Homogeneous models for 5-dimensional para-CR manifolds with Levi form degenerate in one direction

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- More specifically one considers

$$M_{2n-1} = \{\mathbb{R}^n \times \mathbb{R}^n \ni (x, \bar{x}) \mid \Phi(x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n) = 0\}$$

  modulo (local) diffeomorphisms $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$preserving the split of $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.
- The lowest dimension, $n = 2$. Such para-CR structures (if nondegenerate) are related to 2nd-order ODEs, considered modulo point transformations of variables. Sort of dull.
- Today, the next dimension, $n = 3$. 5-dimensional para-CR structures.
- A 5-dim para-CR structure can be defined as a graph of a function z of five variables, $z = z(x, y, \bar{x}, \bar{y}, \bar{z})$, where $(x, y, z, \bar{x}, \bar{y}, \bar{z})$ are coordinates in $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$.
- This in turn, can be considered as a *general solution* of a system of two PDEs for a function $z = z(x, y)$ on the plane (x, y), in which $(\bar{x}, \bar{y}, \bar{z})$ are constants of integration and parametrize the solution space. As in the following example.

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Example:

- Take $(x-\bar{x})^2+(y-\bar{y})(z-\bar{z})=0$, and solve it for z obtaining: $z=-\frac{(x-\bar{x})^2}{y-\bar{y}}+\bar{z}$. Now think about (x,y) as independent variables, and $(\bar{x},\bar{y},\bar{z})$ as parameters. Obviously $\boxed{z_{xxx}=0}$. Also, because $z_y=\frac{(x-\bar{x})^2}{(y-\bar{y})^2}$ and $z_x=\frac{-2(x-\bar{x})}{(y-\bar{y})}$ we have $\boxed{z_y=\frac14 z_x^2}$. So, a para-CR structure defined by the cone $(x-\bar{x})^2+(y-\bar{y})(z-\bar{z})=0$ in $\mathbb{R}^3\times\mathbb{R}^3$ defines a system of PDEs on the plane

$$\boxed{z_{xxx}=0\qquad\&\qquad z_y=\frac14 z_x^2}\qquad\text{for}\qquad z=z(x,y).$$

- Conversely, $z_{xxx}=0$ solves as $\boxed{z=\alpha(y)x^2+\beta(y)x+\gamma(y)}$, and $z_y=\frac14 z_x^2$ gives sucessively: $\alpha'=\alpha^2$, hence $\boxed{\alpha=\frac{-1}{y-\bar{y}}}$, $\beta'=\frac{-\beta}{y-\bar{y}}$, hence $\boxed{\beta=\frac{2\bar{x}}{y-\bar{y}}}$, $\gamma'=\frac{\bar{x}^2}{(y-\bar{y})^2}$, hence $\boxed{\gamma=\frac{-\bar{x}^2}{y-\bar{y}}+\bar{z}}$. This finally gives $z=\frac{-\bar{x}^2}{y-\bar{y}}+\bar{z}+\frac{2x\bar{x}}{y-\bar{y}}-\frac{x^2}{y-\bar{y}}$, i.e. the cone

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- In general, we consider the following system of PDEs on the plane

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- Fact: The general solution of (S) depends on 3 parameters $(\bar{x}, \bar{y}, \bar{z})$, and has the form $z = z(x, y; \bar{x}, \bar{y}, \bar{z})$ if and only if

$$(IC) \qquad \boxed{\triangle H = D^3 G}$$

  where $D = \partial_x + p\partial_z + r\partial_p + H\partial_r$, $\triangle = \partial_y + G\partial_z + DG\partial_p + D^2 G\partial_r$, and we have introduced $p = z_x$, $r = z_{xx}$.
- General solutions of systems (S) give examples of 5-dim para-CR structures. We prefer the PDE point of view, and we will stick to this in the following. In particular, in this point of view, para-CR transformations for hypersurfaces in $(x, y, z, \bar{x}, \bar{y}, \bar{z})$ are *point transformations of variables* of (S).

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- Fact: The general solution of (S) depends on 3 parameters $(\bar{x}, \bar{y}, \bar{z})$, and has the form $z = z(x, y; \bar{x}, \bar{y}, \bar{z})$ if and only if

$$(IC) \qquad \boxed{\triangle H = D^3 G}$$

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$$\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix} \mapsto \begin{pmatrix} f_1 & 0 & 0 & 0 & 0 \\ f_2 & \rho e^{\phi} & f_4 & 0 & 0 \\ f_5 & f_6 & f_7 & 0 & 0 \\ \bar{f}_2 & 0 & 0 & \rho e^{-\phi} & \bar{f}_4 \\ \bar{f}_5 & 0 & 0 & \bar{f}_6 & \bar{f}_7 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix}.$$

- We study systems of PDEs on the plane

$$(S) \qquad \boxed{z_{xxx} = H(x,y,z,p,r) \qquad \& \qquad z_y = G(x,y,z,p)} \qquad \text{for} \qquad z(x,y)$$

such that

$$(IC) \qquad \boxed{\triangle H = D^3 G} \qquad \& \qquad (2NG) \qquad \boxed{G_{pp} \neq 0},$$

with $D = \partial_x + p\partial_z + r\partial_p + H\partial_r$, $\triangle = \partial_y + G\partial_z + DG\partial_p + D^2G\partial_r$, and $p = z_x$, $r = z_{xx}$, considered modulo *point transformations of variables*.

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- Symmetries: A vector field X on $M_5 \ni (x, y, z, p, r)$ is a *symmetry* of the para-CR structure as defined in $(S) - (IC) - (2NG)$ if and only if

$$
\begin{aligned}
&(\mathcal{L}_X \omega^1) \wedge \omega^1 = 0, && \\
&(\mathcal{L}_X \omega^2) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, && (\mathcal{L}_X \omega^4) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0, \\
&(\mathcal{L}_X \omega^3) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = 0, && (\mathcal{L}_X \omega^5) \wedge \omega^1 \wedge \omega^4 \wedge \omega^5 = 0.
\end{aligned}
$$

- Goal: Find all homogeneous models, i.e. find the PDEs $(S) - (IC) - (2NG)$ such that their corresponding para-CR structures have *at least five* symmetries X_1, X_2, X_3, X_4 and X_5 such that $X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \neq 0$.
- Method: Cartan's equivalence and reduction methods (part of SCREAM; if you know what I mean ;)).

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$$(\mathcal{L}_X\omega^3)\wedge\omega^1\wedge\omega^2\wedge\omega^3 = 0, \qquad (\mathcal{L}_X\omega^5)\wedge\omega^1\wedge\omega^4\wedge\omega^5 = 0.$$

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&(\mathcal{L}_X\omega^3)\wedge\omega^1\wedge\omega^2\wedge\omega^3 = 0, \qquad && (\mathcal{L}_X\omega^5)\wedge\omega^1\wedge\omega^4\wedge\omega^5 = 0.
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$$(\mathcal{L}_X\omega^3)\wedge\omega^1\wedge\omega^2\wedge\omega^3 = 0, \qquad (\mathcal{L}_X\omega^5)\wedge\omega^1\wedge\omega^4\wedge\omega^5 = 0.$$

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$$(\mathcal{L}_X\omega^2)\wedge\omega^1\wedge\omega^2\wedge\omega^3 = 0, \qquad (\mathcal{L}_X\omega^4)\wedge\omega^1\wedge\omega^4\wedge\omega^5 = 0,$$
$$(\mathcal{L}_X\omega^3)\wedge\omega^1\wedge\omega^2\wedge\omega^3 = 0, \qquad (\mathcal{L}_X\omega^5)\wedge\omega^1\wedge\omega^4\wedge\omega^5 = 0.$$

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- Symmetries: A vector field X on $M_5 \ni (x, y, z, p, r)$ is a *symmetry* of the para-CR structure as defined in $(S)-(IC)-(2NG)$ if and only if

$$(\mathcal{L}_X\omega^1)\wedge\omega^1 = 0,$$

$$(\mathcal{L}_X\omega^2)\wedge\omega^1\wedge\omega^2\wedge\omega^3 = 0, \qquad (\mathcal{L}_X\omega^4)\wedge\omega^1\wedge\omega^4\wedge\omega^5 = 0,$$

$$(\mathcal{L}_X\omega^3)\wedge\omega^1\wedge\omega^2\wedge\omega^3 = 0, \qquad (\mathcal{L}_X\omega^5)\wedge\omega^1\wedge\omega^4\wedge\omega^5 = 0.$$

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- Method: Cartan's equivalence and reduction methods (part of SCREAM; if you know what I mean ;)).

- Theorem: Given a para-CR structure represented by the forms $\omega^1 = \mathrm{d}z - p\mathrm{d}x - G\mathrm{d}y$, $\omega^2 = \mathrm{d}p - r\mathrm{d}x - DG\mathrm{d}y$, $\omega^3 = \mathrm{d}r - H\mathrm{d}x - D^2G\mathrm{d}y$, $\omega^4 = \mathrm{d}x$ and $\omega^5 = \mathrm{d}y$ with $G_r = 0$, $G_{pp} \neq 0$ and $D^3G = \triangle H$ it is always possible to force the lifted coframe $\theta^1 = f_1\omega^1$, $\theta^2 = f_2\omega^1 + \rho e^{\phi}\omega^2 + f_4\omega^3$, $\theta^3 = f_5\omega^1 + f_6\omega^2 + f_7\omega^3$, $\theta^4 = \bar{f}_2\omega^1 + \rho e^{-\phi}\omega^4 + \bar{f}_4\omega^5$, $\theta^5 = \bar{f}_5\omega^1 + \bar{f}_6\omega^2 + \bar{f}_7\omega^3$ to satisfy the following EDS:

$$
\begin{aligned}
\mathrm{d}\theta^1 &= \Omega_1 \wedge \theta^1 + \theta^2 \wedge \theta^4,\\
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\mathrm{d}\theta^3 &= 2\theta^3 \wedge \Omega_2 - \theta^2 \wedge \Omega_3 + Q\theta^1 \wedge \theta^3 + \frac{e^{3\phi}}{27\rho^3}\mathbf{A}\,\theta^1 \wedge \theta^4 + \frac{e^{-\phi}}{3\rho}\mathbf{C}\,\theta^2 \wedge \theta^3,\\
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$\mathbf{A} = (-\tfrac{1}{2})\,[\,9D^2H_r - 27DH_p - 18H_rDH_r + 18H_pH_r + 4H_r^3 + 54H_z\,]$, *Wuenschmann* 1905

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- Theorem: Given a para-CR structure represented by the forms $\omega^1 = \mathrm{d}z - p\mathrm{d}x - G\mathrm{d}y$, $\omega^2 = \mathrm{d}p - r\mathrm{d}x - DG\mathrm{d}y$, $\omega^3 = \mathrm{d}r - H\mathrm{d}x - D^2G\mathrm{d}y$, $\omega^4 = \mathrm{d}x$ and $\omega^5 = \mathrm{d}y$ with $G_r = 0$, $G_{pp} \neq 0$ and $D^3G = \triangle H$ it is always possible to force the lifted coframe $\theta^1 = f_1\omega^1$, $\theta^2 = f_2\omega^1 + \rho e^{\phi}\omega^2 + f_4\omega^3$, $\theta^3 = f_5\omega^1 + f_6\omega^2 + f_7\omega^3$, $\theta^4 = \bar{f}_2\omega^1 + \rho e^{-\phi}\omega^4 + \bar{f}_4\omega^5$, $\theta^5 = \bar{f}_5\omega^1 + \bar{f}_6\omega^2 + \bar{f}_7\omega^3$ to satisfy the following EDS:

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\begin{aligned}
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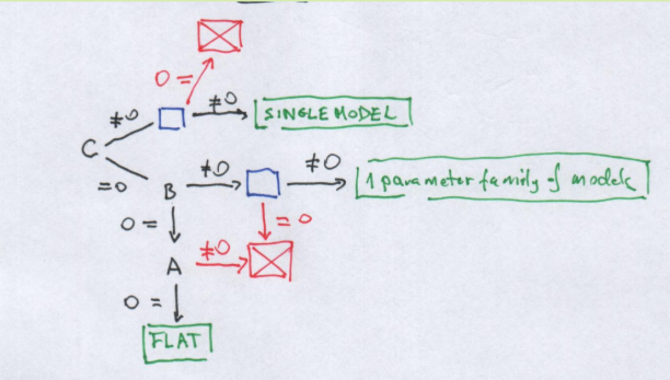
- Method ?: Elie Cartan's *reduction procedure* applied to the EDS from the last Theorem. It required quite a gymnastics!
- Structure?:



- Cartan's reduction produces eventually the homogeneous models in terms of *Maurer-Cartan systems* for invariant forms on the maximal symmetry group of the model. We get:

In the case **C** $\neq 0$, we have 2 models, depending on this if $\epsilon = 1$ or -1:

$$
\begin{aligned}
d\theta^1 =& \epsilon\Big(-6\theta^1\wedge\theta^3 + \tfrac{1}{2}\theta^1\wedge\theta^4 - \tfrac{3}{2}\theta^1\wedge\theta^5\Big) + \theta^2\wedge\theta^4,\\
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& \tfrac{1}{32}\theta^1\wedge\theta^4 - \tfrac{1}{8}\theta^1\wedge\theta^5 + \theta^3\wedge\theta^4,\\
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Symmetry algebra of dimension 5; unique homogeneous model.

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- **Question:** Can these abstract systems be realized as PDEs $(S) - (IC) - (2NG)$?
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In their Acta Mathematica paper Fels and Kaup in 2008 classified all 5-dimensional homogeneous degenerate CR manifolds. If one looks at their para-CR version, with the degeneracy as in this report, one finds the following homogeneous models, which are 'tubes over the following affine surfaces'

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$$\left(z_x^2 + f(z_x)^2\right)\exp\left(2c\arctan\frac{cz_x - f(z_x)}{z_x + cf(z_x)}\right) = 1 + c^2, \ h(z_x) = \frac{(c^2-3)z_x - 4cf(z_x)}{(f(z_x) - cz_x)^2}; \ s = \frac{-\frac{3}{2}(c^2-3)}{(2c(9+c^2))^{\frac{2}{3}}}, \ c > 0.$$

In their Acta Mathematica paper Fels and Kaup in 2008 classified all 5-dimensional homogeneous degenerate CR manifolds. If one looks at their para-CR version, with the degeneracy as in this report, one finds the following homogeneous models, which are 'tubes over the following affine surfaces'

- $M = \{\mathbb{R}^3 \ : \ xy + z^2 = 0\}$. Our flat model $z_y = \frac{1}{4} z_x^2$, $z_{xxx} = 0$.
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Case 1 $\quad M_\alpha = \{\mathbb{R}^3 \ : \ \begin{pmatrix} r \\ re^t \\ re^{\alpha t} \end{pmatrix}, r, t \in \mathbb{R}\}, \alpha > 2.$

Case 2 $\quad M = \{\mathbb{R}^3 \ : , \begin{pmatrix} r \\ rt \\ re^t \end{pmatrix} \ r, t \in \mathbb{R}\}.$

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It turns out, that the surfaces given by cases 1, 2, 3 above are in one-to-one correspondence with OUR 1-PARAMETER FAMILY OF MODELS! . Our real parametr s should be split as follows
$]-\infty, -3(2)^{-5/3}[\ \cup \ \{-3(2)^{-5/3}\} \ \cup \ ]-3(2)^{-5/3}, \infty[.$

Case 1 corresponds to $s < -3(2)^{-5/3}$ and $z_y = \frac{1}{4} z_{xx}^b$, $z_{xxx} = (2-b)\frac{z_{xx}^2}{z_x}$, $1 < b < 2$; $s = \frac{-\frac{3}{2}\left(1-b+b^2\right)}{\left((b-2)(2b-1)\right)^{\frac{2}{3}}}$.

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THANK YOU!