{-1},$$

we call a local C^1 diagonalization for X_0.

The following lemma shows that each matrix with only simple eigenvalues possesses a local C^1 diagonalization.

LEMMA 2.7 *Let $X_0, \Lambda_0, T_0 \in M_d(\mathbb{C})$ and suppose that T_0 is invertible, $\Lambda_0 = \mathrm{diag}\,(\lambda_{01}, \ldots, \lambda_{0d})$ with $\lambda_{0j} \neq \lambda_{0j'}$ for $j \neq j'$ and $X_0 = T_0\Lambda_0T_0^{-1}$. Then there exist an open neighborhood U of X_0 in $M_d(\mathbb{C})$ and holomorphic functions (as functions of d^2 complex variables—entries of a matrix from $M_d(\mathbb{C})$) $\mathcal{T}, \mathcal{D} : U \to M_d(\mathbb{C})$ such that $\mathcal{T}(X_0) = T_0, \mathcal{D}(X_0) = \Lambda_0$, and the matrices $\mathcal{T}(X)$ are invertible, $\mathcal{D}(X)$ are diagonal and $X = \mathcal{T}(X)\mathcal{D}(X)\mathcal{T}(X)^{-1}$ for $X \in U$. In particular $(U, \mathcal{D}, \mathcal{T})$ is a local C^1 diagonalization for X_0.*

The proof can be found in Appendix.

2.3 Explicit diagonalization in special cases

Here, we present some explicit expressions for diagonalization of 2×2 matrices from three particular classes.

2.3.1 Negative discriminant matrices. Let X be a 2×2 real matrix with $\mathrm{discr}\,X < 0$. By $\boldsymbol{\lambda}(X)$ let us denote the eigenvalue of X given by

$$\boldsymbol{\lambda}(X) = \frac{1}{2}(\mathrm{tr}\,X + i\sqrt{-\mathrm{discr}\,X}). \tag{2.12}$$

Observe that $X_{12}, X_{21} \neq 0$, since

$$\operatorname{discr} X = (X_{11} - X_{22})^2 + 4X_{12}X_{21}. \tag{2.13}$$

Thus, we can define $\mathbf{z}(X) = (\boldsymbol{\lambda}(X) - X_{11})X_{12}^{-1}$, and the diagonalizing matrix for X

$$\mathbf{S}(X) = \begin{pmatrix} 1 & 1 \\ \mathbf{z}(X) & \overline{\mathbf{z}(X)} \end{pmatrix}. \tag{2.14}$$

We have

$$\det \mathbf{S}(X) = \frac{-2i \operatorname{Im} \boldsymbol{\lambda}(X)}{X_{12}}, \tag{2.15}$$

hence $S(X)$ is invertible and we have

$$(\mathbf{S}(X))^{-1} X \mathbf{S}(X) = \operatorname{diag}(\boldsymbol{\lambda}(X), \overline{\boldsymbol{\lambda}(X)}). \tag{2.16}$$

2.3.2 Positive discriminant matrices. Let X be a 2×2 real matrix with $\operatorname{discr} X > 0$. By $\boldsymbol{\nu}_{\pm}(X)$ let us denote the eigenvalues of X given by

$$\boldsymbol{\nu}_{\pm}(X) = \frac{1}{2}(\operatorname{tr} X \pm \sqrt{\operatorname{discr} X}). \tag{2.17}$$

We define also two sets of "subscripts" related to X

$$\boldsymbol{\Omega}_{\pm}(X) = \{s \in \{1,2\} : X_{ss} \neq \boldsymbol{\nu}_{\pm}(X)\}.$$

Observe that the both sets are nonempty. Suppose, on the contrary, that $X_{11} = \boldsymbol{\nu}_{+}(X) = X_{22}$ or the same for the "—" case. Then for $\sigma = +$ or $-$ we have $0 = \det(X - \boldsymbol{\nu}_{\sigma}(X)I) = -X_{21}X_{12}$, and hence by equation (2.13) $\operatorname{discr} X = (X_{11} - X_{22})^2 = 0$, which is in contradiction with the assumption that $\operatorname{discr} X > 0$.

Moreover, for any $j = 1,2$ we can easily obtain the following estimate

$$|X_{12}X_{21}| \geq \min\{|\boldsymbol{\nu}_{+}(X) - X_{jj}|^2, |\boldsymbol{\nu}_{-}(X) - X_{jj}|^2\} \tag{2.18}$$

Let $s_{\pm} \in \boldsymbol{\Omega}_{\pm}(X)$. We define a diagonalizing matrix $\mathbf{T}(X)$ for X determined by the choice of s_+ and s_-. When this choice is known, we shall also use the shorter notation $\mathbf{T}(X)$. For instance, if we choose $s_+ = s_- = 1$, then

$$\mathbf{T}(X) = \begin{pmatrix} \frac{X_{12}}{\boldsymbol{\nu}_{+}(X) - X_{11}} & \frac{X_{12}}{\boldsymbol{\nu}_{-}(X) - X_{11}} \\ 1 & 1 \end{pmatrix}.$$

In the general case, the entries of $\mathbf{T}(X)$ are given by

$$(\mathbf{T}(X))_{ij} = \begin{cases} 1 & \text{for } (i - s_+, j = 1) \text{ or } (i - s_-, j = 2) \\ \frac{X_{i\hat{i}}}{\boldsymbol{\nu}_{+}(X) - X_{ii}} & \text{for } i = s_+, j = 1 \\ \frac{X_{i\hat{i}}}{\boldsymbol{\nu}_{-}(X) - X_{ii}} & \text{for } i = s_-, j = 2, \end{cases} \tag{2.19}$$

where for $i \in \{1,2\}$ we define $\hat{i}$ by $\hat{i} \in \{1,2\}$, $\hat{i} \neq i$. Since the columns of $\mathbf{T}(X)$ are eigenvectors of X, we easily obtain

$$(\mathbf{T}(X))^{-1} X \mathbf{T}(X) = \operatorname{diag}(\boldsymbol{\nu}_{+}(X), \boldsymbol{\nu}_{-}(X)).$$

Using the formula for $\mathbf{T}(X)$ we can compute

$$\det \mathbf{T}(s_+, s_-, X) = (-1)^{s_+}\sqrt{\operatorname{discr} X}$$

$$\times \begin{cases} (\boldsymbol{\nu}_+(X) - X_{s_+s_+})^{-1} & \text{for } s_+ \neq s_- \\ X_{s_+\hat{s}_+}(\boldsymbol{\nu}_+(X) - X_{s_+s_+})^{-1}(\boldsymbol{\nu}_-(X) - X_{s_+s_+})^{-1} & \text{for } s_+ = s_-. \end{cases} \tag{2.20}$$

2.3.3 Matrices with a special symmetry. We diagonalize here matrices of the form

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$, with $|b| < |\operatorname{Im} a|$. For such a,b let us denote

$$\mathbf{w}(a,b) = \frac{ib}{\operatorname{Im} a + \sigma\sqrt{(\operatorname{Im} a)^2 - |b|^2}},$$

with $\sigma = \operatorname{sgn}(\operatorname{Im} a)$, $\boldsymbol{\rho}(a,b) = \operatorname{Re} a + \sigma i\sqrt{(\operatorname{Im} a)^2 - |b|^2}$. Using the assumptions on a,b it can be easily checked that $|\mathbf{w}(a,b)| \neq 1$, and

$$\begin{pmatrix} 1 & \mathbf{w}(a,b) \\ \overline{\mathbf{w}(a,b)} & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{w}(a,b) \\ \overline{\mathbf{w}(a,b)} & 1 \end{pmatrix} = \operatorname{diag}(\boldsymbol{\rho}(a,b), \overline{\boldsymbol{\rho}(a,b)}).$$

Moreover, we have

$$\frac{|b|}{2|\operatorname{Im} a|} \leq |\mathbf{w}(a,b)| \leq \frac{|b|}{|\operatorname{Im} a|}. \tag{2.21}$$

2.4 Change of variables

Consider a sequence $\{A(n)\}_{n\geq n_0}$ of complex $d\times d$ matrices and the equation (1.1) for a $\mathbb{C}^d$ vector sequence $\{x(n)\}_{n\geq n_0}$. Let $\{S(n)\}_{n\geq N}$, with $N \geq n_0$, be a sequence of complex $d\times d$ invertible matrices. If $y(n) = S(n)x(n), \quad n \geq N$, then equation (1.1) restricted to $n \geq N$ is equivalent to

$$y(n+1) = \tilde{A}(n)y(n), \quad n \geq N, \tag{2.22}$$

where

$$\tilde{A}(n) = S(n+1)A(n)(S(n))^{-1}, \quad n \geq N. \tag{2.23}$$

Such transformation from equation (1.1) to (2.22) we call in this paper the change of variables by $\{S(n)\}_{n\geq N}$.

In the next sections, for each of the considered cases, we use one or several properly chosen changes of variables to obtain finally the equation with $\tilde{A}(n)$ of the form for which the asymptotic results are already known. Such procedure is a standard tool for asymptotic studies, see e.g. [2,8].

3. Asymptotic behavior of solutions for negative discriminant case

We consider here the equation (1.1) with

$$A(n) = V(n) + R(n), \quad n \geq n_0,$$

where $\{V(n)\}_{n\geq n_0}$ and $\{R(n)\}_{n\geq n_0}$ are sequences of 2×2 complex matrices satisfying

$$\{V(n)\}_{n\geq n_0} \text{ is a } D^k \text{ sequence of real matrices,} \tag{3.1}$$

$$\limsup_{n\to+\infty} \operatorname{discr} V(n) < 0, \tag{3.2}$$

$$\{R(n)\}_{n\geq n_0} \in l^1. \tag{3.3}$$

We shall describe a procedure of successive changes of variables for this case. Note that the case $k = 1$ is covered e.g. by Theorem 2.6 of [9]. Thus, assume that $k \geq 2$ here. By equation (3.2) there exist $\delta > 0$ and $N_0 \geq n_0$ such that $\operatorname{discr} V(n) \leq -\delta$ for $n \geq N_0$. For $n \geq N_0$ denote

$$\lambda_+^{(0)}(n) = \boldsymbol{\lambda}(V(n)),\ \lambda_-^{(0)}(n) = \overline{\lambda_+^{(0)}(n)},\ \Lambda^{(0)}(n) = \operatorname{diag}(\lambda_+^{(0)}(n), \lambda_-^{(0)}(n)), \tag{3.4}$$

and

$$S^{(0)}(n) = \mathbf{S}(V(n)) \tag{3.5}$$

(see equation (2.14)). Using equation (2.13) we obtain

$$|V_{12}(n)| > \delta(4M)^{-1}, \tag{3.6}$$

where $M > 0$ is such that $|V_{jj'}(n)| \leq M$ for $j, j' = 1, 2$, $n \geq N_0$. By equation (2.16) we have

$$V(n) = S^{(0)}(n)\Lambda^{(0)}(n)(S^{(0)}(n))^{-1}, \quad n \geq N_0. \tag{3.7}$$

Moreover, using Lemmas 2.2 and 2.3, by equations (2.12), (2.14), (2.15) and (3.6), we obtain $\{S^{(0)}(n)\}_{n\geq N_0}$, $\{\Lambda^{(0)}(n)\}_{n\geq N_0} \in D^k$ and $\{(S^{(0)}(n))^{-1}\}_{n\geq N_0}$ is bounded. We also have $\overline{S^{(0)}(n)} = S^{(0)}(n)E$ for $n \geq n_0$. Thus, the pair of sequences $\{\Lambda^{(0)}(n)\}_{n\geq N_0}$, $\{S^{(0)}(n)\}_{n\geq N_0}$ satisfies $\mathrm{GSC}_{k,0}$.

We shall now inductively apply the following general step. Let $j \in \{0, \dots k-2\}$ and assume that $\{V^{(j)}(n)\}_{n\geq N_j}$, $\{R^{(j)}(n)\}_{n\geq N_j}$, $\{\Lambda^{(j)}(n)\}_{n\geq N_j}$, $\{S^{(j)}(n)\}_{n\geq N_j}$ are 2×2 complex matrix sequences such that the pair $\{\Lambda^{(j)}(n)\}_{n\geq N_j}$, $\{S^{(j)}(n)\}_{n\geq N_j}$ satisfies $\mathrm{GSC}_{k,j}$ and

$$V^{(j)}(n) = S^{(j)}(n)\Lambda^{(j)}(n)(S^{(j)}(n))^{-1}, \quad n \geq N_j. \tag{3.8}$$

For the equation

$$x^{(j)}(n+1) = \big(V^{(j)}(n) + R^{(j)}(n)\big)x^{(j)}(n), \quad n \geq N_j$$

we change the variables by $\{(S^{(j)}(n-1))^{-1}\}_{n\geq N_j+1}$ and setting $x^{(j+1)}(n) = (S^{(j)}(n-1))^{-1}$ $x^{(j)}(n)$ we obtain

$$x^{(j+1)}(n+1) = \big(V^{(j+1)}(n) + R^{(j+1)}(n)\big)x^{(j+1)}(n), \quad n \geq N_j + 1,$$

where (see equation (2.23))

$$V^{(j+1)}(n) = \Lambda^{(j)}(n), \tag{3.9}$$

$$R^{(j+1)}(n) = (S^{(j)}(n))^{-1} R^{(j)}(n) S^{(j)}(n-1). \tag{3.10}$$

Using Lemma 2.5 we can represent $V^{(j+1)}(n)$ in the form analogous to equation (3.8), i.e. there exists $N_{j+1} \geq N_j + 1$ and the pair of sequences $\{\Lambda^{(j+1)}(n)\}_{n \geq N_{j+1}}$, $\{S^{(j+1)}(n)\}_{n \geq N_{j+1}}$ satisfying $\mathrm{GSC}_{k,j+1}$ such that

$$V^{(j+1)}(n) = S^{(j+1)}(n)\Lambda^{(j+1)}(n)(S^{(j+1)}(n))^{-1}, \quad n \geq N_{j+1},$$

and moreover

$$S^{(j+1)}(n) \to I. \tag{3.11}$$

This makes it possible to repeat the single step "until $j = k - 2$".

Let us apply the above procedure to our equation (1.1), i.e. denote $V^{(0)}(n) = V(n)$, $R^{(0)}(n) = R(n)$ and $x^{(0)}(n) = x(n)$ for $n \geq N_0$. By our previous considerations the procedure can be started. After $k - 1$ steps we obtain the equation

$$x^{(k-1)}(n+1) = \big(V^{(k-1)}(n) + R^{(k-1)}(n)\big)x^{(k-1)}(n), \quad n \geq N_{k-1}, \tag{3.12}$$

which is equivalent to equation (1.1) restricted to $n \geq N_{k-1}$, if we substitute

$$x^{(k-1)}(n) = (U(n-1))^{-1}x(n), \quad n \geq N_{k-1}, \tag{3.13}$$

where for $n \geq N_{k-1} - 1$

$$U(n) = \prod_{j=0}^{k-2} S^{(j)}(n). \tag{3.14}$$

Moreover, we have

$$V^{(k-1)}(n) = S^{(k-1)}(n)\Lambda^{(k-1)}(n)(S^{(k-1)}(n))^{-1}, \quad n \geq N_{k-1}, \tag{3.15}$$

and

$$R^{(k-1)}(n) = (U(n))^{-1}R(n)U(n-1), \quad n \geq N_{k-1}, \tag{3.16}$$

where the pair $\{\Lambda^{(k-1)}(n)\}_{n \geq N_{k-1}}$, $\{S^{(k-1)}(n)\}_{n \geq N_{k-1}}$ satisfy $\mathrm{GSC}_{k,k-1}$,

$$S^{(k-1)}(n) \to I, \tag{3.17}$$

and

$$U(n) = S^{(0)}(n)U'(n), \quad U'(n) \to I. \tag{3.18}$$

It is crucial now that

$$\{S^{(k-1)}(n)\}_{n \geq N_{k-1}} \in D^1 \tag{3.19}$$

(since $D^{k,k-1} = D^1$)!. Therefore, to study equation (3.12) we can use some known "D^1-Levinson" type results, e.g. the results of [9] (see [2]) and we obtain the following theorem.

THEOREM 3.1 *Assume that equations* (3.1)–(3.3) *hold with* $k \geq 2$. *Let* $N \geq N_{k-1}$ *and for* $n \geq N$ *let* $\lambda_{\pm}^{(k-1)}(n)$ *be the eigenvalues of* $V^{(k-1)}(n)$ *with* $\operatorname{Im} \lambda_{+}^{(k-1)}(n) > 0$ *and* $\lambda_{-}^{(k-1)}(n) = \overline{\lambda_{+}^{(k-1)}(n)}$ (*where* $V^{(k-1)}(n)$ *and* N_{k-1} *are introduced above*). *Suppose that* $\det(V(n) + R(n)) \neq 0$ *for* $n \geq N$. *Then the equation*

$$x(n+1) = (V(n) + R(n))x(n), \quad n \geq N \tag{3.20}$$

has a base of solutions $\{x_+(n)\}_{n \geq N}$, $\{x_-(n)\}_{n \geq N}$ *of the form*

$$x_{\pm}(n) = \left(\prod_{s=N}^{n-1} \lambda_{\pm}^{(k-1)}(s) \right) v_{\pm}(n), n \geq N, \tag{3.21}$$

where $v_{\pm}(n)$ *are* $\mathbb{C}^2$ *vectors such that*

$$v_{\pm}(n) = S^{(0)}(n-1)e_{\pm}(n),\ e_+(n) \rightarrow e_1,\ e_-(n) \rightarrow e_2, \tag{3.22}$$

with $S^{(0)}(n)$ *given by equation* (3.5). *Moreover,* $\inf_{n \geq N} \lambda_{+}^{(k-1)}(n) > 0$ *and*

$$\lambda_{+}^{(k-1)} = \boldsymbol{\lambda}(V(n)) + \xi(n), \quad \text{with } \{\xi(n)\}_{n \geq N} \in l^k. \tag{3.23}$$

Proof. Observe that $\Lambda^{(k-1)}(n) = \operatorname{diag}\left(\lambda_{+}^{(k-1)}(n), \lambda_{-}^{(k-1)}(n)\right)$ for $n \geq N$. Hence, by equations (3.3),(3.15)–(3.17),(3.19), we can use [9, Corolary 2.2] and we obtain the existence of two solutions of the equation (3.12) restricted to $n \geq N$ of the form $\left(\prod_{s=N}^{n-1} \lambda_{\pm}^{(k-1)}(s)\right) e_{\pm}(n)$, with $e_{\pm}(n)$ as in equation (3.22). Thus by equations (3.13) and (3.18) and by the previous considerations we obtain the formula for solutions of equation (3.20). The linear independence of the solutions follows immediately from equations (3.21) and (3.22) and the fact that e_1, e_2 are linearly independent $\mathbb{C}^2$ vectors. To prove equation (3.23) it is sufficient to use equation (3.9) for $j = 0, \ldots, k-2$, the formula

$$(S^{(j)}(n))^{-1} S^{(j)}(n-1) = I - (S^{(j)}(n))^{-1} (\Delta S^{(j)})(n-1),$$

and the explicit formula for the eigenvalues of a 2×2 matrix. □

Remarks 2.2

1. The analog asymptotic result for the case $k = 1$ (see [9, Theorem 1.6]) is simpler. The appropriate formulas have the form

$$x_{\pm}(n) = \left(\prod_{s=N}^{n-1} \lambda_{\pm}^{(0)}(s) \right) v_{\pm}(n), \quad n \geq N, \tag{3.24}$$

where $\lambda_{\pm}^{(0)}(n)$ are the eigenvalues of $V^{(0)}(n) = V(n)$:

$$\lambda_{+}^{(0)}(n) = \boldsymbol{\lambda}(V(n)),\ \lambda_{-}^{(0)}(n) = \overline{\lambda_{+}^{(0)}(n)}, \tag{3.25}$$

and

$$v_{\pm}(n) \rightarrow v_{\infty\pm}, \tag{3.26}$$

where

$$v_{\infty+} = (1, \mathbf{z}(V_{\infty})),\ v_{\infty-} = \overline{v_{\infty+}}, \tag{3.27}$$

with V_{∞} being the limit of $V(n)$.

2. If $\{V(n)\}_{n\geq n_0} \in D^1$ then it is a convergent sequence, but D^k sequences for $k > 1$ need not to converge in general. If we additionally assume in Theorem 3.1 that $V(n) \to V_\infty$ then equation (3.2) means that $\operatorname{discr} V_\infty < 0$, and by equations (3.22) and (3.5) we have equations (3.26) and (3.27) similarly as in the case $k = 1$.
3. In the case $k = 2$, the numbers $\lambda_\pm^{(1)}(n)$ can be also explicitly computed. By equation (3.9) they are the eigenvalues of $V^{(1)}(n) = \Lambda^{(0)}(n)(S^{(0)}(n))^{-1}S^{(0)}(n-1)$, i.e.

$$\lambda_\pm^{(1)}(n) = \frac{1}{2}\left(\operatorname{tr} V^{(1)}(n) \pm i\sqrt{-\operatorname{discr} V^{(1)}(n)}\right), \quad n \geq N. \tag{3.28}$$

Moreover, using $\Delta\{\mathbf{z}(V(n))\}_{n\geq N_0} \in l^2$, after some simple computations we get

$$\lambda_\pm^{(1)}(n) = \lambda_\pm^{(0)}(n)(1 - \eta_\pm(n))(1 + r_\pm(n)) \tag{3.29}$$

with $\lambda_\pm^{(0)}(n)$ defined by equation (3.25), $r_\pm \in l^1$, and $\eta_\pm \in l^2$ given by

$$\eta_+(n) = \frac{\mathbf{z}(V(n)) - \mathbf{z}(V(n-1))}{2\operatorname{Im}\mathbf{z}(V(n))}, \quad \eta_-(n) = \overline{\eta_+(n)}. \tag{3.30}$$

Thus, by equation (3.21) we get a base of solutions $\{x_+(n)\}_{n\geq N}$, $\{x_-(n)\}_{n\geq N}$ of equation (3.20) such that for some $N' \geq N$

$$x_\pm(n) = \left(\prod_{s=N'}^{n-1} \lambda_\pm^{(0)}(s)\right)\left(\prod_{s=N'}^{n-1}(1 - \eta_\pm(s))\right)v_\pm(n), \quad n \geq N', \tag{3.31}$$

with $v_\pm(n)$ as in equation (3.22).

The above formula shows, that the scalar term in the asymptotics is essentially changed in comparison with the "D^1—case" described by equation (3.24). The correction is essential, providing that $\sum_{n=N'}^{+\infty}\eta_\pm(n)$ diverges (see [5, Theorem 8.12]), which is the typical situation for our "D^2—case".

4. Asymptotics for positive discriminant case

We present here a theorem formulated for general dimension d of the system (1.1). As a special case, for $d = 2$ and positive discriminant limit of the sequence $\{A(n)\}_{n\geq n_0}$, we get a result which can be treated as an analog of the theorem from the previous section. Below we use the notion of local C^1 diagonalization (see Definition 2.6).

THEOREM 4.1 *Let V_∞ be a $d\times d$ complex matrix having d nonzero eigenvalues with pairwise different absolute values, and let $(U, \mathcal{D}, \mathcal{T})$ be a local C^1 diagonalization for V_∞. Suppose that $\{V(n)\}_{n\geq n_0}$, $\{R(n)\}_{n\geq n_0}$ are sequences of complex matrices satisfying*

$$V(n) \in U, \quad n \geq n_0, \tag{4.1}$$

$$V(n) \to V_\infty, \tag{4.2}$$

$$\{V(n+1) - V(n)\}_{n\geq n_0} \in l^2, \tag{4.3}$$

$$\{R(n)\}_{n\geq n_0} \in l^1. \tag{4.4}$$

Denote for $n \geq n_0$

$$\operatorname{diag}(\lambda_{1\infty}, \ldots, \lambda_{d\infty}) = \mathcal{D}(V_\infty), \quad \Lambda(n) = \operatorname{diag}(\lambda_1(n), \ldots, \lambda_d(n)) = \mathcal{D}(V(n)), \tag{4.5}$$

$$S(n) = \mathcal{T}(V(n)) \tag{4.6}$$

and

$$\xi_j(n) = -\big((S(n+1))^{-1}(S(n+1) - S(n))\Lambda(n)\big)_{jj}, \quad j = 1, \ldots, d. \tag{4.7}$$

If for $n \geq n_0$

$$\det(V(n) + R(n)) \neq 0, \quad \lambda_j(n) + \xi_j(n) \neq 0, \quad j = 1, \ldots, d, \tag{4.8}$$

then the equation

$$x(n+1) = (V(n) + R(n))x(n), \quad n \geq n_0 \tag{4.9}$$

has a base of solutions $\{x_j(n)\}_{n \geq n_0}$, $j = 1, \ldots, d$, *of the form*

$$x_j(n) = \left(\prod_{s=n_0}^{n-1} (\lambda_j(s) + \xi_j(s)) \right) v_j(n), \quad j = 1, \ldots, d, n \geq n_0, \tag{4.10}$$

where $v_j(n)$ *are* C^d *vectors such that*

$$v_j(n) \to v_{j\infty}, \quad j = 1, \ldots, d, \tag{4.11}$$

with $v_{j\infty}$ *being an eigenvector of* V_∞ *for the eigenvalue* $\lambda_{j\infty}$. *Moreover,* $\{\xi_j(n)\}_{n \geq n_0} \in l^2$.

Proof. Let us change the variables in equation (4.9) by $\{(S(n))^{-1}\}_{n \geq n_0}$. The equation for $y(n) = (S(n))^{-1}x(n)$ has the form

$$y(n+1) = (\Lambda(n) + W(n))y(n), \quad n \geq n_0 \tag{4.12}$$

where

$$W(n) = -(S(n+1))^{-1}(S(n+1) - S(n))\Lambda(n) + (S(n+1))^{-1}R(n)S(n). \tag{4.13}$$

Observe that by equations (4.1) and (4.2), all the $V(n)$-s are contained in a compact subdomain of $\mathcal{T} : U \to M_d(\mathbb{C})$. Since $\mathcal{T}$ is a C^1 function, by equation (4.6) there exists $K > 0$ such that

$$\|S(n+1) - S(n)\| \leq K\|V(n+1) - V(n)\|, \quad n \geq n_0.$$

Thus, by equation (4.3) $\{S(n+1) - S(n)\}_{n \geq n_0} \in l^2$. Moreover, by equations (4.5) and (4.6),

$$S(n) \to S_\infty,\ (S(n))^{-1} \to (S_\infty)^{-1},\ \Lambda(n) \to \Lambda_\infty, \tag{4.14}$$

where $S_\infty = \mathcal{T}(V_\infty)$, $\Lambda_\infty = \mathcal{D}(V_\infty)$. Thus $\{\xi_j(n)\}_{n \geq n_0} \in l^2$, and by equations (4.4) and (4.13), we have $\{W(n)\}_{n \geq n_0} \in l^2$, and $|\lambda_j(n)(\lambda_{j'}(n))^{-1}| \to |\lambda_{j\infty}(\lambda_{j'\infty})^{-1}| \neq 1$, $j \neq j'$. Therefore, by [2, Theorem 3.3] there exists $N \geq n_0$ and a sequence of invertible matrices $\{B(n)\}_{n \geq N}$ such that

$$B(n) \to I \tag{4.15}$$

and that changing variables in equation (4.12) by this sequence we obtain

$$w(n+1) = (\Lambda(n) + \operatorname{Diag} W(n) + R'(n))w(n), \quad n \geq N \tag{4.16}$$

for $w(n) = B(n)y(n)$, with $\{R'(n)\}_{n \geq N} \in l^1$. Now, using equation (4.13), we can write equation (4.16) in the form

$$w(n+1) = (\tilde{\Lambda}(n) + R''(n))w(n), \quad n \geq N, \tag{4.17}$$

where $\tilde{\Lambda}(n) = \Lambda(n) - \operatorname{Diag}\left[(S(n+1))^{-1}(S(n+1) - S(n))\Lambda(n)\right]$ and

$$R''(n) = \operatorname{Diag}\left[(S(n+1))^{-1}R(n)S(n)\right] + R'(n).$$

By equation (4.7) $\tilde{\Lambda}(n) = \operatorname{diag}(\lambda_1(n) + \xi_1(n), \ldots, \lambda_d(n) + \xi_d(n))$. Moreover, $\lambda_j(n) + \xi_j(n) \to \lambda_{j\infty}$ and $\{R''(n)\}_{n \geq N} \in l^1$. Thus, the new system (4.17) is an l^1 perturbation of a diagonal system, and we can apply here a "diagonal-Levinson" type result. For instance, we can use [9, Theorem 1.2] (since the assumptions (3.24) and (3.25) of [9] are satisfied; alternatively see [2]: Lemma 2.1 + Remark 2.2 (1)). We obtain the existence of solutions

$$w_j(n) = \left(\prod_{s=N}^{n-1} (\lambda_j(s) + \xi_j(s))\right) v_j{}'(n), \quad j = 1, \ldots, d, \quad n \geq N,$$

with $v_j{}'(n) \to e_j$. Now, using equations (4.8),(4.14) and (4.15), we obtain the formula (4.10) for solutions of equation (4.9), and their linear independence easily follows from the independence of eigenvectors of V_∞. □

Remark 4.2 The oscillation assumptions in Theorems 3.1 and 4.1 for the case $k = 2$ are similar, but different. The condition $\{(\Delta V)(n)\}_{n \geq n_0} \in l^2$ is weaker than the condition $\{V(n)\}_{n \geq n_0} \in D^2$, yet in Theorem 4.1 we additionally assume that $V(n)$ is convergent, which is not necessary in Theorem 3.1.

Under the assumptions of the above theorem, a local C^1 diagonalization for V_∞ always exists by Lemma 2.7, nevertheless the general asymptotic formulas obtained in the theorem may be not explicit enough. Sometimes, making some extra assumptions on the matrix V_∞ and on the sequence $\{V(n)\}_{n \geq n_0}$ we can find the formula for the scalar term $\lambda_j(n) + \xi_j(n)$ in the asymptotics (4.10) in a more explicit form. Below we do this for $d = 2$ with "positive discriminant assumption", using the formulas introduced in Section 2.3.2.

Corollary 4.3 *Let $d = 2$ and assume equations* (4.2)–(4.4). *Suppose that $V(n)$ are real matrices with* $\operatorname{discr} V(n) > 0$ *for $n \geq n_0$, and that*

$$\operatorname{discr} V_\infty > 0, \quad \operatorname{tr} V_\infty \neq 0, \quad \det V_\infty \neq 0. \tag{4.18}$$

Define

$$\lambda_{\infty\pm} = \boldsymbol{\nu}_\pm(V_\infty), \quad \lambda_\pm(n) = \boldsymbol{\nu}_\pm(V(n)), \quad n \geq n_0, \tag{4.19}$$

and choose $s_\pm \in \{1, 2\}$ such that $s_\pm \in \boldsymbol{\Omega}_\pm(V_\infty)$. Then there exists $N \geq n_0$ such that for $n \geq N$

$$s_\pm \in \boldsymbol{\Omega}_\pm(V(n)), \tag{4.20}$$

and $1+\delta_\pm(n)\neq 0$, $\det(V(n)+R(n))\neq 0$, *where for any n satisfying equation* (4.20)

$$\delta_+(n) = -\frac{S_{22}(n+1)(S_{11}(n+1)-S_{11}(n)) - S_{12}(n+1)(S_{21}(n+1)-S_{21}(n))}{\det S(n+1)}$$

$$\delta_-(n) = \frac{S_{21}(n+1)(S_{12}(n+1)-S_{12}(n)) - S_{11}(n+1)(S_{22}(n+1)-S_{22}(n))}{\det S(n+1)}, \tag{4.21}$$

with $S(n)=\mathrm{T}(s_+,s_-,V(n))$. *If N is as above, then the equation* $x(n+1)=(V(n)+R(n))x(n)$, $n\geq N$, *has a base of solutions* $\{x_+(n)\}_{n\geq n_0}$, $\{x_-(n)\}_{n\geq n_0}$ *of the form*

$$x_\pm(n) = \left(\prod_{s=N}^{n-1}(\lambda_\pm(s)\right)\left(\prod_{s=N}^{n-1}(1+\delta_\pm(s)\right)v_\pm(n), \quad n\geq N, \tag{4.22}$$

where $v_\pm(n)$ *are* $\mathbb{C}^2$ *vectors such that*

$$v_\pm(n)\to v_{\pm\infty}, \tag{4.23}$$

with $v_{\pm\infty}$ *being the first and the second column of the matrix* $\mathrm{T}(s_+,s_-,V_\infty)$*, respectively. Moreover,* $\{\delta_\pm(n)\}_{n\geq N}\in l^2$.

Proof. The existence of N follows immediately from equations (4.2) and (4.4) and the fact that $\lambda_j(n)+\xi_j(n)$ converge to the j-th eigenvalue of V_∞, i.e. to $\lambda_{\infty+}$ or $\lambda_{\infty-}$. These eigenvalues are nonzero and have different absolute values by equation (4.18). Now the proof follows easily from Theorem 4.3 and the explicit formula for a local C^1 diagonalization for V_∞. This diagonalization can be defined by an analytic extension of the formulas from Section 2.3.2 (note that these formulas refer to the real matrix X case). □

Remarks 4.4 Similarly, as in the previous section (see Remarks 3.2) equation (4.22) shows, that the scalar term in the asymptotics is essentially changed in comparison with the corresponding"D^1—case" (see [9, Th 1.5]).

5. Double eigenvalues—perturbations of *I*

In general, when the limit of $A(n)$ has double eigenvalue, the asymptotic studies are more difficult. In this section, the matrix sequence $\{A(n)\}_{n\geq n_0}$ in the equation (1.1) is a perturbation of I of the form

$$A(n) = I + \frac{1}{\mu(n)}V(n) + R(n). \tag{5.1}$$

The scalar sequence μ satisfies

$$\mu(n)>0, \quad n\geq n_0, \quad \mu(n)\to+\infty, \tag{5.2}$$

$$\sum_{n=n_0}^{+\infty}\frac{(\mu(n+1)-\mu(n))^2}{\mu(n)} < +\infty, \tag{5.3}$$

$$\sum_{n=n_0}^{+\infty}\frac{1}{\mu(n)} = +\infty. \tag{5.4}$$

Observe, that by equation (5.3) $(\mu(n+1)/\mu(n)) \to 1$. Hence, μ is a shiftable weight sequence (see Section 2.1), and moreover, for any sequence $\{u(n)\}_{n\geq N}$ we have

$$\{u(n)\}_{n\geq N} \in l^p(\mu) \quad \text{iff} \quad \{u(n+1)\}_{n\geq N} \in l^p(\mu). \tag{5.5}$$

Note that the conditions (5.2)–(5.4) are satisfied for instance by sequences of the form $\mu(n) = n^\alpha$, $n \geq 1$ for $0 < \alpha < 1$.

For the matrix part of the perturbation we shall assume here that $\{R(n)\}_{n\geq n_0} \in l^1$ and $\{V(n)\}_{n\geq n_0} \in D^2(\mu)$. We present two theorems with different assumptions on the sign of the discriminant of $V(n)$.

5.1 Perturbations with negative discriminant

In this section, we study the case with the negative sign of $V(n)$ in equation (5.1). We use here the functions $\boldsymbol{\lambda}$, $\mathbf{z}$, $\mathbf{S}$ defined in Section 2.3.1.

THEOREM 5.1 *Suppose that equations* (5.1)–(5.4) *hold and that* $\{V(n)\}_{n\geq n_0}$ *and* $\{R(n)\}_{n\geq n_0}$ *are sequences of* 2×2 *complex matrices satisfying*

$$\{V(n)\}_{n\geq n_0} \text{ is a } D^2(\mu) \textit{ sequence of real matrices}, \tag{5.6}$$

$$\limsup_{n\to+\infty} \operatorname{discr} V(n) < 0, \tag{5.7}$$

$$\{R(n)\}_{n\geq n_0} \in l^1. \tag{5.8}$$

Assume also $\operatorname{discr} V(n) < 0$, $n \geq n_0$, *and for* $n \geq n_0$ *define* $\lambda(n) = \boldsymbol{\lambda}(V(n))$, $z(n) = \mathbf{z}(V(n))$. *For* $n \geq n_0 + 1$ *set*

$$r(n) = z(n) - z(n-1), \quad r'(n) = \frac{r(n)}{2\operatorname{Im} z(n)},$$

$$a(n) = \lambda(n) + ir'(n)(\lambda(n) + \mu(n)), \quad b(n) = \overline{ir'(n)}(\lambda(n) + \mu(n)).$$

Then there exists $N \geq n_0 + 1$ *such that for* $n \geq N$

$$\operatorname{Im} a(n) > |b(n)|, \tag{5.9}$$

and

$$\det A(n) \neq 0. \tag{5.10}$$

If N is as above then the equation

$$x(n+1) = A(n)x(n), \quad n \geq N \tag{5.11}$$

has a base of solutions $\{x_+(n)\}_{n\geq N}$, $\{x_-(n)\}_{n\geq N}$ *of the form*

$$x_\pm(n) = \left(\prod_{s=N}^{n-1}\left(1 + \frac{\rho_\pm(s)}{\mu(s)}\right)\right) v_\pm(n), \quad n \geq N, \tag{5.12}$$

where

$$\rho_+(n) = \mathrm{Re}\,(a(n)) + i\sqrt{(\mathrm{Im}\,a(n))^2 - |b(n)|^2},\ \rho_-(n) = \overline{\rho(n)}$$

and $v_\pm(n)$ *are* $\mathbb{C}^2$ *vectors such that*

$$v_\pm(n) = S(n-1)e_\pm(n),\ e_+(n) \to e_1,\ e_-(n) \to e_2, \tag{5.13}$$

with $S(n) = \mathrm{S}(V(n))$. *Moreover,*

We need the following lemma here:

$$\rho_+(n) - \lambda(n) \to 0. \tag{5.14}$$

LEMMA 5.2 *Assume equations* (5.2)–(5.4). *If* u *is a* $D^2(\mu)$ *sequence, then* $\Delta u(n)\mu(n) \to 0$.

Proof. Denote $t = \Delta u$. We have

$$t(n)\mu(n) = t(n_0)\mu(n_0+1) + \sum_{k=n_0+1}^{n} (\Delta t)(k-1)\mu(k) + \sum_{k=n_0+1}^{n-1} (\Delta\mu)(k)t(k),$$

and both sums on the RHS are convergent. The first, since $\Delta^2 u \in l^1(\mu)$, and by equation (5.5). The second, since

$$(\Delta\mu)(k)t(k) = \frac{(\Delta\mu)(k)}{\sqrt{\mu(k)}}[t(k)\sqrt{\mu(k)}],$$

and using equation (5.3) and $t = \Delta u \in l^2(\mu)$ we see that the above is a product of two l^2 sequences. Thus, $t(n)\mu(n) = (t(n))/(\mu(n))^{-1} \to q$ for some $q \in \mathbb{C}$. Suppose that $\mathrm{Re}\,q \neq 0$. Then using $(\mathrm{Re}\,t(n))/(\mu(n))^{-1} \to \mathrm{Re}\,q$ and equation (5.4), by the comparative test of convergence (the signum of $\mathrm{Re}\,t(n)$ is constant for large n since $\mathrm{Re}\,q \neq 0$), we obtain the divergence of $\sum_{k=n_0}^{n-1} \mathrm{Re}\,t(k) = \mathrm{Re}\,u(n) - \mathrm{Re}\,u(n_0)$ to $+\infty$ or $-\infty$. And so we get a contradiction with the boundedness of u. Thus, $\mathrm{Re}\,q = 0$, and proceeding analogically for $\mathrm{Im}\,q$ we get the assertion of the lemma. □

Proof of Theorem 5.1. We shall frequently use here the property (5.5), but to shorten the argumentation, we shall not refer to it. Denote $\lambda = \{\lambda(n)\}_{n \geq n_0}$, $z = \{z(n)\}_{n \geq n_0}$, $r = \{r(n)\}_{n \geq n_0+1}$, $r' = \{r'(n)\}_{n \geq n_0+1}$, $a = \{a(n)\}_{n \geq n_0+1}$, $b = \{b(n)\}_{n \geq n_0+1}$. Using Lemmas 2.2 and 2.3 we obtain

$$\lambda, z \in D^2(\mu) \tag{5.15}$$

and

$$\inf_{n \geq n_0} \mathrm{Im}\,\lambda(n) > 0,\ \inf_{n \geq n_0} \mathrm{Im}\,z(n) > 0. \tag{5.16}$$

We also have

$$\{S(n)\}_{n \geq n_0},\ \{(S(n))^{-1}\}_{n \geq n_0} \in l^\infty. \tag{5.17}$$

By Lemma 5.2 we get

$$r(n)\mu(n) \to 0. \tag{5.18}$$

Now, by equations (5.2), (5.15), (5.16) and (5.18), we see at once that

$$b(n) \to 0 \tag{5.19}$$

and there exist $N \geq n_0 + 1$, $C, \delta > 0$ such that for $n \geq N$

$$|a(n)| \leq C, \ \operatorname{Im} a(n) > \delta, \tag{5.20}$$

and equations (5.9) and (5.10) hold. Thus, we also have

$$1 + \frac{\rho_\pm(n)}{\mu(n)} \neq 0, \quad n \geq N, \tag{5.21}$$

since $\operatorname{Im} \rho_\pm(n) \neq 0$. Moreover, by equations (5.18)–(5.20), we get equation (5.14).

Let us change the variables in equation (5.11) by $\{(S(n-1))^{-1}\}_{n \geq N}$. The equation for

$$y(n) = (S(n-1))^{-1} x(n) \tag{5.22}$$

has the form

$$y(n+1) = H(n)y(n), \quad n \geq N, \tag{5.23}$$

where by equation (2.23) and by the diagonalization formula $(S(n))^{-1}V(n) = \Lambda(n)(S(n))^{-1}$, with $\Lambda(n) = \operatorname{diag}(\lambda(n), \overline{\lambda(n)})$ (see equation (2.16)), we have

$$H(n) = \left(I + \frac{1}{\mu(n)}\Lambda(n)\right)(S(n))^{-1}S(n-1) + \tilde{R}(n) = I + \frac{1}{\mu(n)}\begin{pmatrix} a(n) & b(n) \\ \overline{b(n)} & \overline{a(n)} \end{pmatrix} + \tilde{R}(n),$$

where $\tilde{R}(n) = (S(n))^{-1}R(n)S(n-1)$. By equations (5.8) and (5.17) we have

$$\{\tilde{R}(n)\}_{n \geq N} \in l^1. \tag{5.24}$$

Using equation (5.9) we can define $w(n) = \mathbf{w}(a(n), b(n))$, $n \geq N$, and by the diagonalization formulas from Section 2.3.3 we have

$$\begin{pmatrix} a(n) & b(n) \\ \overline{b(n)} & \overline{a(n)} \end{pmatrix} = W(n)\operatorname{diag}(\rho_+(n), \rho_-(n))(W(n))^{-1}, \quad n \geq N,$$

with

$$W(n) = \begin{pmatrix} 1 & w(n) \\ \overline{w(n)} & 1 \end{pmatrix}. \tag{5.25}$$

We shall prove that

$$\{W(n)\}_{n \geq N} \in D^1, W(n) \to I. \tag{5.26}$$

First, let us note that having equation (5.26) we can use [9, Corollary 1.2] for the equation (5.23), since by equation (5.10) $\det H(n) \neq 0$,

$$H(n) = W(n)\operatorname{diag}\left(1 + \frac{\rho_+(n)}{\mu(n)}, \overline{1 + \frac{\rho_+(n)}{\mu(n)}}\right)(W(n))^{-1} + \tilde{R}(n)$$

with $0 \neq 1 + (\rho_+(n))/(\mu(n)) \to 1$, and equation (5.24) holds. This way we obtain the existence of solutions of equation (5.23)

$$y_\pm(n) = \left(\prod_{s=N}^{n-1}\left(1 + \frac{\rho_\pm(s)}{\mu(s)}\right)\right) e_\pm(n), \quad n \geq N,$$

with $e_\pm(n)$ as in equation (5.13). Now, using equation (5.22) we get the asymptotic formula for $x_\pm$ and their linear independence as in the assertion of the theorem.

It remains only to prove equation (5.26). By equations (5.19), (5.20) and (2.21) we have $w(n) \to 0$, hence, by equation (5.25), it suffices to show that

$$\{w(n)\}_{n \geq N} \in D^1. \tag{5.27}$$

By the definition of $w(n)$ there exists N' such that $w(n) = ib(n)g(n), \quad n \geq N'$, where by equations (5.19) and (5.20) $g(n) = f(\operatorname{Im} a(n), |b(n)|)$, with $f : U \to \mathbb{R}$, $U = \{(x, y) \in \mathbb{R}^2 : \delta < x < C, |y| < (1/2)x\}$, $f(x, y) = \left(x + \sqrt{x^2 - y^2}\right)^{-1}$. The partial derivatives of f are bounded and thus f is a bounded Lipschitz function. Therefore, $\{g(n)\}_{n \geq N'}$ is bounded and there exists C_1 such that

$$|g(n+1) - g(n)| \leq C_1(|\operatorname{Im} a(n+1) - \operatorname{Im} a(n)| + |b(n+1) - b(n)|), \quad n \geq N'.$$

Hence, by $|w(n+1) - w(n)| \leq |g(n+1)||b(n+1) - b(n)| + |b(n)||g(n+1) - g(n)|$ and equation (5.19) there exist $C_2, C_3 > 0$ such that for $n \geq N'$

$$|w(n+1) - w(n)| \leq C_2|b(n+1) - b(n)| + C_3|b(n)||a(n+1) - a(n)|. \tag{5.28}$$

Thus, to show equation (5.27) is sufficient to prove that

$$b \in D^1 \tag{5.29}$$

and that

$$r'\Delta a \in l^1(\mu), \tag{5.30}$$

since by equations (5.2) and (5.15) $|b(n)| \leq 2|r'(n)|\mu(n)$ for large n. Note first that by equations (5.15) and (5.16)

$$r, r' \in l^2(\mu). \tag{5.31}$$

Similarly, using $(\Delta r)(n) = (\Delta^2 z)(n-1)$ we have

$$\Delta r \in l^1(\mu) \subset l^2(\mu) \tag{5.32}$$

(the last inclusion is a consequence of equation (4.2)), and thus also $\Delta^2 r \in l^1(\mu)$. So we have

$$r \in D^2(\mu). \tag{5.33}$$

Observe also that $r' = rs$, where $s \in D^2(\mu)$ by equations (5.15) and (5.16) and Lemma 2.2. Thus, by equations (5.31) and (5.32) and the Schwarz inequality we obtain

$$\Delta r' \in l^1(\mu). \tag{5.34}$$

For the proof of equation (5.30) let us write $a(n)$ in the form $a(n) = a'(n) + ir'(n)\mu(n)$, where

$$a'(n) = \lambda(n)\left(1 + i\frac{r(n)}{2\,\mathrm{Im}\,z(n)}\right).$$

By Lemma 2.2, equations (5.15) and (5.33) we have $\{a'(n)\}_{n\geq n_0} \in D^2(\mu)$, and thus, by the Schwarz inequality and equation (5.31), the component a' in a can be omitted in the proof of equation (5.30) and we are reduced to proving $r'\Delta(r'\mu) \in l^1(\mu)$. Since $r'(n)\mu(n) \to 0$ by equation (5.18), it suffices to show

$$\Delta(r'\mu) \in l^1. \tag{5.35}$$

We have $(\Delta(r'\mu))(n) = [r'(n+1)\sqrt{\mu(n)}][((\Delta\mu)(n))/(\sqrt{\mu(n)}) + \mu(n)(\Delta r')(n)$, hence by the Schwarz inequality, equations (5.3),(5.31) and (5.34) we obtain equation (5.35).

The last part of the proof is the proof of equation (5.29). We have $\Delta b = i\overline{\Delta(r'\mu)} + i\Delta(\overline{r'}\lambda)$ and $\overline{\Delta(r'\mu)} \in l^1$ by equation (5.35). Moreover, using the Schwarz inequality, equations (5.15), (5.31) and (5.34) we get $\Delta(\overline{r'}\lambda) \in l^1(\mu) \subset l^1$, which proves equation (5.29). □

5.2 Perturbations with positive discriminant

Here, we study some positive discriminant assumptions on $V(n)$ in equation (5.1).

We use here the functions $\boldsymbol{\nu}_{\pm}$, $\boldsymbol{\Omega}_{\pm}$, $\mathbf{T}$ defined in Section 2.3.2.

THEOREM 5.3 *Suppose that equations* (5.1)–(5.4) *hold and that the sequences* $\{V(n)\}_{n\geq n_0}$, $\{R(n)\}_{n\geq n_0}$ *of* 2×2 *complex matrices satisfy equations* (5.6) *and* (5.8)*, and*

$$\liminf_{n\to+\infty} \operatorname{discr} V(n) > 0. \tag{5.36}$$

Assume also that the numbers $s_+, s_- \in \{1,2\}$ *fulfil*

$$\liminf_{n\to+\infty} |\boldsymbol{\nu}_\sigma(V(n)) - V_{s_\sigma s_\sigma}(n)| > 0, \quad \sigma = +,-. \tag{5.37}$$

Then there exists $N \geq n_0 + 1$ *such that*

$$s_{\pm} \in \boldsymbol{\Omega}_{\pm}(V(n))\,\textit{for}\, n \geq N-1, \tag{5.38}$$

$$\operatorname{discr} V(n) > 0,\ \det A(n) \neq 0\,\textit{for}\, n \geq N, \tag{5.39}$$

and

$$\inf_{n\geq N} \operatorname{discr} P(n) > 0,\ \boldsymbol{\nu}_{\pm}(P(n)) \neq -\mu(n)\,\textit{for}\, n \geq N, \tag{5.40}$$

where

$$P(n) = \Lambda(n) + (\Lambda(n) + \mu(n))Q(n), \tag{5.41}$$

with $\Lambda(n) = \operatorname{diag}(\nu_+(n), \nu_-(n))$, $\nu_{\pm}(n) = \boldsymbol{\nu}_{\pm}(V(n))$, $Q(n) = -(S(n))^{-1}(\Delta S)(n-1)$ *for* $n \geq N$, *and* $S(n) = \mathrm{T}(s_+, s_-, V(n))$ *for* $n \geq N-1$. *If N is as above then the equation*

$$x(n+1) = A(n)x(n), \quad n \geq N \tag{5.42}$$

has a base of solutions $\{x_+(n)\}_{n\geq N}$, $\{x_-(n)\}_{n\geq N}$ *of the form*

$$x_\pm(n) = \left(\prod_{s=N}^{n-1}\left(1+\frac{\rho_\pm(s)}{\mu(s)}\right)\right)\psi_\pm(n), \quad n\geq N, \tag{5.43}$$

where $\rho_\pm(n) = \boldsymbol{\nu}_\pm(P(n))$ *and* $\psi_\pm(n)$ *are* $\mathbb{C}^2$ *vectors such that*

$$\psi_\pm(n) = S(n-1)e_\pm(n),\ e_+(n)\to e_1,\ e_-(n)\to e_2. \tag{5.44}$$

Moreover,

$$\rho_\pm(n) - \nu_\pm(n)\to 0. \tag{5.45}$$

Proof. As in the previous proof, we shall frequently use here the property (5.5), without referring to it. The existence of N for which equations (5.38) and (5.39) hold is clear, thus, for n large enough, say, for $n\geq N'$, the matrices $S(n)$, $\Lambda(n)$, $Q(n)$, $P(n)$ are well defined. Denote $\Lambda = \{\Lambda(n)\}_{n\geq N'}$, $S=\{S(n)\}_{n\geq N'}$, $S^{-1}=\{S^{-1}(n)\}_{n\geq N'}$, $Q=\{Q(n)\}_{n\geq N'}$, $P=\{P(n)\}_{n\geq N'}$ and $\mathcal{E}=\{(\Lambda(n)+\mu(n))Q(n)\}_{n\geq N'}$. Using Lemmas 2.2 and 2.3 and equation (5.36) we get

$$\Lambda, S, S^{-1}\in D^2(\mu). \tag{5.46}$$

For Λ the above follows from equation (2.17), for S—from equation (2.19), and for S^{-1}—from equations (2.18) and (2.20) (the last is needed only in the case $s_+=s_-$, with $j=s_+$). Hence, using Lemma 5.2 and equation (5.2), we obtain

$$\mathcal{E}(n)\to 0, \tag{5.47}$$

and therefore,

$$\operatorname{discr} P(n) - \operatorname{discr} V(n) = \operatorname{discr}(\Lambda(n)+\mathcal{E}(n)) - \operatorname{discr}\Lambda(n)\to 0. \tag{5.48}$$

Thus, by equation (5.36), and by equation (5.2) we obtain the existence of N which satisfies also equation (5.40).

Observe that if $A,B\in D^2(\mu)$, then by the Schwarz inequality $A\Delta B\in D^1(\mu)$, since we have

$$\Delta(A\Delta B)(n) = (\Delta A)(n)(\Delta B)(n+1) + A(n)(\Delta^2 B)(n).$$

In particular, by equation (5.46) we get

$$Q\in l^2(\mu)\cap D^1(\mu). \tag{5.49}$$

Now, observe that equations (5.47) and (5.48) proves equation (5.45). Moreover,

$$\begin{aligned}\liminf_{n\to+\infty}(\boldsymbol{\nu}_+(P(n)) - P_{22}(n)) &= \liminf_{n\to+\infty}(\rho_+(n)-\nu_-(n)-\mathcal{E}_{22}(n)) \\ &= \liminf_{n\to+\infty}(\nu_+(n)-\nu_-(n)-\mathcal{E}_{22}(n)+\rho_+(n)-\nu_+(n)) \\ &= \liminf_{n\to+\infty}\operatorname{discr} V(n),\end{aligned} \tag{5.50}$$

and analogically

$$\liminf_{n\to+\infty}(P_{11}(n)-\boldsymbol{\nu}_-(P(n))) = \liminf_{n\to+\infty}\operatorname{discr} V(n). \tag{5.51}$$

Thus, there exists $N_1\geq N$ such that $2\in\boldsymbol{\Omega}_+(P(n))$, $1\in\boldsymbol{\Omega}_-(P(n))$ for $n\geq N_1$ and we can define a diagonalizing sequence $W=\{W(n)\}_{n\geq N_1}$ by $W(n)=\mathbf{T}(2,1,P(n))$, $n\geq N_1$.

By equations (5.47), (5.50), (5.51), (5.36) and (2.19) we have

$$W(n) = \begin{pmatrix} 1 & u(n) \\ w(n) & 1 \end{pmatrix} \to I, \tag{5.52}$$

with

$$w(n) = \frac{\mathcal{E}_{21}(n)}{\rho_+(n) - P_{22}(n)},\ u(n) = \frac{\mathcal{E}_{12}(n)}{\rho_-(n) - P_{11}(n)}, \quad n \geq N_1, \tag{5.53}$$

and

$$P(n) = W(n)\,\mathrm{diag}\,(\rho_+(n), \rho_-(n))(W(n))^{-1}, \quad n \geq N_1. \tag{5.54}$$

We shall prove now that

$$W \in D^1. \tag{5.55}$$

By equations (5.52) and (5.53), using $P_{11}(n) = \mu_+(n) + \mathcal{E}_{11}(n)$, $P_{22}(n) = \mu_-(n) + \mathcal{E}_{22}(n)$ and the formula

$$\Delta\left(\frac{a}{b}\right)(n) = \frac{(\Delta a)(n)b(n) - a(n)(\Delta b)(n)}{b(n+1)b(n)},$$

we see that it is enough to prove the following three statements:

$$\mathcal{E} \in D^1, \tag{5.56}$$

$$(\Delta\rho_\pm)\mathcal{E} \in l^1, \tag{5.57}$$

$$(\Delta\nu_\pm)\mathcal{E} \in l^1, \tag{5.58}$$

where $\rho_\pm = \{\rho_\pm(n)\}_{n\geq N}$, $\nu_\pm = \{\nu_\pm(n)\}_{n\geq N}$. We have $\mathcal{E} = \Lambda Q + \mu Q$, and

$$(\Delta(\Lambda Q))(n) = \Lambda(n+1)(\Delta Q)(n) + (\Delta\Lambda)(n)Q(n),$$

$$(\Delta(\mu Q))(n) = \mu(n+1)(\Delta Q)(n) + \frac{(\Delta\mu)(n)}{\sqrt{\mu(n)}}\sqrt{\mu(n)}Q(n),$$

hence using equations (5.3), (5.46) and (5.49), the Schwarz inequality and $l^p(\mu) \subset l^p$, we get

$$\Lambda Q, \mu Q \in D^1 \tag{5.59}$$

and thus also equation (5.56). Using the similar arguments and the estimate

$$\|(\Delta x)(n)\|\mathcal{E}(n)\| \leq \|\Delta x(n)\|Q(n)\|\Lambda(n)\| + \|(\Delta x)(n)\sqrt{\mu(n)}\|\sqrt{\mu(n)}Q(n)\|$$

we get

$$x \in D^2(\mu) \Rightarrow \|(\Delta x)\|\mathcal{E}\| \in l^1. \tag{5.60}$$

In particular, using equation (5.60) for $x = \nu_\pm$, we get equation (5.58). To prove equation (5.57) let us observe first that there exists $C \geq 0$ such that

$$|(\Delta\rho_\pm)(n)| \leq C\|(\Delta P)(n)\|, \quad n \geq N. \tag{5.61}$$

The above follows from $\rho_\pm(n) = \boldsymbol{\nu}_\pm(P(n))$, from equation (5.40), and from the fact that the sum and the superposition of Lipschitz functions are also Lipschitz functions (we use this for the functions $M_2(\mathbb{R}) \ni X \to \mathrm{tr}\,X$, $D \ni X \to \mathrm{discr}\,X$ and $[\epsilon; K] \ni t \to \sqrt{t}$, for a bounded domain $D \subset M_2(\mathbb{R})$ and for $0 < \epsilon < K$). Thus, by equation (5.61), it suffices to prove

$\|(\Delta P)\|\mathcal{E}\| \in l^1$, and hence, using $P = \Lambda + \Lambda Q + \mu Q$ and equations (5.46), (5.49) and (5.60) we see that it is enough to prove $\Delta(\mu Q) \in l^1$, which follows from equation (5.59). This finishes the proof of equation (5.55).

Now we proceed in a similar manner as in the proof of the previous theorem. We change the variables in equation (5.42) by $\{(S(n-1))^{-1}\}_{n\geq N}$. The equation for $y(n) = (S(n-1))^{-1}x(n)$ has the form

$$y(n+1) = H(n)y(n), \quad n \geq N, \tag{5.62}$$

where

$$H(n) = \left(I + \frac{1}{\mu(n)}P(n)\right) + \tilde{R}(n) \tag{5.63}$$

where $\tilde{R}(n) = (S(n))^{-1}R(n)S(n-1)$. Thus, we have $\{\tilde{R}(n)\}_{n\geq N} \in l^1$, and by equation (5.54) for $n \geq N_1$

$$H(n) = W(n)\mathrm{diag}\left(1 + \frac{\rho_+(n)}{\mu(n)}, 1 + \frac{\rho_-(n)}{\mu(n)}\right)(W(n))^{-1} + \tilde{R}(n).$$

Hence, we can use [9, Theorem 1.4] (see also the results in [2]) to the equation (5.62). For $n \geq N$ we have $\det H(n) \neq 0$ by equation (5.39) and $1 + (\rho_\pm(n))/(\mu(n)) \neq 0$ by equation (5.40). Moreover, the main assumption ("dichotomy condition") of [9, Theorem 1.4] is satisfied for the solution "y_-" since for n large enough $|1 + (\rho_-(n))/(\mu(n))||1 + (\rho_+(n))/(\mu(n))|^{-1} \leq 1$. For the second solution "y_+" we need the above inequality and

$$\prod_{n=N}^{+\infty}\left|1 + \frac{\rho_-(n)}{\mu(n)}\right|\left|1 + \frac{\rho_+(n)}{\mu(n)}\right|^{-1} = 0. \tag{5.64}$$

But equation (5.64) follows immediately from $\sum_{n=N}^{+\infty} \frac{\rho_-(n)-\rho_+(n)}{\mu(n)} = -\infty$, being a consequence of the equality $\rho_-(n) - \rho_+(n) = -\mathrm{discr}\,P(n)$ and of the conditions (5.4) and (5.40). In this way, similarly as in the proof of the previous theorem, we obtain the existence of solutions of equation (5.62), and then, by the change of variables, the asymptotic formula (5.43) and the linear independence of solutions. □

6. Applications for studies of generalized eigenvectors of some Jacobi operators

In this section, we intend to illustrate the abstract results from the previous sections with some examples. These examples refer to the generalized eigenvectors of some Jacobi operators. We show here some asymptotic results which can be obtained by the theorems proved in Sections 3–5, but which do not follow directly from the other discrete versions of the Levinson theorem, e.g. the theorems proved in [2,3,9].

Let us consider a Jacobi matrix, i. e. an infinite tridiagonal matrix of the form

$$\mathcal{J} = \begin{pmatrix} q_1 & w_1 & & & \\ w_1 & q_2 & w_2 & & \\ & w_2 & q_3 & w_3 & \\ & & w_3 & q_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

where w_n, q_n are real coefficients (weights and diagonals, respectively), and $w_n \neq 0$. More precisely, for a complex sequence $u = \{u_n\}_{n \geq 1}$ we define

$$(\mathcal{J}u)_n := w_{n-1}u_{n-1} + q_n u_n + w_n u_{n+1}, \quad n \in \mathbb{N}, \tag{6.1}$$

with the convention that $w_j = u_j := 0$, if $j < 1$. The Jacobi operator J is the operator in the Hilbert space $l^2(\mathbb{N})$ defined by $\mathcal{J}$ on the maximal domain $D(J) := \{u \in l^2(\mathbb{N}) : \mathcal{J}u \in l^2(\mathbb{N})\}$, i.e. $Ju = \mathcal{J}u$ for $u \in D(J)$. Let $\lambda \in \mathbb{C}$. A scalar sequence $u = \{u_n\}_{n \geq 1}$ we call a generalized eigenvector of J for λ, if

$$(\mathcal{J}u)_n := \lambda u_n \text{ for any } n \geq 2. \tag{6.2}$$

Note that to be the eigenvector of J (not only "generalized"), u should satisfy the above equation also for $n = 1$, and it should be a (nonzero) sequence from $l^2(\mathbb{N})$. Nevertheless, properties of generalized eigenvectors have strong relations with some spectral properties of J. For instance, the subordination theory of Gilbert, Pearson and Khan (see [16]) is an example of such a relation. Some spectral results obtained by the subordination theory and by the asymptotic analysis of generalized eigenvectors have been presented in [7,9].

To study the asymptotic behavior of the solutions of equation (5.2) it is convenient to rewrite this equation in the equivalent $\mathbb{C}^2$ vector form

$$x(n+1) = B_n(\lambda)x(n), \quad n \geq 2, \tag{6.3}$$

where $B_n(\lambda)$ is the transfer matrix given by

$$B_n(\lambda) = \begin{pmatrix} 0 & 1 \\ -\frac{w_{n-1}}{w_n} & \frac{\lambda - q_n}{w_n} \end{pmatrix}, \tag{6.4}$$

and the equivalence of equations (5.2) and (5.3) is established by the substitutions

$$x(n) := \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix} \in \mathbb{C}^2 \text{ for } n \geq 2,\ u_n := (x(n+1))_1 \text{ for } n \geq 1. \tag{6.5}$$

Another equivalent $\mathbb{C}^2$ vector form of equation (6.2) can be obtained when instead of single $B_n(\lambda)$'s we use products of two neighbor transfer matrices. Thus, define

$$\tilde{B}_n(\lambda) = B_{2n}(\lambda)B_{2n-1}(\lambda), \quad n \geq 2. \tag{6.6}$$

The equation

$$\tilde{x}(n+1) = \tilde{B}_n(\lambda)\tilde{x}(n), \quad n \geq 2 \tag{6.7}$$

is equivalent to equation (6.2) by the substitutions

$$\tilde{x}(n) := x(2n-1) = \begin{pmatrix} u_{2n-2} \\ u_{2n-1} \end{pmatrix} \in \mathbb{C}^2, \quad n \geq 2, \tag{6.8}$$

$$u_n := (x(n+1))_1 = \begin{cases} (\tilde{x}(l))_1 & \text{for } n = 2l-2 \\ ((B_{n+1}(\lambda))^{-1}\tilde{x}(l))_1 & \text{for } n = 2l-3 \end{cases}, \quad n \geq 1.$$

The first example illustrates Theorems 3.1 and 4.1.

Example 6.1 Let $w_n > 0$, $q_n \in \mathbb{R}$ for $n \in \mathbb{N}$. Assume that $\{(w_{n-1})/(w_n)\}_{n\geq 2}$, $\{(q_n)/(w_n)\}_{n\geq 1} \in D^2$ and

$$w_n \longrightarrow +\infty,\ \frac{w_{n-1}}{w_n} \longrightarrow 1,\ \frac{q_n}{w_n} \longrightarrow a,\ |a| \neq 2.$$

With these assumptions, for any $\lambda \in \mathbb{C}$ we obtain $\{B_n(\lambda)\}_{n\geq 2} \in D^2$ and $B_n(\lambda) \longrightarrow B_\infty$, where

$$B_\infty = \begin{pmatrix} 0 & 1 \\ -1 & -a \end{pmatrix},\ \operatorname{discr}(B_\infty) = a^2 - 4.$$

We can consider the equation (6.3) as equation (1.1) setting $A(n) := B_n(\lambda)$ and $n_0 = 2$.

Case 1. $|a| < 2$.

Applying Theorem 3.1, Remarks 3.2 no. 2 and 3 and the formula (6.5) we get the existence of two linearly independent solutions u^+, u^- of equation (6.2) having the form

$$u_n^\pm = \left(\prod_{s=N}^{n} \lambda_\pm^{(0)}(s)\right)\left(\prod_{s=N}^{n} (1 - \eta_\pm(s))\right)\psi_\pm(n),$$

with $\psi_\pm(n) \longrightarrow 1$,

$$\lambda_\pm^{(0)}(n) = \frac{1}{2}\left(\frac{\lambda - q_n}{w_n} \pm i\sqrt{4\frac{w_{n-1}}{w_n} - \left(\frac{\lambda - q_n}{w_n}\right)^2}\right), \qquad \eta_\pm(n) = \frac{\lambda_\pm^{(0)}(n) - \lambda_\pm^{(0)}(n-1)}{2\operatorname{Im}\lambda_+^{(0)}(n)}$$

for $n \geq N$, with some N large enough (where N and $\psi_\pm(n)$ also depend on λ).

It is a well-known fact based on subordination theory that in this case J is an absolutely continuous operator (i.e. J has only purely absolutely continuous spectrum), provided that it is selfadjoint (see [14, Theorem 3.1]). The same fact can also be easily obtained from the above asymptotic formula for $u^\pm$. However, the details on $\lambda_\pm^{(0)}$ and $\eta_\pm$ obtained here are in some sense "too strong" (to get the absolute continuity of J, it would be enough to know that $\lambda_-^{(0)}(n) = \overline{\lambda_+^{(0)}(n)}$ and $\eta_-(n) = \overline{\eta_+(n)}$) and they could be used to study some more delicate spectral properties of J.

Case 2. $|a| > 2$.

Applying Theorem 4.1, Corollary 4.3 and the formula (6.5) we get the existence of two linearly independent solutions u^+, u^- of equation (6.2) having the form

$$u_n^\pm = \left(\prod_{s=N}^{n} \lambda_\pm(s)\right)\left(\prod_{s=N}^{n} (1 + \delta_\pm(s))\right)\psi_\pm(n),$$

where $\psi_\pm(n) \longrightarrow 1$ and

$$\lambda_\pm(n) = \boldsymbol{v}_\pm(B_n(\lambda) = \frac{1}{2}\left(\frac{\lambda - q_n}{w_n} \pm \beta(n)\right),$$

$$\delta_+(n) = \frac{-\lambda_-(n+1)(\lambda_+(n+1) - \lambda_+(n))}{\lambda_+(n)\beta(n+1)},$$

$$\delta_-(n) = \frac{\lambda_+(n+1)(\lambda_-(n+1) - \lambda_-(n))}{\lambda_-(n)\beta(n+1)},$$

with

$$\beta(n) = \sqrt{\left(\frac{\lambda - q_n}{w_n}\right)^2 - 4\frac{w_{n-1}}{w_n}}$$

for $n \geq N$, with some N large enough, where N and $\psi_{\pm}(n)$ also depend on λ.

The case considered here is the so-called dominating diagonal case, and it is well known that J is selfadjoint and has purely discrete spectrum (see [14]). Observe that

$$\lambda_{\pm}(n) \longrightarrow \lambda_{\pm\infty} := \frac{1}{2}(-a \pm \sqrt{a^2 - 4}).$$

Moreover, $|\lambda_{+\infty}| < 1 < |\lambda_{-\infty}|$ for $a > 2$, and $|\lambda_{\infty}| < 1 < |\lambda_{+\infty}|$ for $a < -2$. Thus, using the above asymptotic formulas we see that if $\lambda \in \sigma(J)$, then for $a > 2$ ($a < -2$) the solution u^+ (u^-) is the eigenvector of J for λ. So, we have obtained quite precise asymptotics for eigenvectors of J. Note, that they are much more precise than the information which can be obtained on the basis of the Poincaré–Perron type theorems (see [5]).

A simple but concrete example of weights and diagonals satisfying the general conditions formulated here (including the selfadjointness of J), and additionally satisfying $\{B_n(\lambda)\}_{n\geq 2} \notin D^1$ (i.e. the "D^1-methods of [9] does not work) can be defined as follows:

$$w_n = n^{\alpha},\ q_n = \left(a + n^{-\frac{1}{2}}\sin(n^p)\right)n^{\alpha}, \quad n \geq 1,$$

where $|a| \neq 2$, $0 < \alpha \leq 1$, $(1/2) < p < (3/4)$. The above is a consequence of the fact that $\{n^{-(1/2)}\sin(n^p)\}_{n\geq 1} \in D^2 \backslash D^1$ (some more general classes of sequences from $D^2 \backslash D^1$ are given by the formulas $n^{-\beta}\sin(n^p)$ and $n^{-\beta}\cos(n^p)$, where $0 < \beta < p < (1/2)(\beta + 1)$).

The second example illustrates Theorems 5.1 and 5.3.

Example 6.2 Assume that $w_n = n^{\alpha} + c_n r_n, q_n = 0, n \in \mathbb{N}$, where $0 < \alpha < 1$, $\{c_n\}_{n\geq 1}$ is a 2-periodic sequence and $r_{2n} = 1, r_{2n+1} = \sin(n^{\gamma})$, with $0 < \gamma < (1 - \alpha)/2$ and let α, γ, c_1,c_2 be such that $w_n \neq 0$ for any n.

Contrary to the previous example, here the asymptotic behavior of the generalized eigenvectors of J strongly depends on the spectral parameter λ. We shall not write down the explicit asymptotic formulas, which can be easily obtained by Theorems 5.1 or 5.3 (depending on λ) and by the substitution formula (6.8), but we limit ourselves just to summarize the spectral consequences of these asymptotic results (with these assumptions J is selfadjoint).

To study the generalized eigenvectors of J we analyze here equation (6.7). We get

$$\tilde{B}_n(\lambda) = -\left(I + \frac{1}{\mu(n)}V(n)\right)$$

where $\mu(n) = w_{2n-1}$ for $n \geq 2$ and where $\{V_n\}_{n\geq 2}$ is a $D^2(\mu)$ matrix sequence satisfying

$$\liminf_{n\to+\infty} \operatorname{discr} V(n) = d_-^2 - 4\lambda^2,\ \limsup_{n\to+\infty} \operatorname{discr} V(n) = d_+^2 - 4\lambda^2,$$

with

$$d_- = \begin{cases} |c_2| - |c_1| & \text{for } |c_2| > |c_1| \\ 0 & \text{for } |c_2| \leq |c_1| \end{cases},$$

$$d_+ = |c_1| + |c_2|$$

The above is a consequence of the fact, that the set of the limit points of the sequence $\{\sin(n^\gamma)\}_{n\geq 1}$ is equal to $[-1;1]$ (see [18, van der Corput theorem]).

Consider two cases.

Case 1. $|\lambda| > d_+$.

In this case, $\limsup_{n\to+\infty} \operatorname{discr} V(n) < 0$, and we can use Theorem 5.1. We obtain two linearly independent solutions with the scalar terms differing only in the complex conjugation. Combining this with the subordination theory and the generalized Behncke Stolz Lemma (see [19, Theorem 1.1]) we can prove that J is absolutely continuous in $\mathbb{R}\backslash[-d_+;d_+]$ and that $\mathbb{R}\backslash(-d_+;d_+)$ is contained in the absolutely continuous spectrum of J.

Case 2. $|\lambda| < d_-$.

In this case, $\liminf_{n\to+\infty} \operatorname{discr} V(n) > 0$. Moreover, if $\lambda \neq 0$, then the condition (5.37) holds for some numbers $s_+, s_- \in \{1,2\}$, and we can apply Theorem 5.3. This allows to prove that there exists a generalized eigenvector of J for λ which is in $l^2(\mathbb{N})$. Using now the subordination theory we can prove that J is pure point in $(-d_-;d_-)$ (i.e. the image of the spectral projection for J on this interval is contained in the closed span of all the eigenvectors of J).

Note that we usually get "a region of uncertainty", which appears when $d_- < d_+$, i.e. when $c_1 \neq 0$. In this region, our abstract results do not rather give us any asymptotic information on the generalized eigenvectors of J. The appearance of the region of uncertainty is the main difference between this example and the examples studied in [7,9]. Note also that the region where we can prove the pure pointness is nonempty iff $|c_2| > |c_1|$.

We stress that by the definition, the fact that J is pure point in a subset of $\mathbb{R}$ does not mean that there exists an eigenvalue of J in this subset—its intersection with the spectrum of J can be, e.g. empty.

The details related to this example, as well as some generalizations, will be presented in [10].

Acknowledgements

The idea of using weighted Stolz classes we owe to Serguei Naboko. We also would like to thank him for the proof of Lemma 4.2. Research supported by the KBN grant 5 P03A/026/21.

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Appendix

We give here some longer proofs of some results formulated in Section 2.

We start from Lemma 2.2.

The proof is based on an integral formula for $\Delta^k f(x)$. We need some extra notation to write this formula in a possibly short form. Fix $j = 1, 2, \ldots$. By $d^j t$ we denote the integration by the Lebesgue measure in $\mathbb{R}^j$, $d^{(j)}f(u)$ is the j-th order differential of the function f at the point u and $d^{(j)}f(u)(h)$ is the value of this differential at the system h of j vectors from X (i.e. $h = (h_1, \ldots, h_j) \in X^j$). For $\alpha \in \mathbb{N}^j$ we set $|\alpha| = \alpha_1 + \ldots + \alpha_j$ and we define $\omega^j, \gamma_s^j, 1_s^j \in \mathbb{N}^j$ for $s = 1, \ldots, j$ by

$$\omega^j = (1, \ldots, 1), \quad (1_s^j)_m = \begin{cases} 1 & \text{for} \quad m = s \\ 0 & \text{for} \quad m \neq s \end{cases}, \quad (\gamma_s^j)_m = \begin{cases} 1 & \text{for} \quad m < s \\ 0 & \text{for} \quad m \geq s \end{cases}$$

for $m = 1, \ldots, j$. We denote also

$$A_k^j := \{\alpha \in \mathbb{N}^j : k \leq |\alpha|, 1 \leq \alpha_s \leq k \quad \text{for} \quad s = 1, \ldots, j\}. \tag{A.1}$$

For a set Y and $n_0 \in \mathbb{N}$ by $\mathrm{Seq}_{n_0}(Y)$ we denote the set of all sequences in Y with the starting index equal to n_0. Let $\alpha \in \mathbb{N}^j$, the operators $T_\alpha, \partial^\alpha : \mathrm{Seq}_{n_0}(X^j) \longrightarrow \mathrm{Seq}_{n_0}(X^j)$ are given by

$$(T_\alpha y)(n) = (y_1(n + \alpha_1), \ldots, y_j(n + \alpha_j)),$$

$$(\partial^\alpha y)(n) = ((\Delta^{\alpha_1} y_1)(n), \ldots, (\Delta^{\alpha_j} y_j)(n)).$$

We also define $\Delta^\alpha : \mathrm{Seq}_{n_0}(X) \to \mathrm{Seq}_{n_0}(X^j)$ by

$$(\Delta^\alpha x)(n) = ((\Delta^{\alpha_1} x)(n), \ldots, (\Delta^{\alpha_j} x)(n)), \quad n \geq n_0.$$

We have

$$\partial^\alpha T_\beta = T_\beta \partial^\alpha, \quad \partial^\alpha \Delta^\beta = \Delta^{\alpha+\beta}. \tag{A.2}$$

For $F : X^j \to X'$ and $y \in \mathrm{Seq}_{n_0}(X^j)$ denote by $F(y)$ the element of $\mathrm{Seq}_{n_0}(X')$ given by $(F(y))(n) = F(y(n))$, $n \geq n_0$. It can be easily proved by induction that for F being j-linear operator the following "discrete Leibnitz formula" holds

$$\Delta F(y) = \sum_{s=1}^{j} F(T_{\gamma_s^j} \partial^{1_s^j} y). \tag{A.3}$$

Assume that X, X', U and K are as in the Lemma 1.2.

Lemma A.1 *For any $k = 1, 2, \ldots$ there exist a finite set I_k and functions r_k, α_k, β_k defined on I_k such that for any $j \in I_k$*

$$r_k(j) \in \{1, \ldots, k\}, \quad \alpha_k(j) \in A_k^{r_k(j)}, \beta_k(j) \in \mathbb{N}^{r_k(j)}$$

and there exist polynomials v_{kj}, w_{kjl} of $r_k(j)$ real variables for $l = 0, \ldots, r_k(j)$, satisfying: for any $x \in \mathrm{Seq}_{n_0}(K)$ and any C^k function $f : U \to X'$

$$(\Delta^k f(x))(n) = \sum_{j \in I_k} \int_{[0;1]^{r_k(j)}} v_{kj}(t) d^{(r_k(j))} f(a_{xtkj}(n)) \big((T_{\beta_k(j)} \Delta^{\alpha_k(j)} x)(n) \big) d^{r_k(j)} t \tag{A.4}$$

for $n \geq n_0$, where for $j \in I_k$, $t \in [0;1]^{r_k(j)}$, $n \geq n_0$

$$a_{xtkj}(n) = \sum_{l=0}^{r_k(j)} w_{kjl}(t)(\Delta^l x)(n) \in K. \tag{A.5}$$

Proof. The proof is by induction on k. If $k = 1$ then we have

$$\begin{aligned}(\Delta f(x))(n) &= f(x(n+1)) - f(x(n)) \\ &= \int_{[0;1]} d^{(1)} f(x(n) + t(\Delta x)(n))((\Delta x)(n)) \mathrm{d}t,\end{aligned}$$

which proves the assertion for $k = 1$. Assume that the assertion holds for some $k \geq 1$, and that f is a C^{k+1} function. Let us first choose $j \in I_k$ and $t \in [0;1]^{r_k(j)}$, $n \geq n_0$. Using (A.3), we obtain

$$\begin{aligned}
&d^{(r_k(j))} f(a_{xtkj}(n+1)) \big((T_{\beta_k(j)} \Delta^{\alpha_k(j)} x)(n+1) \big) - d^{(r_k(j))} f(a_{xtkj}(n)) \big((T_{\beta_k(j)} \Delta^{\alpha_k(j)} x)(n) \big) \\
&\quad = d^{(r_k(j))} f(a_{xtkj}(n+1)) \big((T_{\beta_k(j)} \Delta^{\alpha_k(j)} x)(n+1) \big) - d^{(r_k(j))} f(a_{xtkj}(n)) \big((T_{\beta_k(j)} \Delta^{\alpha_k(j)} x)(n+1) \big) \\
&\qquad + \big(\Delta \big[d^{(r_k(j))} f(a_{xtkj}(n)) \big(T_{\beta_k(j)} \Delta^{\alpha_k(j)} x \big) \big] \big)(n) \\
&\quad = \int_{[0;1]} d^{(r_k(j)+1)} f \big(a_{xtkj}(n) + t' \big(\Delta a_{xtkj} \big)(n) \big) \quad \big((T_{\beta_k(j)} \Delta^{\alpha_k(j)} x)(n+1), \big(\Delta a_{xtkj} \big)(n) \big) \mathrm{d}t' \\
&\qquad + \sum_{s=1}^{r_k(j)} d^{(r_k(j))} f(a_{xtkj}(n)) \Big(\Big(T_{\gamma_s^{r_k(j)}} \partial^{1_s^{r_k(j)}} T_{\beta_k(j)} \Delta^{\alpha_k(j)} x \Big)(n) \Big).
\end{aligned}$$

Thus, by equations (A.2), (A.4), (A.5)

$$
\begin{aligned}
(\Delta^{k+1} f(x))(n) &= (\Delta^k f(x))(n+1) - (\Delta^k f(x))(n) \\
&= \sum_{j\in I_k} \int_{[0;1]^{r_k(j)}} v_{kj}(t'') \int_{[0;1]} d^{(r_k(j)+1)} f\big(a_{xt''kj}(n) + t'(\Delta a_{xt''kj})(n)\big) \\
&\qquad \big((T_{\beta_k(j)} \Delta^{\alpha_k(j)} x)(n+1), (\Delta a_{xt''kj})(n)\big) d^1 t' d^{r_k(j)} t'' \\
&\quad + \sum_{j\in I_k} \sum_{s=1}^{r_k(j)} \int_{[0;1]^{r_k(j)}} v_{kj}(t) d^{(r_k(j))} f(a_{xtkj}(n)) \quad \Big((T_{\gamma_s^{r_k(j)}} \partial^{1_s^{r_k(j)}} T_{\beta_k(j)} \Delta^{\alpha_k(j)} x)(n)\Big) d^{r_k(j)} t \\
&= \sum_{j\in I_k} \sum_{l=0}^{r_k(j)} \int_{[0;1]^{r_k(j)+1}} \tilde{v}_{kjl}(t) d^{(r_k(j)+1)} f(\tilde{a}_{xtkj}(n)) \quad \Big((T_{\tilde{\beta}_{kl}(j)} \Delta^{\tilde{\alpha}_{kl}(j)} x)(n)\Big) d^{r_k(j)+1} t \\
&\quad + \sum_{j\in I_k} \sum_{s=1}^{r_k(j)} \int_{[0;1]^{r_k(j)}} v_{kj}(t) d^{(r_k(j))} f(a_{xtkj}(n)) \quad \Big(\Big(T_{\gamma_s^{r_k(j)} + \beta_k(j)} \Delta^{1_s^{r_k(j)} + \alpha_k(j)} x\Big)(n)\Big) d^{r_k(j)} t,
\end{aligned}
$$

where for $t = (t', t'') \in [0;1] \times [0;1]^{r_k(j)}$, $l = 0, \ldots, r_k(j)$ $\tilde{v}_{kjl}(t) = v_{kj}(t'') w_{kjl}(t'')$,

$$
\begin{aligned}
\tilde{a}_{xtkj}(n) &= a_{xt''kj}(n) + t'(\Delta a_{xt''kj})(n) \\
&= \sum_{l=0}^{r_k(j)} w_{kjl}(t'')[(\Delta^l x)(n) + t'(\Delta^{l+1} x)(n)],
\end{aligned}
\tag{A.6}
$$

and

$$
\tilde{\alpha}_{kl}(j) = (\alpha_k(j), l+1) \in \mathbb{N}^{r_k(j)} \times \mathbb{N}, \quad \tilde{\beta}_{kl}(j) = (\beta_k(j) + \omega^{r_k(j)}, 0) \in \mathbb{N}^{r_k(j)} \times \mathbb{N}. \tag{A.7}
$$

Moreover, by the inductive assumption and by equation (A.6), we have $\tilde{a}_{xtkj}(n) \in K$ since K is convex, and by equations (A.7), (A.1), for any $l = 0, \ldots, r_k(j)$, $s = 1, \ldots, r_k(j)$ we have

$$
\tilde{\alpha}_{kl}(j) \in A_{k+1}^{r_k(j)+1}, \quad \alpha_k(j) + 1_s^{r_k(j)} \in A_{k+1}^{r_k(j)},
$$

which proves the assertion for $k+1$. □

Proof of Lemma 1.2. We have $f(x) \in_k l^\infty$, since K is compact. Choose $m = 1, \ldots, k$. By Lemma A.1, to prove that $\Delta^m(f(x)) \in l^{\frac{k}{m}}(\mu)$ it is sufficient to show that for any $r = 1, \ldots, m$, $\alpha \in A_m^r$, and $\beta \in \mathbb{N}^r$ the sequence $y = \{y(n)\}_{n\geq n_0}$, given by

$$
y(n) = \sup_{u\in K} \|d^{(r)} f(u)((T_\beta \Delta^\alpha x)(n))\|,
$$

is a scalar $l^{(k/m)}(\mu)$ sequence. Using the continuity of $d^{(r)}f$ and the compactness of K we get

$$
y(n) \leq M \prod_{s=1}^{r} \|\big(\Delta^{\alpha_s} x_s\big)(n)\|, \quad n \geq n_0 \tag{A.8}
$$

with some $M < +\infty$, where $x_s(n) = x(n + \beta_s)$. Observe, that x_s is a $D^k(\mu)$ sequence in X for any $s = 1, \ldots, r$, since μ is shiftable. Hence $\Delta^{\alpha_s} x_s$ is a $l^{(k/\alpha_s)}(\mu)$ sequence in X, since by equation (A.1), we have

$$m \le |\alpha|, \quad 1 \le \alpha_s \le m \le k, \quad s = 1, \ldots, r. \tag{A.9}$$

Thus, by equation (A.8), using the Hölder inequality and equation (A.9), we get $y \in l^p(\mu)$, where

$$\frac{1}{p} = \sum_{s=1}^{r} \frac{\alpha_s}{k} = \frac{|\alpha|}{k} \ge \frac{m}{k}.$$

Therefore, $p \le (k/m)$, and $y \in l^{\frac{k}{m}}(\mu)$, since μ is separated from zero. □

The next to prove is Lemma 1.7.

Proof of Lemma 1.7. Denote by v_{0i} the vector being the i-th column of T_0, $i = 1, \ldots, d$. We have $(X_0 - \lambda_{0i})v_{0i} = 0$. We shall prove first that there exist a neighborhood U_i of X_0 and holomorphic functions $v_i : U_i \to C^d$, $\lambda_i : U_i \to C$ such that $\lambda_i(X_0) = \lambda_{0i}$, $\quad v_i(X_0) = v_{0i}$ and $(X - \lambda_i(X))v_i(X) = 0$ for $X \in U_i$. We shall use the implicit function theorem. Since the vector equation $(X - \lambda)v = 0$ is a system of only d corresponding scalar equations, and we are looking for $d + 1$ scalar values (λ and d coordinates of v), we should add one additional "independent" scalar equation. Thus, let us consider the function $F : M_d(\mathbb{C}) \times \mathbb{C}^d \times \mathbb{C} \to \mathbb{C}^d \times \mathbb{C}$ given by the formula

$$F(X, v, \lambda) = ((X - \lambda)v, \gamma v),$$

where $\gamma \in \mathbb{C}^d$ is an arbitrary fixed vector satisfying $\gamma v_{0i} \ne 0$ (with γv being the scalar product of γ and v). Denote by D_0 the differential of F with respect to (v,λ) at the point $(X_0, v_{0i}, \Lambda_{0i})$. D_0 is the linear transformation of $\mathbb{C}^d \times \mathbb{C}$ given by

$$D_0(h) = ((X_0 - \lambda_{0i})h_v - h_\lambda v_{0i}, \gamma h_v),$$

where $h = (h_v, h_\Lambda) \in \mathbb{C}^d \times \mathbb{C}$. If $h \in \operatorname{Ker} D_0$, then $(X_0 - \lambda_{0i})h_v = h_\lambda v_{0i}$ and thus $(\Lambda_0 - \lambda_{0i})(T_0)^{-1}h_v = h_\lambda (T_0)^{-1} v_{0i} = h_\lambda e_i$. Comparing the i-th coordinate of the RHS and the LHS of the last equation, we obtain $h_\lambda = 0$. Hence $h_v \in \operatorname{Ker}(X_0 - \lambda_{0i})$, that is $h_v = c v_{0i}$ for some $c \in \mathbb{C}$. But $h \in \operatorname{Ker} D_0$ means also that $0 = \gamma h_v = c\gamma v_{0i}$, thus by our assumption on γ we obtain $c = 0$, and consequently $h_v = 0$. Therefore, $h = 0$, which yields the invertibility of D_0. The existence of U_i and the appropriate holomorphic functions v_i, λ_i follows from the implicit function theorem for the equation $F(X, v, \mu) = (0, \gamma v_{0i})$.

Now, we can define $\mathcal{T}(X)$ to be the matrix with the i-th column equal to $v_i(X)$, $i = 1, \ldots, d$, $U := \{X \in U_1 \cap \ldots \cap U_d : \det \mathcal{T}(X) \ne 0\}$ and $\mathcal{D}(X) := \operatorname{diag}(\lambda_1(X), \ldots, \lambda_d(X))$. □