Homogeneous models for 5-dimensional para-CR manifolds with Levi form degenerate in one direction

Joël Merker and Paweł Nurowski

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Para-CR manifolds in brief

- Para-CR structure is a geometric structure which a hypersurface $M^{2n-1} \subset (\mathbb{R}^n \times \mathbb{R}^n)$ acquires from the ambient product space $\mathbb{R}^n \times \mathbb{R}^n$.

- More specifically one considers

$$M_{2n-1} = \{ \mathbb{R}^n \times \mathbb{R}^n \ni (x, \bar{x}) \mid \Phi(x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n) = 0 \}$$

modulo (local) diffeomorphisms $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ preserving the split of $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

- The lowest dimension, $n = 2$. Such para-CR structures (if nondegenerate) are related to 2nd-order ODEs, considered modulo point transformations of variables. Sort of dull.

- Today, the next dimension, $n = 3$. 5-dimensional para-CR structures.

- A 5-dim para-CR structure can be defined as a graph of a function $z$ of five variables, $z = z(x, y, \bar{x}, \bar{y}, \bar{z})$, where $(x, y, z, \bar{x}, \bar{y}, \bar{z})$ are coordinates in $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$.

- This in turn, can be considered as a general solution of a system of two PDEs for a function $z = z(x, y)$ on the plane $(x, y)$, in which $(\bar{x}, \bar{y}, \bar{z})$ are constants of integration and parametrize the solution space. As in the following example.
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Example:

Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:

\[
z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}
\]

Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and \(z_x = \frac{-2(x - \bar{x})}{(y - \bar{y})}\) we have \(z_y = \frac{1}{4}z_x^2\). So, a para-CR structure defined by the cone \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane:

\[
z_{xxx} = 0 \quad \& \quad z_y = \frac{1}{4}z_x^2 \quad \text{for} \quad z = z(x, y).
\]

Conversely, \(z_{xxx} = 0\) solves as \(z = \alpha(y)x^2 + \beta(y)x + \gamma(y)\), and \(z_y = \frac{1}{4}z_x^2\)

\[
\alpha = -\frac{1}{y - \bar{y}}, \quad \beta' = -\frac{\beta}{y - \bar{y}}, \quad \text{hence} \quad \beta = \frac{2\bar{x}}{y - \bar{y}}.
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\[
\gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}, \quad \text{hence} \quad \gamma = -\frac{\bar{x}^2}{y - \bar{y}} + \bar{z}.
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This finally gives

\[
z = -\frac{\bar{x}^2}{y - \bar{y}} + \bar{z} + \frac{2x\bar{x}}{y - \bar{y}} - \frac{x^2}{y - \bar{y}}, \quad \text{i.e. the cone}
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Conversely, \(z_{xxx} = 0\) solves as

\[ z = \alpha(y)x^2 + \beta(y)x + \gamma(y), \]

and \(z_y = \frac{1}{4} z_x^2\) gives successively: \(\alpha' = \alpha^2\), hence \(\alpha = \frac{-1}{y - \bar{y}}\), \(\beta' = \frac{-\beta}{y - \bar{y}}\), hence \(\beta = \frac{2\bar{x}}{y - \bar{y}}\), \(\gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}\), hence \(\gamma = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z}\). This finally gives

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5-dim para-CR geometry as a geometry of PDEs

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Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:

\[ z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z} \]

Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and \(z_x = \frac{-2(x - \bar{x})}{y - \bar{y}}\) we have \(z_y = \frac{1}{4} z_x^2\). So, a para-CR structure defined by the cone \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane

\[ z_{xxx} = 0 \quad \& \quad z_y = \frac{1}{4} z_x^2 \quad \text{for} \quad z = z(x, y). \]

Conversely, \(z_{xxx} = 0\) solves as

\[ z = \alpha(y)x^2 + \beta(y)x + \gamma(y) \]

gives successively: \(\alpha' = \alpha^2\), hence \(\alpha = \frac{-1}{y - \bar{y}}\), \(\beta' = \frac{-\beta}{y - \bar{y}}\), hence \(\beta = \frac{2\bar{x}}{y - \bar{y}}\), \(\gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}\), hence \(\gamma = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z}\). This finally gives

\[ z = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z} + \frac{2\bar{x}x}{y - \bar{y}} - \frac{x^2}{y - \bar{y}}, \text{i.e. the cone} \]

\[(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0.\]
5-dim para-CR geometry as a geometry of PDEs

Example:

- Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:

  \[ z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}. \]

  Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and 

\[ z_x = -\frac{2(x - \bar{x})}{(y - \bar{y})} \]

we have \(z_y = \frac{1}{4} z_x^2\). So, a para-CR structure defined by the cone 

\((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane 

\[ z_{xxx} = 0 \quad \text{&} \quad z_y = \frac{1}{4} z_x^2 \]

for \(z = z(x, y)\).

- Conversely, \(z_{xxx} = 0\) solves as 

\[ z = \alpha(y)x^2 + \beta(y)x + \gamma(y) \]

and \(z_y = \frac{1}{4} z_x^2\) gives successively:

\[ \alpha' = \alpha^2, \quad \beta' = -\frac{\beta}{y - \bar{y}}, \quad \beta = \frac{2\bar{x}}{y - \bar{y}} \]

\[ \gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}, \quad \gamma = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z} \]

This finally gives

\[ z = -\frac{x^2}{y - \bar{y}} + \bar{z} + \frac{2x\bar{x}}{y - \bar{y}} - \frac{x^2}{y - \bar{y}}, \]

i.e. the cone

\[ (x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0. \]
Example:

Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:

\[
    z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}.
\]

Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and \(z_x = -\frac{2(x - \bar{x})}{(y - \bar{y})}\), we have \(z_y = \frac{1}{4}z_x^2\). So, a para-CR structure defined by the cone \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane

\[
    z_{xxx} = 0 \quad \& \quad z_y = \frac{1}{4}z_x^2
\]

for \(z = z(x, y)\).

Conversely, \(z_{xxx} = 0\) solves as \(z = \alpha(y)x^2 + \beta(y)x + \gamma(y)\), and \(z_y = \frac{1}{4}z_x^2\) gives successively: \(\alpha' = \alpha^2\), hence \(\alpha = \frac{-1}{y - \bar{y}}\), \(\beta' = \frac{-\beta}{y - \bar{y}}\), hence \(\beta = \frac{2\bar{x}}{y - \bar{y}}\), \(\gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}\), hence \(\gamma = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z}\). This finally gives

\[
    z = \frac{-x^2}{y - \bar{y}} + \bar{z} + \frac{2x\bar{x}}{y - \bar{y}} - \frac{x^2}{y - \bar{y}}, \text{ i.e. the cone}
\]

\[
    (x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0.
\]
Example:

Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:

\[ z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}. \]

Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and \(z_x = \frac{-2(x - \bar{x})}{(y - \bar{y})}\) we have \(z_y = \frac{1}{4}z_x^2\). So, a para-CR structure defined by the cone \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane

\[
\begin{align*}
  z_{xxx} &= 0 & & \text{for } z = z(x, y).
\end{align*}
\]

Conversely, \(z_{xxx} = 0\) solves as

\[
\begin{align*}
  z &= \alpha(y)x^2 + \beta(y)x + \gamma(y) \quad \text{and } z_y = \frac{1}{4}z_x^2 \quad \text{for } z = z(x, y).
\end{align*}
\]

This finally gives

\[
\begin{align*}
  (x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) &= 0.
\end{align*}
\]
Example:

Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:
\[
z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}.
\]
Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and \(z_x = \frac{-2(x - \bar{x})}{(y - \bar{y})}\) we have \(z_y = \frac{1}{4} z_x^2\). So, a para-CR structure defined by the cone
\[
(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0
\]
in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane
\[
\begin{align*}
z_{xxx} &= 0 & \text{for } & z &= z(x, y).
\end{align*}
\]

Conversely, \(z_{xxx} = 0\) solves as
\[
z = \alpha(y)x^2 + \beta(y)x + \gamma(y),
\]
and \(z_y = \frac{1}{4} z_x^2\) gives successively:
\[
\alpha' = \alpha^2, \quad \beta' = -\frac{\beta}{y - \bar{y}}, \quad \gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2},
\]
\[
\alpha = \frac{-1}{y - \bar{y}}, \quad \beta = \frac{2\bar{x}}{y - \bar{y}}, \quad \gamma = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z}.
\]
This finally gives
\[
z = -\frac{\bar{x}^2}{y - \bar{y}} + \bar{z} + \frac{2\bar{x}x}{y - \bar{y}} - \frac{x^2}{y - \bar{y}}, \quad \text{i.e. the cone}
\]
\[
(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0.
\]
Example:

Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:

\[ z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}. \]

Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and \(z_x = -\frac{2(x - \bar{x})}{(y - \bar{y})}\) we have \(z_y = \frac{1}{4} z_x^2\). So, a para-CR structure defined by the cone \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane

\[
\boxed{z_{xxx} = 0 \quad \& \quad z_y = \frac{1}{4} z_x^2}
\]

for \(z = z(x, y)\).

Conversely, \(z_{xxx} = 0\) solves as

\[
z = \alpha(y)x^2 + \beta(y)x + \gamma(y), \quad \text{and} \quad z_y = \frac{1}{4} z_x^2
\]

gives successively: \(\alpha' = \alpha^2\), hence \(\alpha = \frac{-1}{y - \bar{y}}\), \(\beta' = \frac{-\beta}{y - \bar{y}}\), hence \(\beta = \frac{2\bar{x}}{y - \bar{y}}\), \(\gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}\), hence \(\gamma = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z}\). This finally gives

\[
z = -\frac{x^2}{y - \bar{y}} + \bar{z} + \frac{2x\bar{x}}{y - \bar{y}} - \frac{x^2}{y - \bar{y}}, \quad \text{i.e. the cone}
\]

\[
(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0.
\]
Example:

- Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:

  \[ z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}. \]

  Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and \(z_x = \frac{-2(x - \bar{x})}{(y - \bar{y})}\), we have \(z_y = \frac{1}{4}z_x^2\). So, a para-CR structure defined by the cone \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane for \(z = z(x, y)\).

Conversely, \(z_{xxx} = 0\) solves as \(z = \alpha(y)x^2 + \beta(y)x + \gamma(y)\), and \(z_y = \frac{1}{4}z_x^2\). This gives successively:

- \(\alpha' = \alpha^2\), hence \(\alpha = \frac{-1}{y - \bar{y}}\), \(\beta' = \frac{-\beta}{y - \bar{y}}\), hence \(\beta = \frac{2\bar{x}}{y - \bar{y}}\),

- \(\gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}\), hence \(\gamma = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z}\). This finally gives

  \[ z = -\frac{x^2}{y - \bar{y}} + \bar{z} + \frac{2x\bar{x}}{y - \bar{y}} - \frac{x^2}{y - \bar{y}}, \]

  i.e. the cone

  \[(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0.\]
Example:

Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:
\[
z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}.
\]
Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\).

Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and \(z_x = -\frac{2(x - \bar{x})}{(y - \bar{y})}\) we have \(z_y = \frac{1}{4}z_x^2\). So, a para-CR structure defined by the cone \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane:

\[
z_{xxx} = 0 \quad \& \quad z_y = \frac{1}{4}z_x^2 \quad \text{for} \quad z = z(x, y).
\]

Conversely, \(z_{xxx} = 0\) solves as \(z = \alpha(y)x^2 + \beta(y)x + \gamma(y)\), and \(z_y = \frac{1}{4}z_x^2\)
gives successively:

\[
\alpha' = \alpha^2, \quad \text{hence} \quad \alpha = -\frac{1}{y - \bar{y}}, \quad \beta' = -\frac{\beta}{y - \bar{y}}, \quad \text{hence} \quad \beta = \frac{2\bar{x}}{y - \bar{y}}, \quad \gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}, \quad \text{hence} \quad \gamma = -\frac{\bar{x}^2}{y - \bar{y}} + \bar{z}.
\]

This finally gives

\[
z = -\frac{x^2}{y - \bar{y}} + \bar{z} + \frac{2x\bar{x}}{y - \bar{y}} - \frac{x^2}{y - \bar{y}}, \quad \text{i.e. the cone}
\]

\[
(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0.
\]
Example:

Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:
\[
z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}.
\]
Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and \(z_x = \frac{-2(x - \bar{x})}{(y - \bar{y})}\) we have \(z_y = \frac{1}{4}z_x^2\). So, a para-CR structure defined by the cone \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane
\[
z_{xxx} = 0 \quad \& \quad z_y = \frac{1}{4}z_x^2 \quad \text{for} \quad z = z(x, y).
\]

Conversely, \(z_{xxx} = 0\) solves as \(z = \alpha(y)x^2 + \beta(y)x + \gamma(y)\), and \(z_y = \frac{1}{4}z_x^2\) gives successively: \(\alpha' = \alpha^2\), hence \(\alpha = \frac{-1}{y - \bar{y}}\), \(\beta' = \frac{-\beta}{y - \bar{y}}\), hence \(\beta = \frac{2\bar{x}}{y - \bar{y}}\), \(\gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}\), hence \(\gamma = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z}\). This finally gives
\[
z = -\frac{\bar{x}^2}{y - \bar{y}} + \bar{z} + \frac{2x\bar{x}}{y - \bar{y}} - \frac{x^2}{y - \bar{y}}, \text{ i.e. the cone}
\[
(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0.
\]
5-dim para-CR geometry as a geometry of PDEs

Example:
- Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:
  \[z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}\]. Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and \(z_x = \frac{-2(x - \bar{x})}{(y - \bar{y})}\) we have \(z_y = \frac{1}{4}z_x^2\). So, a para-CR structure defined by the cone \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane

\[
\begin{align*}
z_{xxx} &= 0 & \text{and} & \quad z_y &= \frac{1}{4}z_x^2 \\
\text{for} & \quad z = z(x, y).
\end{align*}
\]

Conversely, \(z_{xxx} = 0\) solves as \(z = \alpha(y)x^2 + \beta(y)x + \gamma(y)\), and \(z_y = \frac{1}{4}z_x^2\) gives successively: \(\alpha' = \alpha^2\), hence \(\alpha = \frac{-1}{y - \bar{y}}\), \(\beta' = -\beta\), hence \(\beta = \frac{2\bar{x}}{y - \bar{y}}\), \(\gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}\), hence \(\gamma = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z}\). This finally gives

\[z = -\frac{x^2}{y - \bar{y}} + \bar{z} + \frac{2x\bar{x}}{y - \bar{y}} - \frac{x^2}{y - \bar{y}},\]
i.e. the cone

\[(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\].
Example:

- Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:
  \[ z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}. \]

Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and 
\[ z_x = \frac{-2(x - \bar{x})}{(y - \bar{y})} \] we have \(z_y = \frac{1}{4} z_x^2\). So, a para-CR structure defined by the cone 
\((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane 
\[ z_{xxx} = 0 \quad \& \quad z_y = \frac{1}{4} z_x^2 \quad \text{for} \quad z = z(x, y). \]

Conversely, \(z_{xxx} = 0\) solves as 
\[ z = \alpha y x^2 + \beta y x + \gamma y, \]
and \(z_y = \frac{1}{4} z_x^2\) gives successively: \(\alpha' = \alpha^2\), hence \(\alpha = -\frac{1}{y - \bar{y}}\), \(\beta' = -\frac{\beta}{y - \bar{y}}\), hence \(\beta = \frac{2\bar{x}}{y - \bar{y}}\), 
\(\gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}\), hence \(\gamma = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z}\). This finally gives 
\[ z = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z} + \frac{2x\bar{x}}{y - \bar{y}} - \frac{x^2}{y - \bar{y}}, \]
i.e. the cone 
\((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\).
Example:

Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:
\[
  z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}.
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Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and \(z_x = \frac{-2(x - \bar{x})}{(y - \bar{y})}\) we have \(z_y = \frac{1}{4}z_x^2\). So, a para-CR structure defined by the cone \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane

\[
  z_{xxx} = 0 \quad & \quad z_y = \frac{1}{4}z_x^2
\]
for \(z = z(x, y)\).

Conversely, \(z_{xxx} = 0\) solves as \(z = \alpha(y)x^2 + \beta(y)x + \gamma(y)\), and \(z_y = \frac{1}{4}z_x^2\) gives successively: \(\alpha' = \alpha^2\), hence \(\alpha = \frac{-1}{y - \bar{y}}\), \(\beta' = -\frac{\beta}{y - \bar{y}}\), hence \(\beta = \frac{2\bar{x}}{y - \bar{y}}\),
\[
\gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}, \text{ hence } \gamma = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z}\]. This finally gives
\[
z = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z} + \frac{2x\bar{x}}{y - \bar{y}} - \frac{x^2}{y - \bar{y}}, \text{ i.e. the cone}
(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0.
Example:

Take \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\), and solve it for \(z\) obtaining:

\[z = -\frac{(x - \bar{x})^2}{y - \bar{y}} + \bar{z}.
\]

Now think about \((x, y)\) as independent variables, and \((\bar{x}, \bar{y}, \bar{z})\) as parameters. Obviously \(z_{xxx} = 0\). Also, because \(z_y = \frac{(x - \bar{x})^2}{(y - \bar{y})^2}\) and \(z_x = \frac{-2(x - \bar{x})}{(y - \bar{y})}\) we have \(z_y = \frac{1}{4}z_x^2\). So, a para-CR structure defined by the cone \((x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0\) in \(\mathbb{R}^3 \times \mathbb{R}^3\) defines a system of PDEs on the plane

\[
\begin{align*}
    z_{xxx} &= 0 & \text{and} & \quad z_y = \frac{1}{4}z_x^2
\end{align*}
\]

for \(z = z(x, y)\).

Conversely, \(z_{xxx} = 0\) solves as \(z = \alpha(y)x^2 + \beta(y)x + \gamma(y)\), and \(z_y = \frac{1}{4}z_x^2\) gives successively: \(\alpha' = \alpha^2\), hence \(\alpha = \frac{-1}{y - \bar{y}}\), \(\beta' = -\frac{\beta}{y - \bar{y}}\), hence \(\beta = \frac{2\bar{x}}{y - \bar{y}}\), \(\gamma' = \frac{\bar{x}^2}{(y - \bar{y})^2}\), hence \(\gamma = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z}\). This finally gives

\[
z = \frac{-\bar{x}^2}{y - \bar{y}} + \bar{z} + \frac{2x\bar{x}}{y - \bar{y}} - \frac{x^2}{y - \bar{y}}, \text{ i.e. the cone}
\]

\[
(x - \bar{x})^2 + (y - \bar{y})(z - \bar{z}) = 0.
\]
In general, we consider the following system of PDEs on the plane

\[
\begin{align*}
S: \quad & z_{xxx} = H(x, y, z, z_x, z_{xx}) \quad \& \quad z_y = G(x, y, z, z_x, z_{xx}) \quad \text{for} \quad z(x, y).
\end{align*}
\]

Fact: The general solution of \((S)\) depends on 3 parameters \((\bar{x}, \bar{y}, \bar{z})\), and has the form \(z = z(x, y; \bar{x}, \bar{y}, \bar{z})\) if and only if

\[
(\text{IC}) \quad \Delta H = D^3 G
\]

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General solutions of systems \((S)\) give examples of 5-dim para-CR structures. We prefer the PDE point of view, and we will stick to this in the following. In particular, in this point of view, para-CR transformations for hypersurfaces in \((x, y, z, \bar{x}, \bar{y}, \bar{z})\) are point transformations of variables of \((S)\).
5-dim para-CR geometry as a geometry of PDEs

- **In general, we consider the following system of PDEs on the plane**

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  (S) \quad z_{xxx} = H(x, y, z, z_x, z_{xx}) \quad \& \quad z_y = G(x, y, z, z_x, z_{xx}) \quad \text{for} \quad z(x, y).
  \]

- **Fact:** The general solution of \((S)\) depends on 3 parameters \((\bar{x}, \bar{y}, \bar{z})\), and has the form \(z = z(x, y; \bar{x}, \bar{y}, \bar{z})\) if and only if

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In the PDE picture these two distributions are the respective annihilators of the following system of 1-forms

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D_1 = \begin{pmatrix}
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Definition of a 5-dimensional para-CR structure locally a'la Elie Cartan: A 5-dimensional para-CR structure is a structure consisting of an equivalence class $[(\omega^1, \omega^2, \omega^3, \omega^4, \omega^5)]$ of coframes $\omega = (\omega^1, \omega^2, \omega^3, \omega^4, \omega^5)$ on $\mathbb{R}^5$ parameterized by $(x, y, z, p, r)$, with an equivalence relation $\sim$ given by

$$\hat{\omega} \sim \omega \iff \begin{pmatrix} \hat{\omega}^1 \\ \hat{\omega}^2 \\ \hat{\omega}^3 \\ \hat{\omega}^4 \\ \hat{\omega}^5 \end{pmatrix} = \begin{pmatrix} f_1 & 0 & 0 & 0 & 0 \\ f_2 & \rho e^\phi & f_4 & 0 & 0 \\ f_5 & f_6 & f_7 & 0 & 0 \\ \hat{f}_2 & 0 & 0 & \rho e^{-\phi} & \hat{f}_4 \\ \hat{f}_5 & 0 & 0 & \hat{f}_6 & \hat{f}_7 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix},$$

with $\omega^1 = dz - pdx - Gdy$, $\omega^2 = dp - rdx - DГdy$, $\omega^3 = dr - Hdx - D^2Гdy$, $\omega^4 = dx$ and $\omega^5 = dy$.

The integrability of $D_1$ and $D_2$ implies that

$$\left(d\omega^1 - L_{11}\omega^2 \wedge \omega^4 - L_{12}\omega^2 \wedge \omega^5 - L_{21}\omega^3 \wedge \omega^4 - L_{22}\omega^3 \wedge \omega^5\right) \wedge \omega^1 \equiv 0,$$

with the $2 \times 2$ matrix $L$ of functions $L_{AB}$ on $M_5$ defined by this condition.

The matrix $L$ is not well defined by the equivalence class of $\omega$, but its signature is! Hence $\det(L) = 0$ or $\det(L) \neq 0$ is a para-CR invariant condition at each point. If $\det(L) \neq 0$, the corresponding para-CR structure is nondegenerate, and it defines one of the parabolic geometries in dimension 5 (flat model - a flying soucer in the attacking mode).

Today: para-CR structures with $L \neq 0$ but $\det(L) \equiv 0$. 5-dimensional para-CR structures with Levi form $L$ degenerate in 1 direction.

In terms of our PDEs this degeneracy means that $G_r \equiv 0$, or $G = (x, y, z, z_x)$.

We also do not want that our para-CR structure is locally equivalent to $(3 - \dim para - CR) \times (\mathbb{R} \times \mathbb{R})$. This results in an assumption about $G_{pp} \neq 0$. 

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\hat{\omega}^2 \\
\hat{\omega}^3 \\
\hat{\omega}^4 \\
\hat{\omega}^5
\end{pmatrix} = \begin{pmatrix}
f_1 & 0 & 0 & 0 & 0 \\
f_2 & \rho e^\phi & f_4 & 0 & 0 \\
f_5 & f_6 & f_7 & 0 & 0 \\
f_2 & 0 & 0 & \rho e^{-\phi} & f_4 \\
f_5 & 0 & 0 & f_6 & f_7
\end{pmatrix} \begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix},
\]

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\[
\begin{pmatrix}
\hat{\omega}^1 \\
\hat{\omega}^2 \\
\hat{\omega}^3 \\
\hat{\omega}^4 \\
\hat{\omega}^5
\end{pmatrix} \sim \begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix} \iff \begin{pmatrix}
f_1 & 0 & 0 & 0 & 0 \\
f_2 & \rho e^\phi & f_4 & 0 & 0 \\
f_3 & f_6 & f_7 & 0 & 0 \\
f_4 & 0 & 0 & \rho e^{-\phi} & f_4 \\
f_5 & \bar{f}_6 & \bar{f}_7 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
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f_5 & f_6 & f_7 & 0 & 0 \\
f_8 & 0 & 0 & \rho \epsilon^{-\phi} & \bar{f}_4 \\
f_9 & \bar{f}_6 & \bar{f}_7 & 0 & \bar{f}_5
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\hat{\omega}^3 \\
\hat{\omega}^4 \\
\hat{\omega}^5 \\
\end{array} \right) = \left( \begin{array}{cccccc}
f_1 & 0 & 0 & 0 & 0 \\
f_2 & \rho e^\phi & f_4 & 0 & 0 \\
f_5 & f_6 & f_7 & 0 & 0 \\
f_8 & 0 & 0 & \rho e^{-\phi} & f_4 \\
f_9 & f_6 & f_7 & 0 & 0 \\
\end{array} \right) \left( \begin{array}{c}
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5-dim para-CR geometry as a geometry of PDEs - the first invariant - signature of the Levi form

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\[
\left(\begin{array}{c}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{array}\right) \sim
\left(\begin{array}{c}
\tilde{\omega}^1 \\
\tilde{\omega}^2 \\
\tilde{\omega}^3 \\
\tilde{\omega}^4 \\
\tilde{\omega}^5
\end{array}\right) \iff
\left(\begin{array}{cccccc}
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$$
\hat{ω} \sim ω \iff \begin{pmatrix}
\hat{ω}^1 \\
\hat{ω}^2 \\
\hat{ω}^3 \\
\hat{ω}^4 \\
\hat{ω}^5
\end{pmatrix} = \begin{pmatrix}
f_1 & 0 & 0 & 0 & 0 \\
f_2 & ρe^φ & f_4 & 0 & 0 \\
f_3 & 0 & f_6 & 0 & 0 \\
f_4 & 0 & 0 & ρe^{1φ} & f_7 \\
f_5 & f_6 & f_7 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
ω^1 \\
ω^2 \\
ω^3 \\
ω^4 \\
ω^5
\end{pmatrix},
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\hat{\omega}^2 \\
\hat{\omega}^3 \\
\hat{\omega}^4 \\
\hat{\omega}^5 \\
\end{pmatrix}
\sim
\begin{pmatrix}
f_1 & 0 & 0 & 0 & 0 \\
f_2 & \rho e^\phi & f_4 & 0 & 0 \\
f_5 & f_6 & f_7 & 0 & 0 \\
\hat{f}_1 & 0 & 0 & \rho e^{-\phi} & \hat{f}_4 \\
\hat{f}_5 & 0 & 0 & \hat{f}_6 & \hat{f}_7 \\
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\[
\begin{pmatrix}
\hat{\omega}^1 \\
\hat{\omega}^2 \\
\hat{\omega}^3 \\
\hat{\omega}^4 \\
\hat{\omega}^5
\end{pmatrix} \sim
\begin{pmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5
\end{pmatrix} \begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix},
\]

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f_1 & 0 & 0 & 0 & 0 \\
f_2 & \rho e^\phi & f_4 & 0 & 0 \\
f_5 & f_6 & f_7 & 0 & 0 \\
f_8 & 0 & 0 & \rho e^{-\phi} & f_9 \\
f_{10} & f_{11} & f_{12} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
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f_3 & f_6 & f_7 & 0 & 0 \\
f_4 & 0 & 0 & \rho e^{-\phi} & \bar{f}_4 \\
f_5 & \bar{f}_2 & 0 & 0 & \rho e^{-\phi}
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f_2 & \rho e^\phi & f_4 & 0 & 0 \\
f_5 & f_6 & f_7 & 0 & 0 \\
\tilde{f}_2 & 0 & 0 & \rho e^{-\phi} & f_4 \\
\tilde{f}_5 & 0 & 0 & \tilde{f}_6 & \tilde{f}_7
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\end{pmatrix}
\sim
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f_1 \\
f_2 \\
f_3 \\
f_4 \\
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We study systems of PDEs on the plane

\((S)\quad z_{xxx} = H(x, y, z, p, r) \quad \& \quad z_y = G(x, y, z, p)\) for \(z(x, y)\)

such that

\((IC)\quad \Delta H = D^3 G\) \quad \& \quad (2NG) \quad G_{pp} \neq 0,

with \(D = \partial_x + p\partial_z + r\partial_p + H\partial_r\), \(\Delta = \partial_y + G\partial_z + DG\partial_p + D^2 G\partial_r\), and \(p = z_x\), \(r = z_{xx}\), considered modulo point transformations of variables.

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\[
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5 \\
\end{pmatrix} \mapsto \begin{pmatrix}
f_1 & 0 & 0 & 0 & 0 \\
f_2 & \rho e^\phi & f_4 & 0 & 0 \\
f_5 & f_6 & f_7 & 0 & 0 \\
f_2 & 0 & 0 & \rho e^{-\phi} & f_4 \\
f_5 & 0 & 0 & f_6 & f_7 \\
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f_2 \\
f_5 \\
f_4 \\
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\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
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f_5 & f_6 & f_7 & 0 & 0 \\
0 & 0 & \rho e^{-\phi} & \tilde{f}_4 & \tilde{f}_7 \\
0 & 0 & \tilde{f}_6 & \tilde{f}_7 & 0 \\
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with \( D = \partial_x + p \partial_z + r \partial_p + H \partial_r, \) \( \triangle = \partial_y + G \partial_z + DG \partial_p + D^2 G \partial_r, \) and \( p = z_x, \)
\( r = z_{xx}, \) considered modulo point transformations of variables.

This is equivalent to study coframes \( \omega^1 = dz - pdx - Gdy, \)
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\omega^4 \\
\omega^5 \\
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
p e^\phi & f_4 & 0 & 0 & 0 \\
0 & f_6 & f_7 & 0 & 0 \\
0 & 0 & e^{-\phi} & f_4 & 0 \\
0 & 0 & 0 & f_6 & f_7 \\
\end{pmatrix}
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5 \\
\end{pmatrix}.
\]
We study systems of PDEs on the plane

\[(S) \quad z_{xxx} = H(x, y, z, p, r) \quad \& \quad z_y = G(x, y, z, p) \quad \text{for} \quad z(x, y)\]

such that

\[(IC) \quad \triangle H = D^3 G \quad \& \quad (2NG) \quad G_{pp} \neq 0,\]

with \(D = \partial_x + p \partial_z + r \partial_p + H \partial_r\), \(\triangle = \partial_y + G \partial_z + DG \partial_p + D^2 G \partial_r\), and \(p = z_x\), \(r = z_{xx}\), considered modulo point transformations of variables.

This is equivalent to study coframes \(\omega^1 = dz - pdx - Gdy\), \(\omega^2 = dp - rdx - DGdy\), \(\omega^3 = dr - Hdx - D^2 Gdy\), \(\omega^4 = dx\) and \(\omega^5 = dy\), with \(D^3 G = \triangle H\), \(G_{pp} \neq 0\), and \(G_r = 0\), given modulo

\[
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}
\mapsto
\begin{pmatrix}
f_1 & 0 & 0 & 0 & 0 & 0 \\
f_2 & p e^\phi & f_4 & 0 & 0 \\
f_5 & f_6 & f_7 & 0 & 0 \\
f_2 & 0 & 0 & p e^{-\phi} & f_4 \\
f_5 & 0 & 0 & f_6 & f_7
\end{pmatrix}
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}.
\]
We study systems of PDEs on the plane

\[ z_{xxx} = H(x, y, z, p, r) \quad \& \quad z_y = G(x, y, z, p) \quad \text{for} \quad z(x, y) \]

such that

\[ \triangle H = D^3 G \quad \& \quad G_{pp} \neq 0, \]

with \( D = \partial_x + p \partial_z + r \partial_p + H \partial_r, \) \( \triangle = \partial_y + G \partial_z + DG \partial_p + D^2 G \partial_r, \) and \( p = z_x, \)
\( r = z_{xx}, \) considered modulo point transformations of variables.

This is equivalent to study coframes

\[
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}
\mapsto
\begin{pmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6 \\
f_7
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\rho e^\phi & f_4 & 0 & 0 & 0 \\
0 & f_6 & f_7 & 0 & 0 \\
0 & 0 & \rho e^{-\phi} & f_4 & f_7 \\
0 & 0 & 0 & f_6 & f_7
\end{pmatrix}
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}.
\]
We study systems of PDEs on the plane

\[(S) \quad z_{xxx} = H(x, y, z, p, r) \quad \& \quad z_y = G(x, y, z, p) \quad \text{for} \quad z(x, y)\]

such that

\[(IC) \quad \Box H = D^3 G \quad \& \quad (2NG) \quad G_{pp} \neq 0,\]

with \(D = \partial_x + p\partial_z + r\partial_p + H\partial_r, \quad \Box = \partial_y + G\partial_z + DG\partial_p + D^2 G\partial_r,\) and \(p = z_x, \quad r = z_{xx},\) considered modulo point transformations of variables.

This is equivalent to study coframes

\[
\begin{align*}
\omega^1 &= dz - pdx - Gdy, \\
\omega^2 &= dp - rdx - DGdy, \\
\omega^3 &= dr - Hdx - D^2 Gdy, \\
\omega^4 &= dx \quad \text{and} \quad \omega^5 = dy
\end{align*}
\]

with \(D^3 G = \Box H, \; G_{pp} \neq 0,\) and \(G_r = 0,\) given modulo

\[
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\begin{array}{ccccc}
\tilde{f}_1 & 0 & 0 & 0 & 0 \\
\tilde{f}_2 & \rho e^{\phi} & \tilde{f}_4 & 0 & 0 \\
\tilde{f}_5 & \tilde{f}_6 & \tilde{f}_7 & 0 & 0 \\
\tilde{f}_8 & 0 & 0 & \rho e^{-\phi} & \tilde{f}_9 \\
\tilde{f}_{10} & \tilde{f}_{11} & \tilde{f}_{12} & \tilde{f}_{13} & \tilde{f}_{14}
\end{array}
\end{pmatrix}
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}.
\]
We study systems of PDEs on the plane

\[(S) \quad z_{xxx} = H(x, y, z, p, r) \quad \& \quad z_y = G(x, y, z, p) \quad \text{for} \quad z(x, y)\]

such that

\[(IC) \quad \Delta H = D^3 G \quad \& \quad (2NG) \quad G_{pp} \neq 0,\]

with $D = \partial_x + p \partial_z + r \partial_p + H \partial_r$, $\Delta = \partial_y + G \partial_z + DG \partial_p + D^2 G \partial_r$, and $p = z_x$, $r = z_{xx}$, considered modulo point transformations of variables.

This is equivalent to study coframes $\omega^1 = dz - pdx - Gdy$, $\omega^2 = dp - rdx - DGdy$, $\omega^3 = dr - Hdx - D^2 Gdy$, $\omega^4 = dx$ and $\omega^5 = dy$, with $D^3 G = \Delta H$, $G_{pp} \neq 0$, and $G_r = 0$, given modulo

\[
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}
\mapsto
\begin{pmatrix}
f_1 & 0 & 0 & 0 & 0 \\
f_2 & f_3 \rho e^\phi & f_4 & 0 & 0 \\
f_5 & f_6 & f_7 & 0 & 0 \\
f_8 & 0 & 0 & f_9 \rho e^{-\phi} & f_{10} \\
f_{11} & f_{12} & f_{13} & f_{14} & f_{15}
\end{pmatrix}
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}.
\]
We study systems of PDEs on the plane

\[
S \quad z_{xxx} = H(x, y, z, p, r) \quad \& \quad z_y = G(x, y, z, p) \quad \text{for} \quad z(x, y)
\]

such that

\[
IC \quad \Delta H = D^3 G \quad \& \quad (2NG) \quad G_{pp} \neq 0,
\]

with \( D = \partial_x + p\partial_z + r\partial_p + H\partial_r \), \( \Delta = \partial_y + G\partial_z + DG\partial_p + D^2 G\partial_r \), and \( p = z_x \), \( r = z_{xx} \), considered modulo \textit{point transformations of variables}.

This is equivalent to study coframes \( \omega^1 = dz - pdx - Gdy \), \( \omega^2 = dp - rdx - DGdy \), \( \omega^3 = dr - Hdx - D^2 Gdy \), \( \omega^4 = dx \) and \( \omega^5 = dy \), with \( D^3 G = \Delta H \), \( G_{pp} \neq 0 \), and \( G_r = 0 \), given modulo

\[
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix} \mapsto \begin{pmatrix}
f_1 \\
f_2 \\
f_5 \\
f_2 \\
f_5
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\rho e^{-\Phi} & f_4 & 0 & 0 & 0 \\
f_6 & f_7 & 0 & 0 & 0 \\
0 & 0 & \rho e^{\Phi} & f_4 & 0 \\
f_6 & f_7 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}.
\]
We study systems of PDEs on the plane

\[(S) \quad z_{xxx} = H(x, y, z, p, r) \quad \& \quad z_y = G(x, y, z, p) \quad \text{for} \quad z(x, y)\]

such that

\[(IC) \quad \Delta H = D^3 G \quad \& \quad (2NG) \quad G_{pp} \neq 0,\]

with \( D = \partial_x + p \partial_z + r \partial_p + H \partial_r, \quad \Delta = \partial_y + G \partial_z + DG \partial_p + D^2 G \partial_r, \) and \( p = z_x, \) \( r = z_{xx}, \) considered modulo point transformations of variables.

This is equivalent to study coframes \( \omega^1 = dz - pdx - Gdy, \)
\( \omega^2 = dp - rdx - DGdy, \)
\( \omega^3 = dr - Hdx - D^2 Gdy, \)
\( \omega^4 = dx \) and \( \omega^5 = dy, \) with \( D^3 G = \Delta H, \) \( G_{pp} \neq 0, \) and \( G_r = 0, \) given modulo

\[
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}
\mapsto
\begin{pmatrix}
f_1 & 0 & 0 & 0 & 0 \\
f_2 & \rho e^\phi & f_4 & 0 & 0 \\
f_5 & f_6 & f_7 & 0 & 0 \\
\bar{f}_2 & 0 & 0 & \rho e^{-\phi} & \bar{f}_4 \\
\bar{f}_5 & 0 & 0 & \bar{f}_6 & \bar{f}_7
\end{pmatrix}
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}.
\]
We study systems of PDEs on the plane

\[(S) \quad z_{xxx} = H(x, y, z, p, r) \quad & \quad z_y = G(x, y, z, p) \quad \text{for} \quad z(x, y)\]

such that

\[(IC) \quad \Delta H = D^3 G \quad & \quad (2NG) \quad G_{pp} \neq 0,\]

with \(D = \partial_x + p \partial_z + r \partial_p + H \partial_r, \quad \Delta = \partial_y + G \partial_z + DG \partial_p + D^2 G \partial_r, \quad \text{and} \quad p = z_x, \quad r = z_{xx}, \quad \text{considered modulo point transformations of variables.}\)

This is equivalent to study coframes \(\omega_1 = dz - pdx - Gdy,\)

\(\omega_2 = dp - rdx - DGdy, \quad \omega_3 = dr - Hdx - D^2 Gdy, \quad \omega_4 = dx \quad \text{and} \quad \omega_5 = dy, \quad \text{with} \quad D^3 G = \Delta H, \quad G_{pp} \neq 0, \quad \text{and} \quad G_r = 0, \quad \text{given modulo} \)

\[
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\omega_4 \\
\omega_5
\end{pmatrix} \mapsto
\begin{pmatrix}
\tilde{f}_1 \\
\tilde{f}_2 \\
\tilde{f}_3 \\
\tilde{f}_4 \\
\tilde{f}_5 \\
\tilde{f}_6 \\
\tilde{f}_7
\end{pmatrix}.
\]
We study systems of PDEs on the plane

\((S)\) \[ z_{xxx} = H(x, y, z, p, r) \quad \& \quad z_y = G(x, y, z, p) \]

for \(z(x, y)\) such that

\((IC)\) \[ \triangle H = D^3 G \quad \& \quad (2NG) \quad G_{pp} \neq 0, \]

with \(D = \partial_x + p\partial_z + r\partial_p + H\partial_r, \triangle = \partial_y + G\partial_z + DG\partial_p + D^2 G\partial_r, \) and \(p = z_x, \) \(r = z_{xx},\) considered modulo point transformations of variables.

This is equivalent to study coframes \(\omega^1 = dz - pdx - Gdy,\)
\(\omega^2 = dp - rdx - DGdy, \omega^3 = dr - Hdx - D^2 Gdy, \omega^4 = dx \) and \(\omega^5 = dy,\) with \(D^3 G = \triangle H, G_{pp} \neq 0, \) and \(G_r = 0,\) given modulo

\[
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}
\rightarrow
\begin{pmatrix}
f_1 & 0 & 0 & 0 & 0 \\
f_2 & \rho e^{\phi} & f_4 & 0 & 0 \\
f_5 & f_6 & f_7 & 0 & 0 \\
f_2 & 0 & 0 & \rho e^{-\phi} & f_4 \\
f_5 & 0 & 0 & f_6 & f_7
\end{pmatrix}
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}.
\]
We study systems of PDEs on the plane

\[(S)\quad z_{xxx} = H(x, y, z, p, r) \quad \& \quad z_y = G(x, y, z, p) \quad \text{for} \quad z(x, y)\]

such that

\[(IC)\quad \Delta H = D^3 G \quad \& \quad (2NG) \quad G_{pp} \neq 0\]

with \(D = \partial_x + p \partial_z + r \partial_p + H \partial_r\), \(\Delta = \partial_y + G \partial_z + DG \partial_p + D^2 G \partial_r\), and \(p = z_x\), \(r = z_{xx}\), considered modulo point transformations of variables.

This is equivalent to study coframes \(\omega^1 = dz - pdx - Gdy\), \(\omega^2 = dp - rdx - DGdy\), \(\omega^3 = dr - Hdx - D^2 Gdy\), \(\omega^4 = dx\) and \(\omega^5 = dy\), with \(D^3 G = \Delta H\), \(G_{pp} \neq 0\), and \(G_r = 0\), given modulo

\[
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix} \mapsto \begin{pmatrix}
f_1 & 0 & 0 & 0 & 0 \\
f_2 & \rho e^\phi & f_4 & 0 & 0 \\
f_5 & f_6 & f_7 & 0 & 0 \\
f_2 & 0 & 0 & \rho e^{-\phi} & f_4 \\
f_5 & 0 & 0 & f_6 & f_7
\end{pmatrix}\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix}.
\]
Main goal

- Symmetries: A vector field $X$ on $M_5 \ni (x, y, z, p, r)$ is a symmetry of the para-CR structure as defined in $(S) - (IC) - (2NG)$ if and only if

  \[
  (\mathcal{L}_X \omega^1) \wedge \omega^1 = 0,
  \]

  \[
  (\mathcal{L}_X \omega^2) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0,
  \]

  \[
  (\mathcal{L}_X \omega^3) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0,
  \]

  \[
  (\mathcal{L}_X \omega^4) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0,
  \]

  \[
  (\mathcal{L}_X \omega^5) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0.
  \]

- Goal: Find all homogeneous models, i.e. find the PDEs $(S) - (IC) - (2NG)$ such that their corresponding para-CR structures have at least five symmetries $X_1, X_2, X_3, X_4$ and $X_5$ such that $X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \neq 0$.

- Method: Cartan’s equivalence and reduction methods (part of SCREAM; if you know what I mean ;) ).
**Main goal**

- **Symmetries:** A vector field $X$ on $M_5 \ni (x, y, z, p, r)$ is a *symmetry* of the para-CR structure as defined in $(S) - (IC) - (2NG)$ if and only if

  \[
  (L_X \omega^1) \wedge \omega^1 = 0, \\
  (L_X \omega^2) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, \\
  (L_X \omega^3) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, \\
  (L_X \omega^4) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0, \\
  (L_X \omega^5) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0.
  \]

- **Goal:** Find all homogeneous models, i.e. find the PDEs $(S) - (IC) - (2NG)$ such that their corresponding para-CR structures have *at least five* symmetries $X_1, X_2, X_3, X_4$ and $X_5$ such that $X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \neq 0$.

- **Method:** Cartan’s equivalence and reduction methods (part of SCREAM; if you know what I mean ;).
Symmetries: A vector field $X$ on $M_5 \ni (x, y, z, p, r)$ is a symmetry of the para-CR structure as defined in $(S)$ – $(IC)$ – $(2NG)$ if and only if

\[
(\mathcal{L}_X \omega^1) \wedge \omega^1 = 0, \\
(\mathcal{L}_X \omega^2) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, \\
(\mathcal{L}_X \omega^3) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, \\
(\mathcal{L}_X \omega^4) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0, \\
(\mathcal{L}_X \omega^5) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0.
\]

Goal: Find all homogeneous models, i.e. find the PDEs $(S)$ – $(IC)$ – $(2NG)$ such that their corresponding para-CR structures have at least five symmetries $X_1$, $X_2$, $X_3$, $X_4$ and $X_5$ such that $X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \neq 0$.

Method: Cartan’s equivalence and reduction methods (part of SCREAM; if you know what I mean ;)).
Symmetries: A vector field $X$ on $M_5 \ni (x, y, z, p, r)$ is a symmetry of the para-CR structure as defined in $(S) - (IC) - (2NG)$ if and only if

\[
\begin{align*}
(L_X \omega^1) \wedge \omega^1 &= 0, \\
(L_X \omega^2) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 &= 0, \\
(L_X \omega^3) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 &= 0, \\
(L_X \omega^4) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 &= 0, \\
(L_X \omega^5) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 &= 0.
\end{align*}
\]

Goal: Find all homogeneous models, i.e. find the PDEs $(S) - (IC) - (2NG)$ such that their corresponding para-CR structures have at least five symmetries $X_1$, $X_2$, $X_3$, $X_4$ and $X_5$ such that $X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \neq 0$.

Method: Cartan’s equivalence and reduction methods (part of SCREAM; if you know what I mean ;)).
Symmetries: A vector field $X$ on $M_5 \ni (x, y, z, p, r)$ is a symmetry of the para-CR structure as defined in $(S) - (IC) - (2NG)$ if and only if

\[
(L_X \omega^1) \wedge \omega^1 = 0,
(L_X \omega^2) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, \quad (L_X \omega^4) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0,
(L_X \omega^3) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, \quad (L_X \omega^5) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0.
\]

Goal: Find all homogeneous models, i.e. find the PDEs $(S) - (IC) - (2NG)$ such that their corresponding para-CR structures have at least five symmetries $X_1, X_2, X_3, X_4$ and $X_5$ such that $X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \neq 0$.

Method: Cartan’s equivalence and reduction methods (part of SCREAM; if you know what I mean ;).
Symmetries: A vector field $X$ on $M_5 \ni (x, y, z, p, r)$ is a symmetry of the para-CR structure as defined in $(S) - (IC) - (2NG)$ if and only if

\[
(\mathcal{L}_X \omega^1) \wedge \omega^1 = 0, \\
(\mathcal{L}_X \omega^2) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, \\
(\mathcal{L}_X \omega^3) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, \\
(\mathcal{L}_X \omega^4) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0, \\
(\mathcal{L}_X \omega^5) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0.
\]

Goal: Find all homogeneous models, i.e. find the PDEs $(S) - (IC) - (2NG)$ such that their corresponding para-CR structures have at least five symmetries $X_1, X_2, X_3, X_4$ and $X_5$ such that $X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \neq 0$.

Method: Cartan's equivalence and reduction methods (part of SCREAM; if you know what I mean ;) ).
Symmetries: A vector field $X$ on $M_5 \ni (x, y, z, p, r)$ is a symmetry of the para-CR structure as defined in $(S) - (IC) - (2NG)$ if and only if
\[
(L_X \omega^1) \wedge \omega^1 = 0, \\
(L_X \omega^2) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, \\
(L_X \omega^3) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, \\
(L_X \omega^4) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0, \\
(L_X \omega^5) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0.
\]

Goal: Find all homogeneous models, i.e. find the PDEs $(S) - (IC) - (2NG)$ such that their corresponding para-CR structures have at least five symmetries $X_1, X_2, X_3, X_4$ and $X_5$ such that $X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \neq 0$.

Method: Cartan’s equivalence and reduction methods (part of SCREAM; if you know what I mean :) ).
Symmetries: A vector field \( X \) on \( M_5 \ni (x, y, z, p, r) \) is a symmetry of the para-CR structure as defined in \((S) - (IC) - (2NG)\) if and only if

\[
\left( \mathcal{L}_X \omega^1 \right) \wedge \omega^1 = 0,
\]
\[
\left( \mathcal{L}_X \omega^2 \right) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0,
\]
\[
\left( \mathcal{L}_X \omega^3 \right) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0,
\]
\[
\left( \mathcal{L}_X \omega^4 \right) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0,
\]
\[
\left( \mathcal{L}_X \omega^5 \right) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0.
\]

Goal: Find all homogeneous models, i.e. find the PDEs \((S) - (IC) - (2NG)\) such that their corresponding para-CR structures have at least five symmetries \( X_1, X_2, X_3, X_4 \) and \( X_5 \) such that \( X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \neq 0 \).

Method: Cartan’s equivalence and reduction methods (part of SCREAM; if you know what I mean ;) ).
Symmetries: A vector field $X$ on $M_5 \ni (x, y, z, p, r)$ is a symmetry of the para-CR structure as defined in $(S) - (IC) - (2NG)$ if and only if

$$(\mathcal{L}_X \omega^1) \wedge \omega^1 = 0,$$

$$(\mathcal{L}_X \omega^2) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0,$$

$$(\mathcal{L}_X \omega^3) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0,$$

$$(\mathcal{L}_X \omega^4) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0,$$

$$(\mathcal{L}_X \omega^5) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0.$$

Goal: Find all homogeneous models, i.e. find the PDEs $(S) - (IC) - (2NG)$ such that their corresponding para-CR structures have at least five symmetries $X_1, X_2, X_3, X_4$ and $X_5$ such that $X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \neq 0$.

Method: Cartan’s equivalence and reduction methods (part of SCREAM; if you know what I mean ;).
Theorem: Given a para-CR structure represented by the forms

\[ \omega^1 = dz - pdx - Gdy, \]
\[ \omega^2 = dp - rdx - DGdy, \]
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Monge 18??

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\[9/16\]
The first step

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9/16
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9/16
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\[ \omega^5 = dy \]

with \( G_r = 0, \ G_{pp} \neq 0 \) and \( D^3G = \triangle H \) it is always possible to force the lifted coframe

\[ \theta^1 = f_1 \omega^1, \theta^2 = f_2 \omega^1 + \rho e^\phi \omega^2 + f_4 \omega^3, \]
\[ \theta^3 = f_5 \omega^1 + f_6 \omega^2 + f_7 \omega^3, \theta^4 = \bar{f}_2 \omega^1 + \rho e^{-\phi} \omega^4 + \bar{f}_4 \omega^5, \theta^5 = \bar{f}_5 \omega^1 + \bar{f}_6 \omega^2 + \bar{f}_7 \omega^3 \]

to satisfy the following EDS:

\[ d\theta^1 = \Omega_1 \wedge \theta^1 + \theta^2 \wedge \theta^4, \]
\[ d\theta^2 = \theta^2 \wedge (\Omega_2 - \frac{1}{2} \Omega_1) - \theta^1 \wedge \Omega_3 + \theta^3 \wedge \theta^4, \]
\[ d\theta^3 = 2\theta^3 \wedge \Omega_2 - \theta^2 \wedge \Omega_3 + Q \theta^1 \wedge \theta^3 + \frac{1}{27\rho^3} A \theta^1 \wedge \theta^4 + \frac{e^{-\phi}}{3\rho} C \theta^2 \wedge \theta^3, \]
\[ d\theta^4 = -\theta^2 \wedge \theta^5 - \theta^4 \wedge (\frac{1}{2} \Omega_1 + \Omega_2) - \theta^1 \wedge \Omega_4, \]
\[ d\theta^5 = -2\theta^5 \wedge \Omega_2 + \theta^4 \wedge \Omega_2 + \frac{1}{27\rho^3} B \theta^1 \wedge \theta^2 + Q \theta^1 \wedge \theta^5 + \frac{e^\phi}{3\rho} \bar{C} \theta^4 \wedge \theta^5. \]

\[ A = (\frac{1}{2}) \left[ 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H^3_r + 54H_z \right], \quad \text{Wuenschmann } 1905 \]
\[ B = \left( \frac{1}{2G_{pp}} \right) \left[ 40G^3_{ppp} - 45G_{pp}G_{ppp}G_{pppp} + 9G^2_{pp}G_{ppppp} \right], \quad \text{Monge } 18?? \]
\[ C = \left( \frac{1}{G_{pp}} \right) \left[ 2G_{ppp} + G_{pp}H_{rr} \right], \quad ?? 201? \]

\( \bar{C} \) vanishes if \( C \equiv 0 \).

Moreover, **vanishing or not of each of** \( A, B \) or \( C \) **is an invariant property** of the corresponding para-CR structure.

**Remark:** We were unable to make normalizations such that the EDS describes a curvature of some Cartan connection. **BUT we did not tried hard.** See the end of the talk.

**Flat model:** \( A = B = C = 0 \), and this is locally equivalent to \( z_{xxx} = 0, \ z_y = \frac{1}{4} z_x^2 \). Symmetry algebra \( sp(4, \mathbb{R}) \simeq so(2, 3) \).
The first step

Theorem: Given a para-CR structure represented by the forms $\omega^1 = dz - pdx - Gdy$, $\omega^2 = dp - rdx - DGdy$, $\omega^3 = dr - Hdx - D^2Gdy$, $\omega^4 = dx$ and $\omega^5 = dy$ with $G_r = 0$, $G_{pp} \neq 0$ and $D^3G = \triangle H$ it is always possible to force the lifted coframe $\theta^1 = f_1\omega^1$, $\theta^2 = f_2\omega^1 + \rho e^\phi \omega^2 + f_4\omega^3$, $\theta^3 = f_5\omega^1 + f_6\omega^2 + f_7\omega^3$, $\theta^4 = \tilde{f}_2\omega^1 + \rho e^{-\phi} \omega^4 + \tilde{f}_4\omega^5$, $\theta^5 = \tilde{f}_5\omega^1 + \tilde{f}_6\omega^2 + \tilde{f}_7\omega^3$ to satisfy the following EDS:

$$
\begin{align*}
\theta^1 & = \Omega_1 \cap \theta^1 + \theta^2 \cap \theta^4, \\
\theta^2 & = \theta^2 \cap (\Omega_2 - \frac{1}{2}\Omega_1) - \theta^1 \cap \Omega_3 + \theta^3 \cap \theta^4, \\
\theta^3 & = 2\theta^3 \cap \Omega_2 - \theta^2 \cap \Omega_3 + Q\theta^1 \cap \theta^3 + \frac{e^3\phi}{27\rho^3} A \theta^1 \cap \theta^4 + \frac{e^{-\phi}}{3\rho} C \theta^2 \cap \theta^3, \\
\theta^4 & = - \theta^2 \cap \theta^5 - \theta^4 \cap (\frac{1}{2}\Omega_1 + \Omega_2) - \theta^1 \cap \Omega_4, \\
\theta^5 & = - 2\theta^5 \cap \Omega_2 + \theta^4 \cap \Omega_2 + \frac{e^3\phi}{27\rho^3} B \theta^1 \cap \theta^2 + Q\theta^1 \cap \theta^5 + \frac{e^\phi}{3\rho} \tilde{C} \theta^4 \cap \theta^5.
\end{align*}
$$

Here

$$
A = \left( - \frac{1}{2} \right) \left[ 9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H^3_r + 54H_2 \right], \quad \text{Wuenschmann 1905}
$$

$$
B = \left( \frac{1}{2G^3_{pp}} \right) \left[ 40G^3_{pp} - 45G_{pp}G_{ppp}G_{pppp} + 9G^2_{pp}G_{ppppp} \right], \quad \text{Monge 18??}
$$

$$
C = \left( \frac{1}{G_{pp}} \right) \left[ 2G_{pp} + G_{pp}Hrr \right], \quad ??? 201?
$$

\(\tilde{C}\) vanishes if \(C \equiv 0\).

Moreover, \textit{vanishing or not of each of} \(A\), \(B\) or \(C\) \textit{is an invariant property} of the corresponding para-CR structure.

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Flat model: \(A = B = C = 0\), and this is locally equivalent to \(z_{xxx} = 0\), \(z_y = \frac{1}{4}z_x^2\). Symmetry algebra \(sp(4, \mathbb{R}) \cong so(2, 3)\).
Method ?: Elie Cartan’s *reduction procedure* applied to the EDS from the last Theorem. It **required quite a gymnastics!**

Structure?:

Cartan’s reduction produces eventually the homogeneous models in terms of *Maurer-Cartan systems* for invariant forms on the maximal symmetry group of the model. We get:
In the case \( C \neq 0 \), we have 2 models, depending on this if \( \epsilon = 1 \) or \(-1\):

\[
d\theta^1 = \epsilon \left( -6 \theta^1 \wedge \theta^3 + \frac{1}{2} \theta^1 \wedge \theta^4 - \frac{3}{2} \theta^1 \wedge \theta^5 \right) + \theta^2 \wedge \theta^4,
\]

\[
d\theta^2 = \epsilon \left( -\frac{1}{16} \theta^1 \wedge \theta^2 - 2 \theta^2 \wedge \theta^3 + \frac{1}{2} \theta^2 \wedge \theta^4 - \theta^2 \wedge \theta^5 \right) - \theta^1 \wedge \theta^3 + \frac{1}{32} \theta^1 \wedge \theta^4 - \frac{1}{8} \theta^1 \wedge \theta^5 + \theta^3 \wedge \theta^4,
\]

\[
d\theta^3 = \epsilon \left( -\frac{3}{16} \theta^1 \wedge \theta^3 + \frac{1}{2} \theta^3 \wedge \theta^4 - \frac{1}{2} \theta^3 \wedge \theta^5 \right) + \frac{1}{32} \theta^2 \wedge \theta^4 - \frac{1}{8} \theta^2 \wedge \theta^5,
\]

\[
d\theta^4 = \epsilon \left( -\frac{1}{8} \theta^1 \wedge \theta^4 + \frac{1}{4} \theta^1 \wedge \theta^5 + 4 \theta^3 \wedge \theta^4 - \frac{1}{2} \theta^4 \wedge \theta^5 \right) - \theta^2 \wedge \theta^5,
\]

\[
d\theta^5 = \epsilon \left( -\frac{1}{16} \theta^1 \wedge \theta^5 + 2 \theta^3 \wedge \theta^5 - \frac{1}{4} \theta^4 \wedge \theta^5 \right).
\]

Symmetry algebra of dimension 5; unique homogeneous model.
In the case $\mathbf{C} \neq 0$, we have 2 models, depending on this if $\epsilon = 1$ or $-1$:

\[
\begin{align*}
\text{d}\theta^1 &= \epsilon ( -6 \theta^1 \wedge \theta^3 + \frac{1}{2} \theta^1 \wedge \theta^4 - \frac{3}{2} \theta^1 \wedge \theta^5 ) + \theta^2 \wedge \theta^4 , \\
\text{d}\theta^2 &= \epsilon ( -\frac{1}{16} \theta^1 \wedge \theta^2 - 2 \theta^2 \wedge \theta^3 + \frac{1}{2} \theta^2 \wedge \theta^4 - \theta^2 \wedge \theta^5 ) - \theta^1 \wedge \theta^3 + \\
&\quad + \frac{1}{32} \theta^1 \wedge \theta^4 - \frac{1}{8} \theta^1 \wedge \theta^5 + \theta^3 \wedge \theta^4 , \\
\text{d}\theta^3 &= \epsilon ( -\frac{3}{16} \theta^1 \wedge \theta^3 + \frac{1}{2} \theta^3 \wedge \theta^4 - \frac{1}{2} \theta^3 \wedge \theta^5 ) + \frac{1}{32} \theta^2 \wedge \theta^4 - \frac{1}{8} \theta^2 \wedge \theta^5 , \\
\text{d}\theta^4 &= \epsilon ( -\frac{1}{8} \theta^1 \wedge \theta^4 + \frac{1}{4} \theta^1 \wedge \theta^5 + 4 \theta^3 \wedge \theta^4 - \frac{1}{2} \theta^4 \wedge \theta^5 ) - \theta^2 \wedge \theta^5 , \\
\text{d}\theta^5 &= \epsilon ( -\frac{1}{16} \theta^1 \wedge \theta^5 + 2 \theta^3 \wedge \theta^5 - \frac{1}{4} \theta^4 \wedge \theta^5 ).
\end{align*}
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\[ d\theta^2 = \epsilon \left( -\frac{1}{16} \theta^1 \wedge \theta^2 - 2\theta^2 \wedge \theta^3 + \frac{1}{2} \theta^2 \wedge \theta^4 - \theta^2 \wedge \theta^5 \right) - \theta^1 \wedge \theta^3 + \frac{1}{32} \theta^1 \wedge \theta^4 - \frac{1}{8} \theta^1 \wedge \theta^5 + \theta^3 \wedge \theta^4, \]
\[ d\theta^3 = \epsilon \left( -\frac{3}{16} \theta^1 \wedge \theta^3 + \frac{1}{2} \theta^3 \wedge \theta^4 - \frac{1}{2} \theta^3 \wedge \theta^5 \right) + \frac{1}{32} \theta^2 \wedge \theta^4 - \frac{1}{8} \theta^2 \wedge \theta^5, \]
\[ d\theta^4 = \epsilon \left( -\frac{1}{8} \theta^1 \wedge \theta^4 + \frac{1}{4} \theta^1 \wedge \theta^5 + 4\theta^3 \wedge \theta^4 - \frac{1}{2} \theta^4 \wedge \theta^5 \right) - \theta^2 \wedge \theta^5, \]
\[ d\theta^5 = \epsilon \left( -\frac{1}{16} \theta^1 \wedge \theta^5 + 2\theta^3 \wedge \theta^5 - \frac{1}{4} \theta^4 \wedge \theta^5 \right). \]

Symmetry algebra of dimension 5; unique homogeneous model.
In the case \( C = 0 \) and \( B \neq 0 \), we have two 1-parameter families of nonequivalent homogeneous models, depending on this if \( \epsilon = 1 \) or \(-1\):

\[
\begin{align*}
\, \, \, d\theta^1 &= - \epsilon \left( \theta^1 \wedge \theta^3 + \theta^1 \wedge \theta^5 \right) + \theta^2 \wedge \theta^4, \\
\, \, \, d\theta^2 &= \epsilon \left( s \theta^1 \wedge \theta^2 - \theta^2 \wedge \theta^5 \right) - s \theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^4, \\
\, \, \, d\theta^3 &= \epsilon \left( \theta^1 \wedge \theta^4 - \theta^3 \wedge \theta^5 \right) - \theta^1 \wedge \theta^2 - s \theta^2 \wedge \theta^4, \\
\, \, \, d\theta^4 &= \epsilon \left( - s \theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^4 \right) + s \theta^1 \wedge \theta^2 - \theta^2 \wedge \theta^5, \\
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\end{align*}
\]

Here every \( s \in \mathbb{R} \) gives a model, and different \( s \) corresponds to the nonequivalent ones. Symmetry algebra of dimension 5.
In the case $C = 0$ and $B \neq 0$, we have two 1-parameter families of nonequivalent homogeneous models, depending on this if $\epsilon = 1$ or $-1$:

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\begin{align*}
\mathrm{d}\theta^1 &= -\epsilon\left(\theta^1 \wedge \theta^3 + \theta^1 \wedge \theta^5\right) + \theta^2 \wedge \theta^4, \\
\mathrm{d}\theta^2 &= \epsilon\left(s \theta^1 \wedge \theta^2 - \theta^2 \wedge \theta^5\right) - s \theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^4, \\
\mathrm{d}\theta^3 &= \epsilon\left(\theta^1 \wedge \theta^4 - \theta^3 \wedge \theta^5\right) - \theta^1 \wedge \theta^2 - s \theta^2 \wedge \theta^4, \\
\mathrm{d}\theta^4 &= \epsilon\left(- s \theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^4\right) + s \theta^1 \wedge \theta^2 - \theta^2 \wedge \theta^5, \\
\mathrm{d}\theta^5 &= \epsilon\left(- \theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^5\right) + \theta^1 \wedge \theta^2 + s \theta^2 \wedge \theta^4.
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d\theta^3 &= \epsilon \left( \theta^1 \wedge \theta^4 - \theta^3 \wedge \theta^5 \right) - \theta^1 \wedge \theta^2 - s \theta^2 \wedge \theta^4, \\
d\theta^4 &= \epsilon \left( -s \theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^4 \right) + s \theta^1 \wedge \theta^2 - \theta^2 \wedge \theta^5, \\
d\theta^5 &= \epsilon \left( -\theta^1 \wedge \theta^4 + \theta^3 \wedge \theta^5 \right) + \theta^1 \wedge \theta^2 + s \theta^2 \wedge \theta^4.
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Here every \( s \in \mathbb{R} \) gives a model, and different \( s \) corresponds to the nonequivalent ones. Symmetry algebra of dimension 5.
In the case $A = B = C = 0$, we have

\[
\begin{align*}
\text{d}\theta^1 &= \theta^2 \wedge \theta^4 - \theta^1 \wedge \Omega_1 \\
\text{d}\theta^2 &= \theta^3 \wedge \theta^4 + \theta^2 \wedge (\Omega_2 - \frac{1}{2} \Omega_1) - \theta^1 \wedge \Omega_3 \\
\text{d}\theta^3 &= 2\theta^3 \wedge \Omega_2 - \theta^2 \wedge \Omega_3 \\
\text{d}\theta^4 &= -\theta^2 \wedge \theta^5 - \theta^4 \wedge (\frac{1}{2} \Omega_1 + \Omega_2) - \theta^1 \wedge \Omega_4 \\
\text{d}\theta^5 &= -2\theta^5 \wedge \Omega_2 + \theta^4 \wedge \Omega_4 \\
\text{d}\Omega_1 &= -\theta^4 \wedge \Omega_3 + \theta^2 \wedge \Omega_4 - \theta^1 \wedge \Omega_5 \\
\text{d}\Omega_2 &= -\theta^3 \wedge \theta^5 - \frac{1}{2} \theta^4 \wedge \Omega_3 - \frac{1}{2} \theta^2 \wedge \Omega_4 \\
\text{d}\Omega_3 &= -(\frac{1}{2} \Omega_1 + \Omega_2) \wedge \Omega_3 + \theta^3 \wedge \Omega_4 - \frac{1}{2} \theta^2 \wedge \Omega_5 \\
\text{d}\Omega_4 &= (\Omega_2 - \frac{1}{2} \Omega_1) \wedge \Omega_4 + \theta^5 \wedge \Omega_3 - \frac{1}{2} \theta^4 \wedge \Omega_5 \\
\text{d}\Omega_5 &= -\Omega_1 \wedge \Omega_5 + 2\Omega_3 \wedge \Omega_4.
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\]

Symmetry algebra of dimension 10; unique model, $sp(4, \mathbb{R})$ symmetry.
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$$
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$$

Symmetry algebra of dimension 10; unique model, $sp(4, \mathbb{R})$ symmetry.
**Question:** Can these abstract systems be realized as PDEs \((S) - (IC) - (2NG)\)?

**Worry:** Mike Eastwood’s talk. When I got these 2 systems with exactly 5 symmetries and showed it to Joël he said: ‘you must have overlooked some models’. Every homogeneous affine surface gives rise to our para-CR by simply extending it as constant along 3-dimensions - ‘tube over an affine surface’. And looking at the classification of affine surfaces, which Mike showed us last week, one sees that one has (0) our flat model, (1) TWO single models and (2) TWO 1-parameter families. And these TWO are not related to our \(\epsilon\). Seems that Mike has more models than we have with Joël.
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Resolution of Joël's worries – homogeneous para-CR of Fels and Kaup

In their Acta Mathematica paper Fels and Kaup in 2008 classified all 5-dimensional homogeneous degenerate CR manifolds. If one looks at their para-CR version, with the degeneracy as in this report, one finds the following homogeneous models, which are 'tubes over the following affine surfaces'

- \( M = \{ \mathbb{R}^3 : xy + z^2 = 0 \} \). Our flat model \( z_y = \frac{1}{4} z_x^2, z_{xxx} = 0 \).

- \( M = \{ \mathbb{R}^3 : \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + r \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}, r, t \in \mathbb{R} \} \). Our single model \( z_y = \frac{1}{4} z_x^2, z_{xxx} = z_{xx} \).

Case 1 \( M_\alpha = \{ \mathbb{R}^3 : \begin{pmatrix} r \\ re_t \\ re_{\alpha t} \end{pmatrix}, r, t \in \mathbb{R} \}, \alpha > 2 \).

Case 2 \( M_\beta = \{ \mathbb{R}^3 : \begin{pmatrix} r \cos t \\ rt \\ re^t \end{pmatrix}, r, t \in \mathbb{R} \} \).

Case 3 \( M_\beta = \{ \mathbb{R}^3 : \begin{pmatrix} r \cos t \\ rt \\ re^t \end{pmatrix}, r, t \in \mathbb{R} \}, \beta > 0 \).

It turns out, that the surfaces given by cases 1, 2, 3 above are in one-to-one correspondence with OUR 1-PARAMETER FAMILY OF MODELS! Our real parametr \( s \) should be split as follows:

\[ s = \begin{cases} -\infty, -3(2)^{-5/3} & \cup \{ -3(2)^{-5/3} \} \cup \{ -3(2)^{-5/3}, \infty \} \\ -3(2)^{-5/3} \end{cases} \]

Case 1 corresponds to \( s < -3(2)^{-5/3} \) and \( z_y = \frac{1}{4} z_x^b, z_{xxx} = (2 - b) \frac{z_{xx}^2}{z_x}, 1 < b < 2 \); \( s = \frac{-3}{2} \left( 1 - b + b^2 \right) \left( (b - 2)(2b - 1) \right)^{2/3} \).

Case 2 corresponds to \( b = 1 \) above; \( s = -3(2)^{-5/3} \).

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Case 1 \( M_\alpha = \{ \mathbb{R}^3 : \begin{pmatrix} r \\ re^t \alpha \\ re^{t \alpha} \end{pmatrix}, \ r, t \in \mathbb{R} \}, \ \alpha > 2 \).

Case 2 \( M = \{ \mathbb{R}^3 : \begin{pmatrix} r \\ rt \\ re^{t} \end{pmatrix}, \ r, t \in \mathbb{R} \} \).

Case 3 \( M_\beta = \{ \mathbb{R}^3 : \begin{pmatrix} r \cos t \\ r \sin t \\ re^{\beta t} \end{pmatrix}, \ r, t \in \mathbb{R} \}, \ \beta > 0 \).

It turns out, that the surfaces given by cases 1, 2, 3 above are in one-to-one correspondence with OUR 1-PARAMETER FAMILY OF MODELS! Our real parameter \( s \) should be split as follows

\[ ] - \infty, -3(2)^{-5/3} [ \cup \{-3(2)^{-5/3}\} \cup ] - 3(2)^{-5/3}, \infty[. \]

Case 1 corresponds to \( s < -3(2)^{-5/3} \) and \( z_y = \frac{1}{4}z_x^b, z_{xxx} = (2 - b)\frac{z_{xx}^2}{z_x}, 1 < b < 2; \ s = -\frac{3}{2} \left( \frac{1-b+b^2}{(b-2)(2b-1)} \right)^{2/3} \).

Case 2 corresponds to \( b = 1 \) above; \( s = -3(2)^{-5/3} \).

Case 3 corresponds to \( s > -3(2)^{-5/3} \), and \( z_y = f(z_x), z_{xxx} = h(z_x)z_{xx}^2 \), where functions \( f \) and \( h \) are given by:

\[
(z_x^2 + f(z_x)^2) \exp \left( 2 \arctan \frac{c z_x - f(z_x)}{z_x + c f(z_x)} \right) = 1 + c^2, \ h(z_x) = \frac{(c^2 - 3)z_x - 4cf(z_x)}{(f(z_x) - cz_x)^2}; \ s = -\frac{3}{2} \left( \frac{c^2 - 3}{2(c(9+c^2))^2} \right), \ c > 0.
\]
In their Acta Mathematica paper Fels and Kaup in 2008 classified all 5-dimensional homogeneous degenerate CR manifolds. If one looks at their para-CR version, with the degeneracy as in this report, one finds the following homogeneous models, which are ‘tubes over the following affine surfaces’

\[ M = \{ \mathbb{R}^3 : xy + z^2 = 0 \} \text{. Our flat model } z_y = \frac{1}{4} z_x^2, z_{xxx} = 0. \]

\[ M = \{ \mathbb{R}^3 : \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + r \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}, r, t \in \mathbb{R} \} \text{. Our single model } z_y = \frac{1}{4} z_x^2, z_{xxx} = z_{xx}. \]

Case 1 \( M_\alpha = \{ \mathbb{R}^3 : \begin{pmatrix} r \\ r e^t \\ r e^{\alpha t} \end{pmatrix}, r, t \in \mathbb{R} \}, \alpha > 2. \)

Case 2 \( M = \{ \mathbb{R}^3 : \begin{pmatrix} r \\ r t \\ r e^t \end{pmatrix}, r, t \in \mathbb{R} \}. \)

Case 3 \( M_\beta = \{ \mathbb{R}^3 : \begin{pmatrix} r \cos t \\ r \sin t \\ r e^{\beta t} \end{pmatrix}, r, t \in \mathbb{R} \}, \beta > 0. \)

It turns out, that the surfaces given by cases 1, 2, 3 above are in one-to-one correspondence with OUR 1-PARAMETER FAMILY OF MODELS! . Our real parameter \( s \) should be split as follows

\[ ] - \infty, -3(2)^{-5/3} \cap \{ -3(2)^{-5/3} \} \cup \{ -3(2)^{-5/3} \} \cup [ ] - 3(2)^{-5/3}, \infty [ . \]

Case 1 corresponds to \( s < -3(2)^{-5/3} \) and \( z_y = \frac{1}{4} z_x^b, z_{xxx} = (2 - b) \frac{z_{xx}^2}{z_x}, 1 < b < 2; s = \frac{-\frac{3}{2} (1 - b + b^2)}{((b-2)(2b-1))^2}. \)

Case 2 corresponds to \( b = 1 \) above; \( s = -3(2)^{-5/3}. \)

Case 3 corresponds to \( s > -3(2)^{-5/3} \), and \( z_y = f(z_x), z_{xxx} = h(z_x) z_{xx}^2 \), where functions \( f \) and \( h \) are given by:

\[ (z_x^2 + f(z_x)^2) \exp \left( 2 \text{arctan} \frac{cz_x - f(z_x)}{z_x + cf(z_x)} \right) = 1 + c^2, h(z_x) = \frac{(c^2 - 3)z_x - 4cf(z_x)}{(f(z_x) - cz_x)^2}; s = \frac{-\frac{3}{2} (c^2 - 3)}{(2c(9+c^2))^2}, c > 0. \]
THANK YOU!