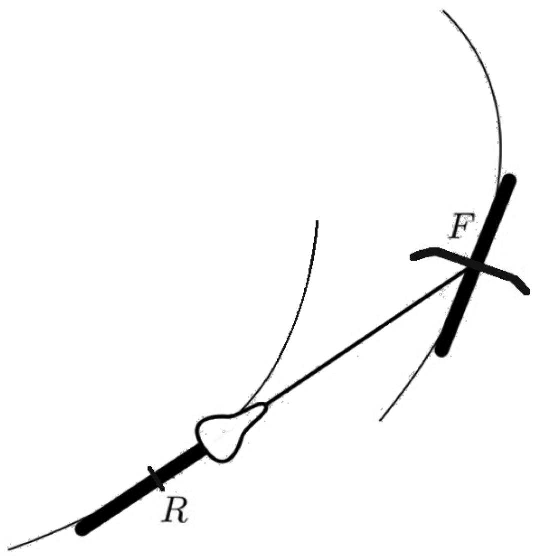
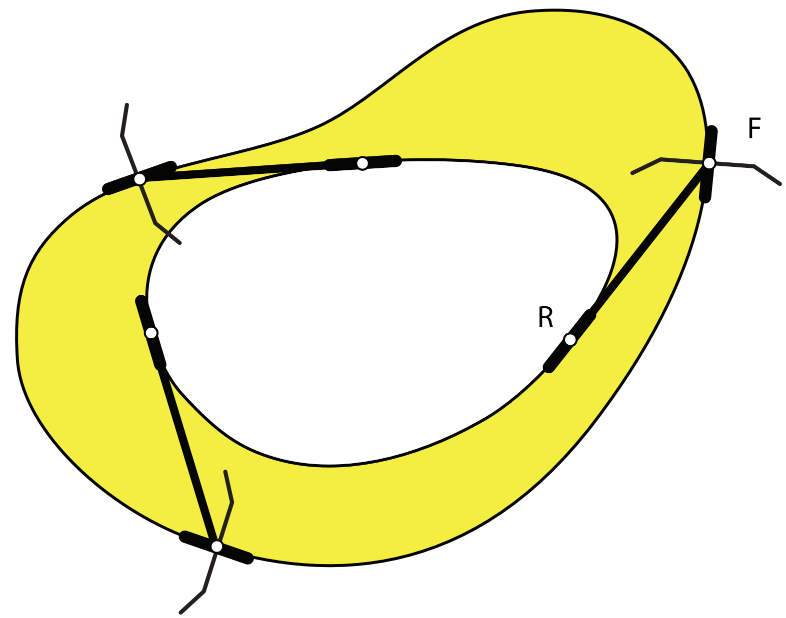
Flavors of bicycle mathematics

Geometry and Differential Equations Seminar, November 2020

Latest paper: G. Bor, M. Levi, and R. Perline
IMRN v. 2020, Issue 9, 2698–2768.

The model





Segment RF moves so that the trajectory of point R is tangent to the segment. *Notation*: γ is the rear wheel track, Γ is the front wheel track. γ may have cusps, when the steering angle is 90° (not recommended in real life!) Likewise, in $\mathbf{R}^n$.

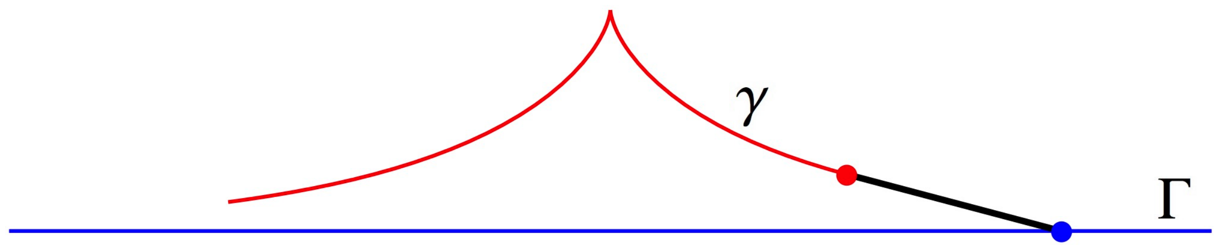
Contact geometry

In the plane, the configuration space of a bicycle is the space of contact elements. It has a non-integrable distribution, a contact structure, given by a "skating" constraint.

Bicycle motion is a smooth horizontal (Legendrian) curve in this contact space. The projection on the front end point is always smooth, and the projection on the rear end point may have cusps.

Likewise, in n-dimensional case, one has a non-integrable n-dimensional distribution in the $(2n-1)$-dimensional configuration space.

Classical connection: the tractrix is the rear wheel track γ, when the front one, Γ, is a straight line.



Claude Perrault

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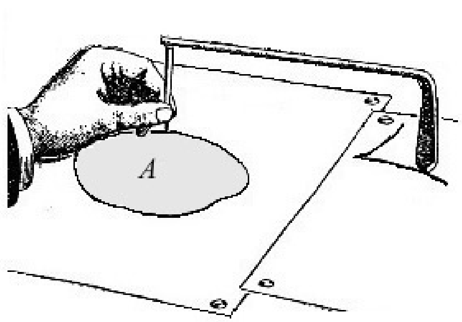
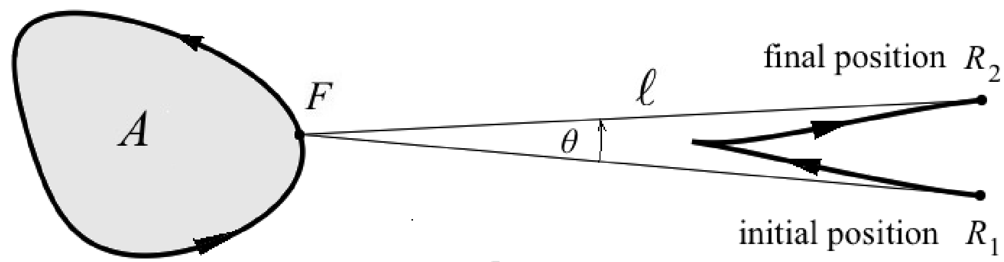
Hatchet, or Prytz, Planimeter: Holger Prytz, a Danish cavalry officer and engineer, 1886, published under the pseudonym "Z".





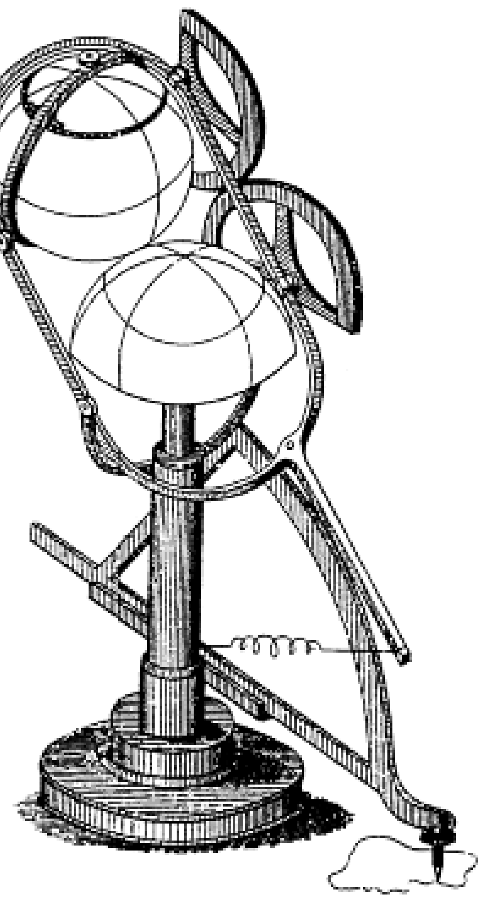
How it works:

$$\text{Area} = \ell^2\theta + O\left(\frac{1}{\ell}\right),$$

actually, a power series in $1/\ell$.

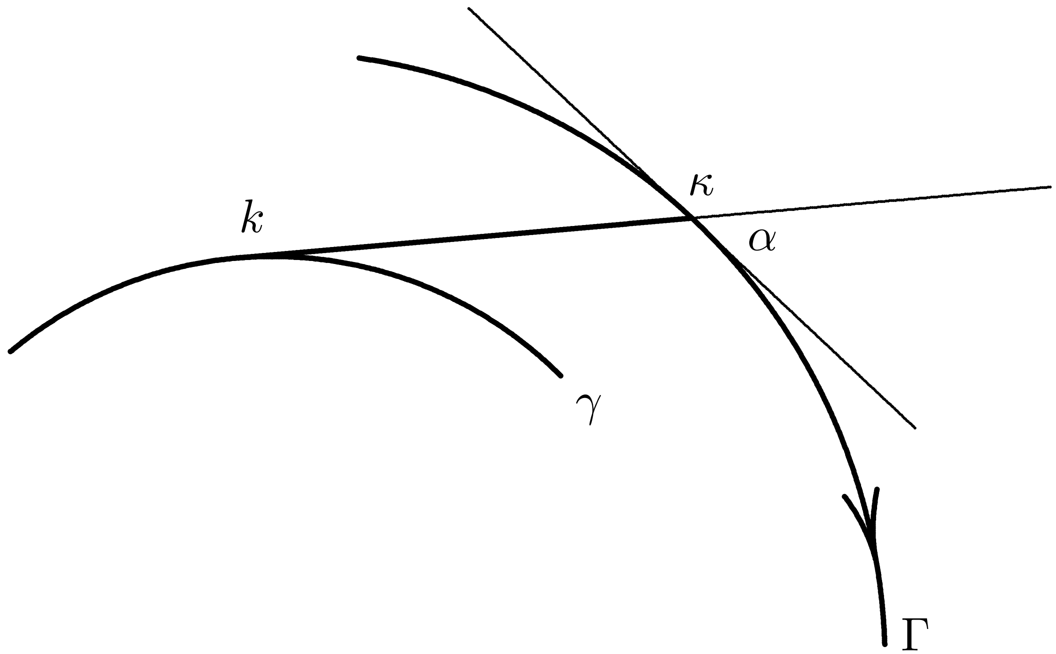
A consequence for parallel parking: maximize the area bounded by the trajectory of the front wheels (remark by Andy Ruina).

Maxwell's Planimeter, 1855



Differential equation and the bicycle monodromy

Determining rear wheel track γ from the front wheel track Γ:



α is the steering angle, x is the arc length parameter on Γ and t on γ, and k and κ are the curvatures of the tracks.

Then

$$\frac{d\alpha(x)}{dx} + \frac{\sin\alpha(x)}{\ell} = \kappa(x), \qquad (*)$$

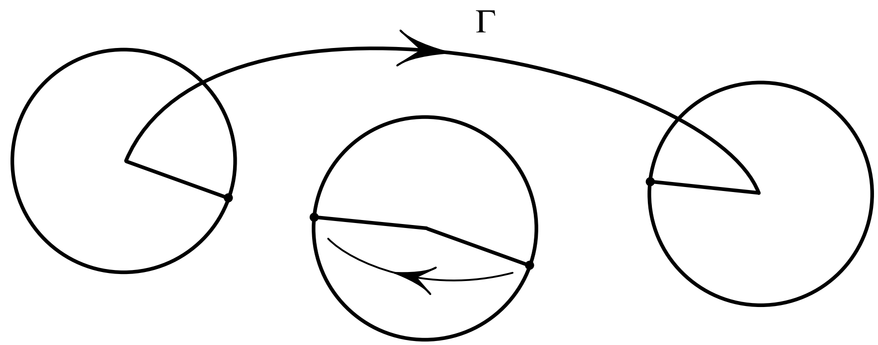
and also,

$$\left|\frac{dt}{dx}\right| = |\cos\alpha|, \qquad k = \frac{\tan\alpha}{\ell}.$$

Cusps $\equiv$ infinite curvature $\equiv \{\alpha = \pi/2\}$.
Consequence: the front wheel goes faster and should wear out sooner (does it, really?)

Equation (*) also describes the overdamped case of the Josephson effect (current through a very narrow insulator separating two superconductors; Nobel Prize 1973). Recent work: V. Buchstaber, A. Glutsyuk, O. Karpov, S. Tertychnyi, and others.

Monodromy M: initial position $\mapsto$ terminal position (in higher dimensions, $M : S^{n-1} \to S^{n-1}$).



Theorem (in dim 2, R. Foote, 1998; M. Levi & S. T., 2009): M *is a Möbius transformation.*

Möbius group $O(n, 1)$: isometries of the hyperbolic space H^n (in the hyperboloid model), acting on the sphere S^{n-1} at infinity.

In dimension 2, if $y = \tan(\alpha/2)$, then (*) becomes a Riccati equation:

$$y'(x) = -\frac{y(x)}{\ell} + \frac{1}{2}(y^2(x) + 1)\kappa(x).$$

In this case, $S^1 = \mathbf{RP}^1$, and M is a real fractional-linear transformation. Concerning the number of fixed points, one has the trichotomy: elliptic, parabolic, hyperbolic.

Likewise, in dimension 3, $S^2 = \mathbf{CP}^1$, and the monodromy is a complex fractional-linear transformation. It always has two fixed points (maybe, coinciding).

Planimeters revisited: what does (a long) bicycle measure in higher dimensions?

The *area bivector* of a curve:

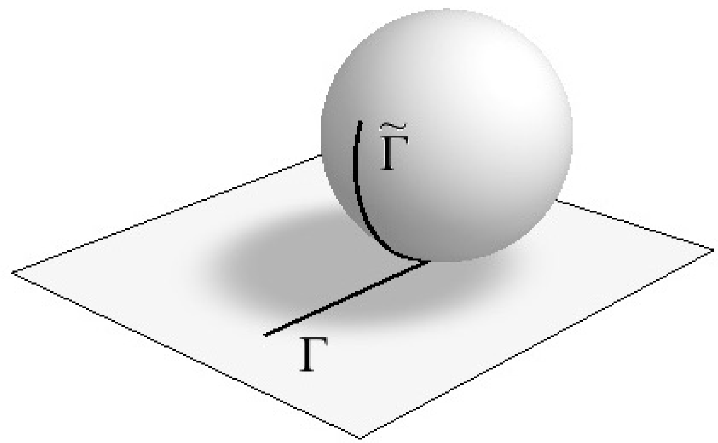
$$A_\Gamma = \frac{1}{2}\int \Gamma(x) \wedge \Gamma'(x)\ dx$$

(thought of as a skew symmetric linear operator).

Theorem: *Upon traversing a curve* Γ*, the bicycle segment* $\mathbf{r}$ *undergoes a net rigid rotation:*

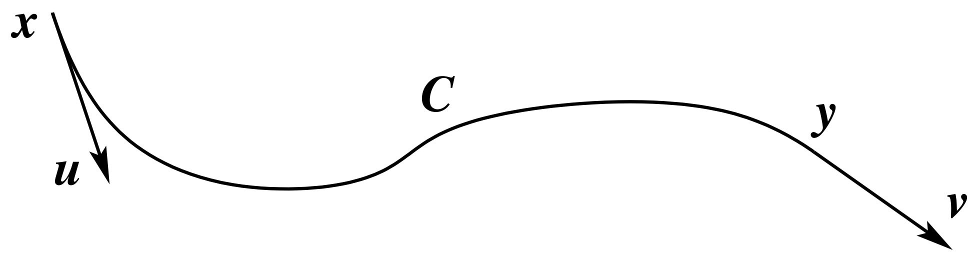
$$\mathbf{r} \mapsto \mathbf{r} + \frac{1}{\ell^2} A_\Gamma(\mathbf{r}) + O\left(\frac{1}{\ell^3}\right).$$

Rolling interpretation. Rolling a ball, without sliding and spinning, along a curve in the plane results in the change of its orientation, an element of $SO(3)$.



The planar curve is developed on the sphere.

Likewise, riding a bike of length ℓ, one "rolls" the hyperbolic plane (of curvature $-1/\ell^2$) along the front wheel track Γ, that is, developes the front track Γ in the hyperbolic plane as the an isometric curve $C \subset H^2$.



Theorem: *The bicycle monodromy is the (unique) isometry of the hyperbolic plane* $M : (y, v) \mapsto (x, u)$.

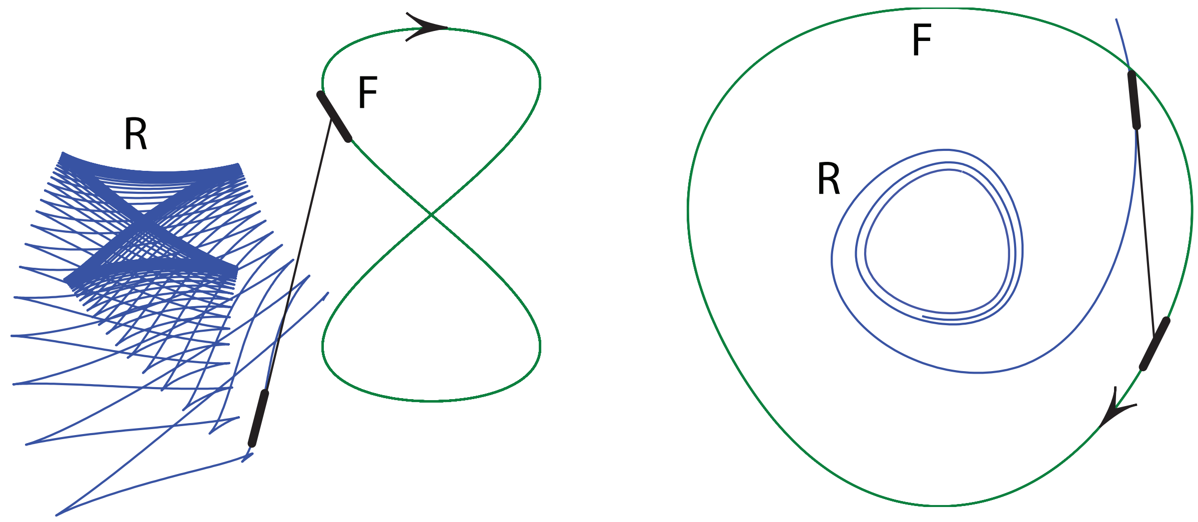
Corollary: *The monodromy is the identity iff the developed curve C is C^1-closed.*

It never happens if Γ is closed and convex: in the Euclidean plane, $\int k\ ds = 2\pi$, and in the hyperbolic plane, $\int k\ ds = 2\pi + A$, due to the Gauss-Bonnet theorem.

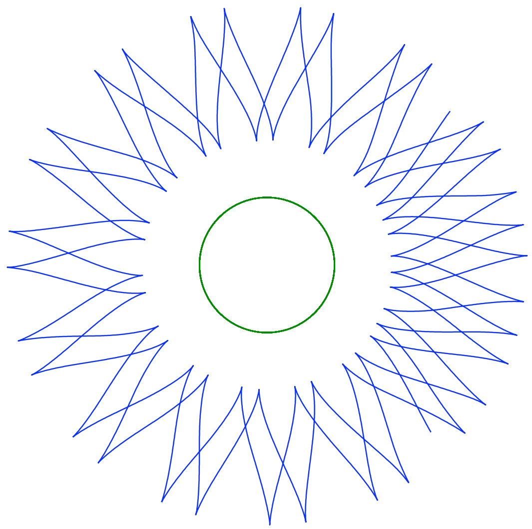
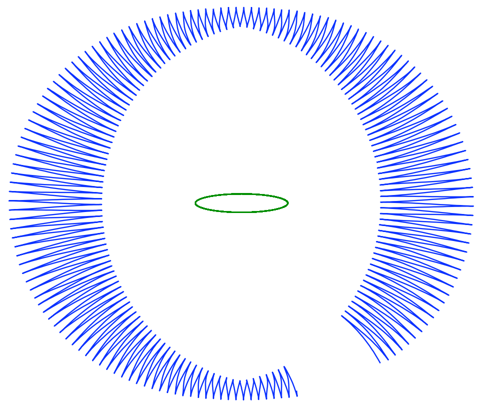
The rolling interpretation holds in all dimensions!

Menzin's conjecture

Hyperbolic monodromy:



Elliptic monodromy:

Theorem: *The monodromy is parabolic iff the signed length of the rear track is zero.* (The sign changes upon passing a cusp).

Menzin's Conjecture (1906): *If* $\ell = 1$ *and* Γ *is an oval of area* $> \pi$*, then the monodromy is hyperbolic.*

> ... the tractrix will approach, asymptotically, a limiting closed curve. From purely empirical observations, it seems that this effect can be obtained so long as the length of arm does not exceed the radius of a circle of area equal to the area of the base curve.

Proved by M. Levi & S.T., 2009 (the heart of the proof is Wirtinger's inequality).

Spherical and hyperbolic versions (S. Howe, M. Pancia, V. Zakharevich, 2011).

The monodromy is still Möbius. *Master equations*:

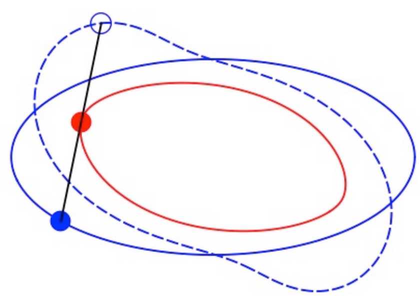
$$\frac{d\alpha(x)}{dx} + \cot \ell \ \sin \alpha(x) = \kappa(x), \ \frac{d\alpha(x)}{dx} + \coth \ell \ \sin \alpha(x) = \kappa(x).$$

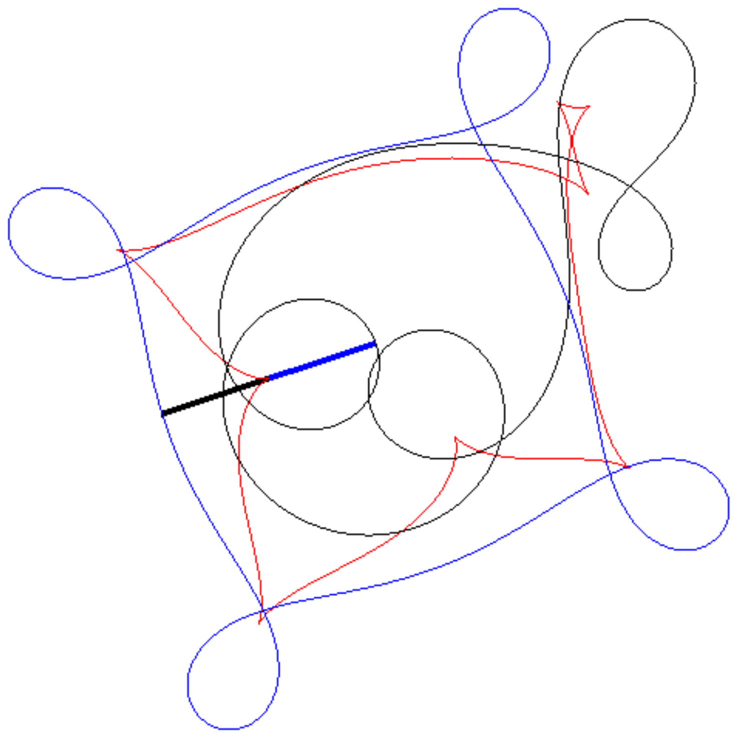
Theorem: *1). in S^2: if Γ is a simple convex curve bounding area $> 2\pi(1 - \cos \ell)$, then the monodromy is hyperbolic;*
2). in H^2: if Γ is a simple horocyclically convex (curvature greater than 1) curve bounding area $> 2\pi(\cosh \ell - 1)$, then the monodromy is hyperbolic.

The areas are those of the disks of radius ℓ.

Bicycle (Darboux, Bäcklund) transformation

The two front tracks that share the rear one: “Pushmi-Pullyu” kinematics.





If two front tracks share the rear track, with the opposite orientations, write: $\mathcal{B}_{2\ell}(\Gamma_1, \Gamma_2)$, the bicycle correspondence.

Equivalently, two points, x_1 and x_2, traverse the curves Γ_1 and Γ_2 in such a way that the distance x_1x_2 is equal to 2ℓ, and the velocity of the midpoint of the segment x_1x_2 is aligned with the segment.

In dimension 2, to get a mapping $\Gamma_1 \mapsto \Gamma_2$, one should assume that Γ_1 has a hyperbolic monodromy; in dimension 3, the monodromy always has a fixed point, and no assumptions are needed.

Properties, valid in all dimensions

Theorem: *If closed curves Γ_1 and Γ_2 are in the bicycle correspondence, then $M_{\Gamma_1,\lambda}$ and $M_{\Gamma_2,\lambda}$ are conjugated for all values of λ.*

Thus the conjugacy invariants of $M_{\Gamma,\lambda}$, as functions of λ (the spectral parameter), are integrals of the bicycle correspondence.

Theorem: (Bianchi permutability). *Let Γ_1, Γ_2 and Γ_3 be three closed curves, such that $\mathcal{B}_\ell(\Gamma_1, \Gamma_2)$ and $\mathcal{B}_\lambda(\Gamma_1, \Gamma_3)$ hold. Then there exists a closed curve Γ_4, such that $\mathcal{B}_\lambda(\Gamma_2, \Gamma_4)$ and $\mathcal{B}_\ell(\Gamma_3, \Gamma_4)$ hold (commutative square).*

In words, loosely: the bicycle transformations with different length parameters commute.

Other integrals of the bicycle transformation:

$$\int_\Gamma \Gamma(t) \wedge \Gamma'(t)\ dt$$

(area bivector), and

$$\int_\Gamma (\Gamma(t) \cdot \Gamma'(t))\ \Gamma(t)\ dt$$

(centroid). Overall,

$$\binom{n}{2} + n = \binom{n+1}{2},$$

the dimension of the group of motions (in agreement with Emmy Noether's theorem).

(Pre)symplectic geometry of the bicycle transformation

Two differential 2-forms on the space of smooth curves in $\mathbf{R}^3$:

$$\omega(u,v) = \int u'(x) \cdot v(x)\ dx, \quad \Omega(u,v) = \int \mathrm{det}(\Gamma'(x), u(x), v(x))\ dx,$$

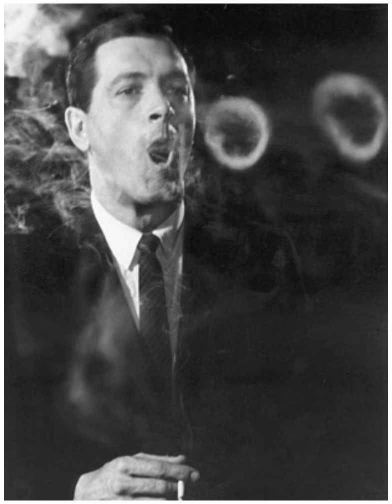
where $u(x), v(x)$ are vector fields along a curve $\Gamma(x)$. Both forms are closed (in fact, exact).

The form ω depends on the metric, but exists in all dimensions; Ω (the Marsden-Weinstein form) depends only on the volume element, but is specifically 3-dimensional.

Theorem: *The bicycle transformation preserves both forms ω and Ω.*

Relation with the filament (binormal, smoke ring, LIE) equation

$$\dot{\Gamma} = \Gamma' \times \Gamma''.$$



Equation introduced by L. Da Rios (a student of Levi-Civita) in 1906 to model the motion of vortices (the same year as Menzin!). It is a well studied completely integrable systems of soliton type.

Theorem: *(i) The filament equation also preserves the 2-forms ω and Ω.*
(ii) The two systems, the bicycle transformation and the filament equation, commute and share integrals.

Integrals of the filament equation F_n:

$$\int 1\ dx,\ \int \tau\ dx,\ \int \kappa^2\ dx,\ \int \kappa^2\tau\ dx,\ \int \left((\kappa')^2+\kappa^2\tau^2-\frac{1}{4}\kappa^4\right)\ dx,\ldots$$

where τ is the torsion and κ is the curvature of a curve. The corresponding commuting Hamiltonian vector fields X_n:

$$-T,\ \kappa B,\ \frac{\kappa^2}{2}T+\kappa' N+\kappa\tau B,\ldots$$

they satisfy the recurrence relation

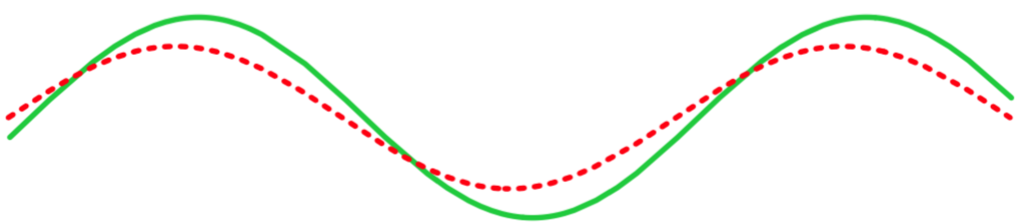
$$\omega(X_{n-1},\cdot)=\Omega(X_n,\cdot)=dF_n.$$

In dimension 2, every other integral is non-trivial.

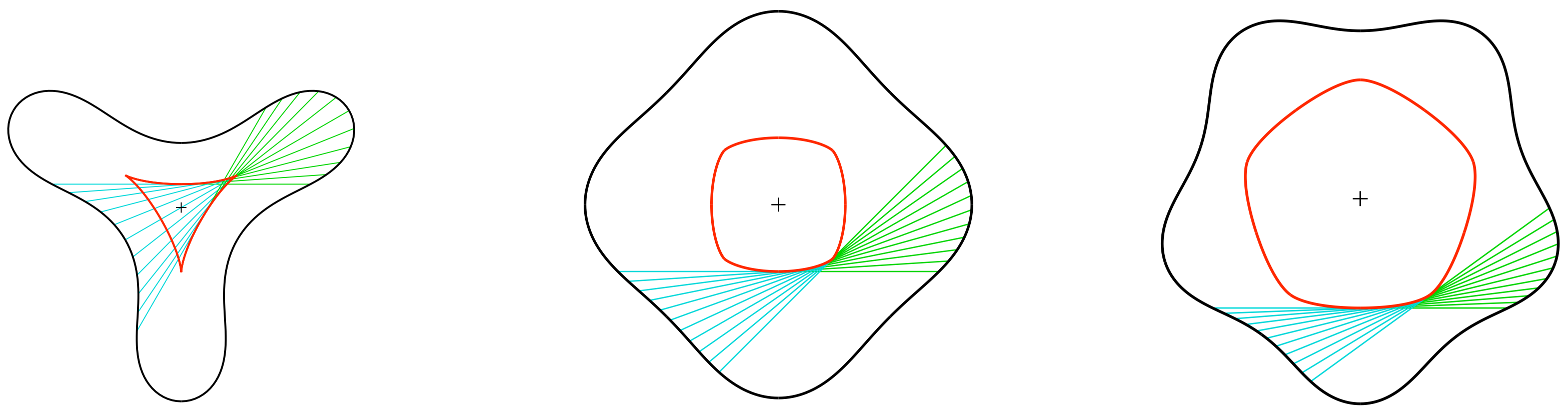
Which way did the bicycle go?

In "The Adventure of the Priory School" by A. Conan Doyle, Sherlock Holmes did not do very well:

> No, no, my dear Watson. The more deeply sunk impression is, of course, hind wheel, upon which the weight rests. You perceive several places where it has passed across and obliterated the more shallow mark of the front one. It was undoubtedly heading away from the school.



Usually, you can tell which way the bicycle went, but sometimes you cannot. Trivial example: concentric circles. But also



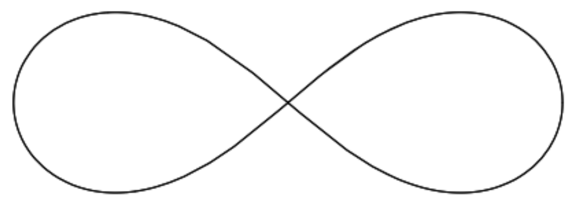
and many more: F. Wegner, *Three Problems – One Solution*
`http://www.tphys.uni-heidelberg.de/~wegner/Fl2mvs/Movies.html`

Ulam's problem: which homogeneous bodies float in equilibrium in all positions? ("Scottish Book", Problem No 19)

In dimension two (floating log), it's the same problem!

The role of relative density is played by the relative length of the arc of Γ, subtended by the moving segment.

Classical elastica: extremize the bending energy $\int k^2 dx$ with fixed length, satisfy the Euler-Lagrange equation $k'' + \frac{1}{2}k^3 + \lambda k = 0$.



Buckled rings (pressurized elastica): relative extrema of the total squared curvature with perimeter and area constraints:

$$k'' + \frac{1}{2}k^3 + \lambda k + \mu = 0,$$

where λ, μ are Lagrange multipliers.

Studied by M. Lévy (1884), G. Halphen (1888), and A. Greenhill (1889).

Wegner's curves: in polar coordinates $r = r(\psi)$,

$$\frac{1}{\sqrt{r^2 + r_\psi^2}} = ar^2 + b + \frac{c}{r^2}$$

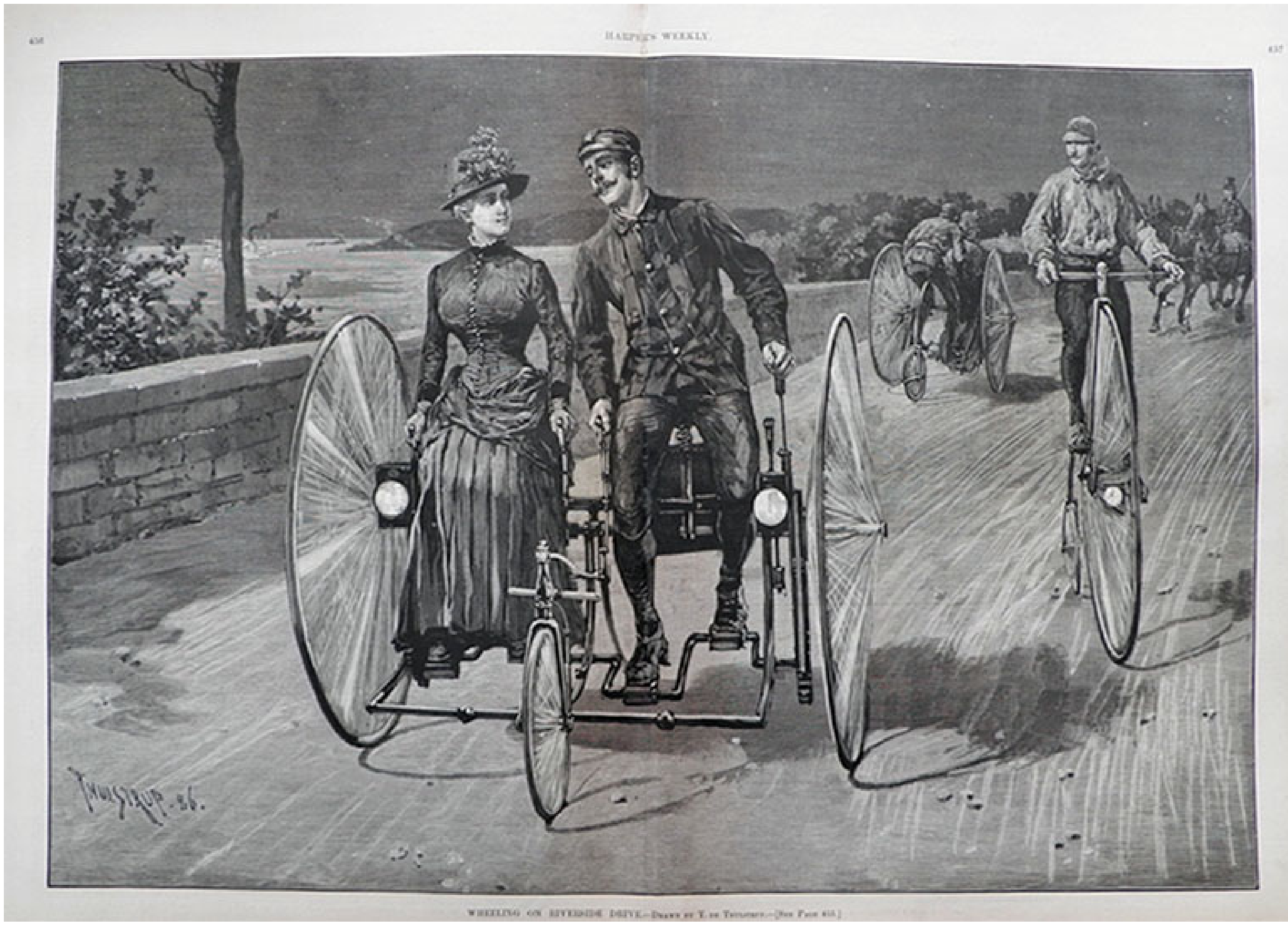
with parameters a, b, c.

Recall $X_2 = \frac{k^2}{2} T + k' N$, the planar filament vector field.

Theorem: *The Wegner curves are solitons: under X_2, they evolve by rigid rotation and parameter shift. They are buckled rings: their curvature satisfies*

$$k'' + \frac{1}{2} k^3 + \lambda k = \mu$$

with $\lambda = 8ac - 2b^2, \mu = 8a$.



Thank you!