STUDYING GEOMETRIC PDEs

or

Mathematical Nobel 2019, Fields Medal 1986, and .... Physics

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Geometry and Diff. Equations Seminar IMPAN, 3 June 2020

- Last year the Abel Prize ("Mathematical Nobel") was awarded to Karen Keskulla Uhlenbeck, for the first time to a woman (the prize is being awarded since 2003).
- In 1986 Simon K. Donaldson received Fields Medal when he was 29.
- In 1954 Cheng Ning Yang and Robert Mills published a 5 pages article which was a key step initializing the construction of a mathematical model of "Mendeleyev table" of elementary particles, called Standard Model.

The main aim of this talk is a soft description of results of K. Uhlenbeck.

I will also try to explain why the above events are closely related.

The crucial results of Uhlenbeck (1982) and Donaldson (1983) were presented at the International Congress of Mathematicians in Warsaw in August 1983.

I attended the Congress but ... missed both lectures.

Abel Prize 2019 for Karen Keskulla Uhlenbeck

Jury announcement: "Karen Uhlenbeck receives the Abel Prize 2019 for her fundamental work in geometric analysis and gauge theory, which has dramatically changed the mathematical landscape. ..."

Also, "for her pioneering achievements in geometric partial differential equations, gauge theory and integrable systems, and for the fundamental impact of her work on analysis, geometry and mathematical physics."

Awarding: Karen Uhlenbeck received the Abel Prize from H.M. King Harald V

Geometric differential equations

Geometric differential equations are equations arising in geometric problems like:

- finding shortest/longest curves in spaces with a metric (e.g. Riemannian manifolds) - geodesics;
- finding surfaces of minimal (locally minimal) area in such spaces - minimal surfaces;
- finding maps between Riemannian manifolds having minimal "energy" - harmonic maps;
- finding fields extremizing action functionals (Lagrangian approach in physics) - Einstein equations, Yang-Mills fields...

Geometric differential equations as Euler-Lagrange eqns

Most of geometric differential equations are Euler-Lagrange equations for integral functionals.
The most elementary are equations for locally shortest curves $\gamma : t \mapsto x(t)$ on a Riemannian manifold M, called geodesics. They are extremals of the length functional

$$\ell(\gamma) = \int_{t_0}^{t_1} \|\dot{x}(t)\| dt,$$

where $\|\dot{x}(t)\| = (g(\dot{x}, \dot{x}))^{1/2}$ is the length, with respect to the metric g on M, of the tangent vector to γ at $x(t)$.
The same curves, having in addition a unique length parametrization, are obtained when we replace $L(\cdot)$ by the energy functional

$$E(\gamma) = \frac{1}{2} \int_{t_0}^{t_1} \|\dot{x}(t)\|^2 dt.$$

Equations for geodesics

The equations for geodesics, defined as local energy minimizers, can also be written with the use of the (Levi-Civita) connection ∇ associated to the metric g. They say that the acceleration (the covariant derivative of the velocity along the geodesic) should vanish:

$$\nabla_{\dot{x}(t)} \dot{x}(t) = 0.$$

In a coordinate system,

$$\ddot{x}^k = \sum_{i,j} \Gamma^k_{ij}(x) \dot{x}^i \dot{x}^j$$

where Γ^k_{ij} are Christoffel symbols of a connection (Levi-Civita connection) defined by the metric g.

Uhlenbeck's fundamental results

- Regularity and singularities of minimal surfaces (with J. Sacks, 1981)
- Existence and regularity of harmonic mappings between Riemannian manifolds (paper with Richard Schoen, 1982)
- "Donaldson-Uhlenbeck-Yau Theorem" - theorem on existence of Hermite-Einstein connection in stable holomorphic vector bundles on compact Kähler manifolds.

- "Existence of Coulomb gauges" - existence of gauges which kill connection divergence
- "Uhlenbeck compactification" - thm on compactness of the space of connections with bounded curvature
- Removal of singularities in Yang-Mills fields

• Last three results were published in 1982 and presented at International Congress of Mathematicians in Warsaw in 1983.

Harmonic maps are generalizations of minimal surfaces.

Let (M, g) and (N, g) be Riemannian manifolds with Riemannian metrics g on M and h on N, with M compact. For a C^1 map $f : M \to N$ we define the energy of f

$$E(f) = \int_M e(f) d\mu_g,$$

where μ_g is the measure on M defined by the metric $g = (g_{ij})$.
Above, the energy density $e(f)$ is the square of the norm of the differential df:

$$e(f)(x) = \frac{1}{2} \|df(x)\|^2.$$

Definition. A map $f : M \to N$ is called harmonic if it is a critical point of the energy functional $E(f)$.
(If $N \subset \mathbb{R}^k$ then one assumes $f \in W^{1,2}(M, \mathbb{R}^k) \cap C^0(M, N)$, where $W^{1,2}$ - Sobolev space.)

Minimal surfaces may also be defined as images of maps $f : M \to N$ with $\dim M = 2$ which locally minimize the energy functional.
In particular, if f is a local minimum of the energy E then it is harmonic.

The following results became classic:

Thm on regularity and singularities of harmonic maps.
An energy-minimizing map $f : M \to N$ is smooth away from a closed subset $S \subset M$ of its singular points .
The set S is discrete in M, if $\dim M = 3$,
and has Hausdorff dimension $\leqslant n - 3$, if $n \geqslant 4$.

The energy functionals used in the theorem was the sum of the energy $E(f)$ and the integral of lower order terms.

A technical results in this paper, needed for the proof of the main result and used later by many authors, was

Small energy regularity theorem. Let $B(n, r)$ denote the ball of radius r in $\mathbb{R}^n$. There are constants $\varepsilon > 0$ and $C > 0$ such that if $f : B(n, 1) \to N$ is energy minimizing such that

$$\int_{B(n,1)} \|\nabla f\|^2 dx \leqslant \varepsilon$$

then

$$\sup_{B(n,1/2)} \|\nabla f\|^2 \leqslant C \int_{B(n,1)} \|\nabla f\|^2 dx.$$

A comment in the paper with Schoen:

"**Our methods work** for functional which are the energy plus lower order terms, and thus **have direct bearing on the question of the existence of global Coulomb gauges in nonabelian gauge theories.**"

- In 1954 C.N. Yang and R. Mills proposed a model for strong interactions in atomic nuclei. Extending a "geometric" approach to quantization of electromagnetism they proposed a similar method where the abelian group $U(1)$ in electromagnetism was replaced with the nonabelian $U(2)$.
- Around 1970 C.N. Yang got acquainted with James Simons (Chair of Math. Dept. at Stony Brook) and, in the course of conversations, they realized that there was a resemblance of the objects in his paper with Mills and some mathematical constructions recognized by Simons as connections on fiber bundles. Then Yang and Wu published a paper which gave a dictionary between the physical and mathematical terms.
- The results were communicated to I. Singer (MIT) and to M. Atiyah who, with other top mathematicians, started to study solutions to Yang-Mills equations and their relation to topology of the underlying manifold.
- At the beginning of 80-ties groundbreaking results were obtained by S. Donaldson, a PhD student of Atiyah. He used solutions of Yang-Mills PDEs for constructing invariants of 4-manifolds.
- Main technical background was provided by results of K. Uhlenbeck and C.H. Taubes.

Let E - a vector bundle over a manifold M^n, $U \times F$- its local trivialization where $U \subset M$ and fiber $F \simeq \mathbb{R}^r$. Consider a section $\Phi : U \to F = \mathbb{R}^r$.

A connection (covariant derivative) $D = d + A$ on the bundle E is a differential operator acting on sections $\Phi : M \to E$ by

$$D\Phi = d\Phi + A\Phi$$

where $d\Phi$, in a given local basis in E, is de Rham differential of Φ and $A = (A^i_j)$ a $r \times r$ matrix with coefficients A^i_j being differential 1-forms on M.

Curvature F of the connection is a matrix with coefficients being differential 2-forms on M,

$$F = dA + A \wedge A \quad (= D \circ D).$$

Fundamental problem: having the curvature F, find (if exist) connection matrices A "continuously depending on F", satisfying the equation

$$dA + A \wedge A = F.$$

Three difficulties: 1. This is a (system of) nonlinear equation(s).
2. The linear part is not elliptic.
3. The regularity of A may be spoiled by a non-regular gauge.

A remedy found by Uhlenbeck: Use suitable gauge ("linear coordinates on E").

A gauge transformation is a linear invertible transformation U on sections of E, depending on points $x \in M$.
If $U(x) : E_x \to E_x$ is such transformation then the connection $D = d + A$ is transformed to $D' = d + A'$, where

$$A' = U^{-1} dU + U^{-1} A U.$$

Idea: Find a gauge transformation U, called Coulomb gauge, such that

$$d^* A' = 0 \quad (\text{equivalently, } \operatorname{div} A' = 0)$$

where $\cdot^*$ denotes Hodge star conjugate operator to d.

In new "coordinates" equations for A are

$$dA = F - A \wedge A, \qquad d^* A = 0.$$

An advantage of such equations: the operator $d \oplus d^*$ is elliptic.
The nonlinearity $A \wedge A$ can be neutralized if A is small (known phenomenon in PDEs).

In the next two theorems we assume: $G \subset SO(r)$ is a compact Lie group, E is a G-vector bundle with inner product in the fibers preserved by G.

Theorem 1 (Uhlenbeck, Comm. Math. Phys. 83, 1982)
Let E - a trivial G-vector bundle over a closed ball $B \subset R^n$. There exists a constant

$$\varepsilon = \varepsilon(n, G) > 0$$

such that if a connection $D = d + A$, $A \in W^{1,n/2}(B)$ (Sobolev space) satisfies the inequality

$$\int_B \|F(A)\|^{n/2} dx < \varepsilon$$

then there exists a gauge transformation U which transforms A to A' such that

$$dA' + A' \wedge A' = F', \quad d^* A' = 0.$$

Additionally,

$$\int_B \|\mathrm{grad}\, A'\|^{n/2} dx \leqslant K \int_B \|F(A')\|^{n/2} dx,$$

where $K = K(n, G) > 0$. If a suplementary condition on a radial component of A' is imposed then A' is unique up to constant gauge transformation.

Theorem 2 (Uhlenbeck, Comm. Math. Phys. 83, 1982)
If (M, g) is a compact oriented Riemann manifold then the space of connection matrices $A \in W^{1,p}(M)$ on a principal G-bundle (G compact) satisfying

$$\int_B \|F(A)\|^p d\mu_g \leqslant C, \qquad p > n/2,$$

is weakly compact.

Precisely, the result says that any sequence of connections $A_i \in W^{1,p}(M)$ with uniform curvature bound as above has a subsequence which, after gauge transformations, converges weakly to a connection $A_\infty \in W^{1,p}(M)$.
A consequence of this is that when A_i are assumed to satisfy Yang-Mills equations then, additionally, A_∞ is smooth and the convergence is C^∞.

These results give basic tools for studying problems related to gauge fields and gauge transformations, in particular problems of classifications of gauge fields with respect to the group of gauge transformations.

In literature results of this type are called:

- „Existence of Coulomb gauge" or „gauge fixing theorems",
- „Uhlenbeck's compactness theorem",
- „Uhlenbeck and Donaldson-Uhlenbeck compactification".

Removable singularities in Yang-Mills fields, Comm. Math. Phys. 83, 1982

(M, g) - Riemannian manifold; μ_g - measure on M induced by the metric g;

$D = d + A$ - a metric connection on a vector bundle with a compact structure group G and fiber L-Lie algebra of G (and the adjoint action of G on L).

The Euler-Lagrange equations for the action functional (energy)

$$S(A) = \int_M \|F(A)\|^2 d\mu_g,$$

are called Yang-Mills equations and can be written as

$$D^* F(A) = 0,$$

where D^* is the adjoint operator to D.

Main results of the paper:

- A field satisfying the Yang-Mills equations in dimension 4 with a point singularity is gauge equivalent to a smooth field if $S(A) < \infty$.
- Additionally, every Yang-Mills field over $\mathbb{R}^4$ with bounded $S(A)$ may be obtained from a field on $S^4 = R^4 \cup \{\infty\}$.
- Coulomb gauges can be constructed for general small fields in arbitrary dimensions including 4.

A revolution in topology of 4-manifolds

- with the use of Uhlenbeck's results

- A breakthrough of S. Donaldson:
  Solutions of Yang-Mills equations can be used for understanding the topology of 4-manifolds.
- The results of Karen Uhlenbeck provided new toolkit for achieving this breakthrough.

The results of the two papers of K. Uhlenbeck, together with Taubes' results in J.Diff. Geom.1982, were crucial for the proof of the Donaldson's results on 4-manifolds (J. Diff. Geom. 1983).

In particular, the compactness of the space of moduli was crucial for defining Donaldson's invariants by integration on a compact space of moduli (Gregory L. Naber, Springer 2011 book, p. 355).

Donaldson's results

In 1981 Michael Freedman used the intersection form to state the following result on topological 4-manifolds. Given any unimodular (i.e., with isomorphic map $v \to Q(v, \cdot)$) symmetric bilinear form Q over the integers, there is a simply connected closed topological 4-manifold M with intersection form Q.

Simon K. Donaldson proved in 1982 (as a postgraduate student) the following result, published in 1983.

If a closed, simply connected topological 4-manifold admits a differentiable structure then its intersection form is diagonalizable (over transformations with integer coefficients). In consequence many topological manifolds do not admit differential structures.

In the words of Atiyah, his result "stunned the mathematical world". Donaldson received Fields Medal in 1986 at the age of 29.

Recall: the intersection form $Q : H^2(M) \times H^2(M) \to \mathbb{Z}$, $Q(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$. If M is differentiable and compact then, with α, β closed 2-forms, then

$$Q(\alpha, \beta) = \int_M \alpha \wedge \beta.$$

Donaldson's results II

In the proof Donaldson used instantons, a particular class of solution to the equations of Yang–Mills gauge theory. Specifically, he constructed invariants of the differentiable structure on the manifold using the moduli space of the instanton solutions under the action of the group of gauge transformations.

Donaldson also derived polynomial invariants from gauge theory. These were new topological invariants sensitive to the underlying smooth structure of the four-manifold. They made it possible to deduce the existence of "exotic" smooth structures. Certain topological four-manifolds could carry an infinite family of different smooth structures.

In particular, it was discovered that the usual space R^4, treated as a topological space, may admit nonequivalent "exotic" smooth structures in addition to the usual one.

This is described in a book of K. Uhlenbeck and D. Freed, "Instanton and Four Manifolds" (1984).

Instantons and moduli spaces

A particular class of solutions of the second order Yang-Mills equations

$$D^* F(A) = 0 \qquad (YM)$$

are instantons, solutions of one of the two first order equations

$$\star F(A) = F(A), \quad \text{or} \quad \star F(A) = -F(A), \qquad (M)$$

where "$\star$" denotes the Hodge (algebraic) operator.
Solutions of them are called self-dual and anti-self-dual instantons, respectively.

Since D^* is the composition $D^* = \star D \star$, using the Bianchi identity $DF(A) = 0$ one sees that instantons satisfy (YM).

Given a principal G-bundle over a manifold M, with the unitary group $G = SU(n)$, Donaldson analyzed the space of orbits of the gauge group (the moduli space)

$$\mathcal{M} = \mathcal{A}(M) \,/\, \mathcal{U}(M),$$

where $\mathcal{A}(M)$ denotes the space of self-dual or anti-self-dual monopoles on M and $\mathcal{U}(M)$ is the group of gauge transformations.
It turned out that for each topological class of the vector bundle $\mathcal{M}$ is a finite dimensional manifold with singularities, which can be analyzed.
This lead to discovering the invariants of the underlying manifold M.

Last part:

How gauge fields appeared in physics

and why the presented math is relevant

(just a three pages "explanation").

Electromagnetic field as a differential 2-form

After special relativity theory has established its reputation it was found that the electric field $E(x) = (E_1, E_2, E_3)$ and magnetic field $B(x) = (B_1, B_2, B_3)$ can be arranged into a 4×4 antisymmetric tensor field

$$F = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

Geometrically, F is a differential 2-form on the space-time $M = \mathbb{R}^4$,

$$F = (F_{\mu\nu}), \quad F = \frac{1}{2} \sum F_{\mu\nu} dx^\mu dx^\nu$$

where (x^0, x^1, x^2, x^3) are coordinates in the space-time M.

The electromagnetic field F has a "potential" A which has a geometric meaning of a connection $\nabla = d + A$ in a 1-dimensional vector bundle over M. F is the curvature of the connection ∇, in terms of differential geometry!

Physical fields in geometric terms

At present there are four known types of forces in nature: electromagnetic, weak, strong, and gravity. The first three are unified in a theory named Standard Model, which explains (up to minor details) appearance and properties of all elementary particles including all fermions and all bosons.

In 1954 Yang and Mills proposed to extend a geometric description of electromagnetism in Minkowski space (special relativity) to a geometric quantum mechanical description of so called weak nuclear forces in atoms (electroweak theory). Instead of the group $U(1) = S^1$ used in electromagnetism, they used the unitary group $SU(2)$ for "gauge symmetries".

The Standard Model of Elementary Particles

Later developments in physics showed that also strong nuclear interactions can be described in a similar way and all physical forces (fields), excluding gravity, can be described in a unified way using more general (gauge) group G.

When strong forces are concerned, an energy functional can be defined as the integral of the square norm of a physical field F responsible for the strong forces. In geometric terms F is a curvature of a connection on a G vector bundle over space-time M. Physical fields satisfy the Euler-Lagrange equations for such energy. For the strong interactions the group is the unitary group $G = SU(3)$. The resulting Euler-Lagrange equations are then called Yang-Mills equations and their solutions are called Yang-Mills fields.

All three types of physical forces: electromagnetic, weak, and strong, can be described by using the group $G = SU(3) \times SU(2) \times U(1)$, called gauge group. The "physical potential" is a connection on a G-vector bundle over M and the field F representing the forces is the curvature of the connection. The physical F has to satisfy the Yang-Mills equations which are the Euler-Lagrange equations for the energy functional. This forms the basis of the Standard Model of particle physics. A quantization procedure using the Feynman path integral leads to a description of known elementary particles.