Deformations of the Veronese embedding and Finsler 2-spheres of constant curvature

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Path geometries

**Setup:** $M$ connected oriented smooth surface

**Path geometry:** Prescription of a path on $M$ for each direction in every tangent space (e.g. geodesics of a Finsler metric, geodesics of a projective structure)

**Projective circle bundle**

$$\pi : SM := (TM \setminus \{0_M\}) / \mathbb{R}^+ \to M$$

**Contact structure**

$$\tau[v] = \{\xi \in T[v]SM : \pi'(\xi) \wedge v = 0\}$$

**Immersed curve** $\gamma : (a, b) \to M$ lifts s.t. $\dot{\delta}(t)$ lies in $\tau$

$$\delta := [\dot{\gamma}] : (a, b) \to SM$$

**Path geometry:** 1-dim distribution $P \to SM$ so that $P + \ker \pi' = \tau$.

**Paths:** Integral curves of $P$ projected to $M$
The dual of a path geometry

**Definition (Bryant).** A generalised path geometry is a 3-manifold $N$ together with an ordered pair $(P, L)$ of transverse 1-dim distributions spanning a contact structure.

**Path geometry:**

$N = SM$, \quad P = \text{“path bundle”}$, \quad L = \text{vertical bundle of projection } SM \to M$

**Definition.** The dual of a generalised path geometry $(N, P, L)$ is the generalised path geometry $(N, L, P)$.

**Question.** Are there (non-trivial global) examples where the dual of a path geometry is again a path geometry?
Projective structures

**Affine connection:** connection $\nabla$ on $TM$, assume $\nabla$ is torsion-free

**Geodesic:** immersed curve $\gamma : I \to M$ s.t.

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0.$$  

**Projective equivalence:** $\nabla \sim \nabla'$ iff $\nabla$ and $\nabla'$ have the same geodesics up to parametrisation.

**Projective structure:** Equivalence class $p$ of connections

**Lemma (Cartan, Eisenhart, Weyl).** $\nabla \sim \nabla' \iff \exists \beta \in \Omega^1(M)$ such that

$$\nabla_x Y - \nabla'_x Y = \beta(X)Y + \beta(Y)X.$$  

Projective surface $(M, p)$ is called **flat** if it is locally diffeomorphic to $S^2$ so that geodesics are mapped onto (segments of) great circles.
Finsler metrics

A **Finsler norm** is a continuous function $F : TM \rightarrow [0, \infty)$ which is smooth away from the zero section and so that

- $F(\lambda v) = \lambda F(v)$ for $\lambda \geq 0$
- $F(v) > 0$ unless $v = 0$
- the symmetric bilinear form
  
  $$g_v(X, Y) = \frac{1}{2} \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} \left[ F(v + sX + tY)^2 \right]$$

  is positive definite.

F is called **reversible** is $F(v) = F(-v)$ for all $v \in TM$

**Length** of immersed curve $\gamma : [a, b] \rightarrow M$, $L(\gamma) := \int_a^b F(\dot{\gamma}(t)) \, dt$ is invariant under orientation preserving reparametrisations

Locally length minimising curves are the **geodesics** of $F$. 
Finsler norm is determined by its **unit tangent bundle**

$$UM := \{ v \in TM : F(v) = 1 \}.$$ 

**Zermelo deformation:** Construct new Finsler metric by translating each fibre of $UM$ with a vector of small enough length.

**Cartan:** $UM$ is equipped with a coframing $(\chi, \eta, \nu)$ which satisfies the structure equations

$$d\chi = -\eta \wedge \nu, \quad d\eta = -\nu \wedge (\chi - I\eta), \quad d\nu = -(K\chi - J\nu) \wedge \eta,$$

for $I, J, K \in C^\infty(UM)$.

**Riemannian case:** $(M,g)$ choose **isothermal coordinates** $(x, y)$

$$g = e^{2u(x,y)}(dx^2 + dy^2)$$

**Coframing**

$$\chi = e^u (\cos \alpha \ dx + \sin \alpha \ dy), \quad \eta = e^u (-\sin \alpha \ dx + \cos \alpha \ dy), \quad \nu = d\alpha + \star du,$$

where $\alpha$ is the **angle coordinate** on the unit tangent bundle.
**Riemannian Finsler metric:** $I \equiv J \equiv 0$ and $K$ is (the pullback to $UM$ of) the Gauss curvature $K_g$.

$K$ is the **Finsler–Gauss curvature** or flag curvature.

**Theorem (Akbar-Zadeh, 1988).** If a Finsler metric on a compact surface has constant negative curvature, then it is Riemannian, and, if it has zero curvature, then it is locally Minkowskian.

**Theorem (Bryant, 2006).** If a reversible Finsler metric on a compact surface has constant positive curvature, then it is Riemannian.

**Fact:** A Zermelo deformation of a constant curvature Finsler metric by a Killing vector field has again constant curvature.

**Example.** (Katok) First example of non-Riemannian $K \equiv 1$ Finsler metric on $S^2$ via Zermelo deformation of constant curvature metric.

**Theorem (Bryant, 1997).** Classification of $K \equiv 1$ Finsler 2-spheres that are projectively flat.
(Generalised) thermostats

**Dual vector fields** $(X, H, V)$ to $(\chi, \eta, \nu)$

\[
[V, X] = H, \quad [V, H] = -X, \quad [X, H] = K_g V
\]

**Tautological bundle** $\tau = \{\eta = 0\}$, **vertical bundle** $\{\chi = \eta = 0\}$

**Thermostat:** flow $\phi$ generated by $X + \lambda V$ for $\lambda \in C^\infty(UM)$

Choice of metric $g$ identifies path geometry $P$ with thermostat.

$\lambda = \lambda(x, y, \alpha)$, 2\pi-periodic in $\alpha$, **Fourier-decomposition** in $\alpha$

**Volume form:** $\Theta = \chi \wedge \eta \wedge \nu$ and **inner product:**

\[
\langle u, v \rangle = \int_{UM} uv \Theta,
\]

**Densely defined operator** $-iV$ is self-adjoint

\[
L^2(UM) = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_m, \quad \mathcal{H}_m = \ker(m \text{Id} + iV)
\]
Examples of thermostats

**Example.** $\alpha \in \Omega^2(M), g \in \text{Riem}(M)$. Consider flow of Hamiltonian vector field $X_\eta$ on $(T^*M, \Omega_0 + \nu^*\alpha)$ generated by Hamiltonian $\eta(\xi) = \frac{1}{2}|\xi|^2_{g^\#}$.

**Magnetic flows** correspond to thermostats of degree 0, i.e. $V\lambda = 0$

$$\pi^*\alpha = \lambda \chi \wedge \eta.$$

**1-forms** $\lambda \in \mathcal{C}^\infty(UM) \cap (\mathcal{H}_{-1} \oplus \mathcal{H}_1) \leftrightarrow \Omega^1(M)$

To $\theta \in \Omega^1(M)$ – thought of as a function $\theta : UM \to \mathbf{R}$ – we associate the thermostat $\phi$ generated by the vector field

$$F = X - V(\theta)V.$$

Orbits of $\phi$ – when projected to $M$ – are reparametrisations of the geodesics of the **Weyl connection** defined by $(g, \theta)$. 
**Weyl connections**

**Weyl connection**: Affine torsion-free connection $\nabla$ preserving a conformal structure $[g]$, i.e. parallel transport maps of $\nabla$ are **angle preserving** w.r.t. $[g],

$$\nabla g = 2\theta \otimes g,$$

Weyl connections are of the form

$$(g, \theta) \nabla = g \nabla + g \otimes \theta^\flat - \theta \otimes \text{Id} - \text{Id} \otimes \theta$$

with $g \in [g]$ and $\theta \in \Omega^1(M)$.

**Weyl structure** is an equivalence class $[(g, \theta)]$ where

$$(g, \theta) \sim (\hat{g}, \hat{\theta}) \iff \hat{g} = e^{2u}g \text{ and } \hat{\theta} = \theta + du, u \in C^\infty(M)$$

Weyl structures are in one-to-one correspondence with Weyl connections

$$[(g, \theta)] \mapsto g \nabla + g \otimes \theta^\flat - \theta \otimes \text{Id} - \text{Id} \otimes \theta$$

Weyl connections with $\theta$ **exact** correspond to Levi-Civita connections
A Weyl structure \([(g, \theta)]\) is called **positive** if \(\text{Sym}(\text{Ric}(g, \theta) \nabla))\) is positive definite.

On oriented surface \(M\)

\[
[(g, \theta)] \text{ is positive } \iff (K_g - \delta_g \theta) dA_g > 0
\]

**Lemma.** For a positive Weyl structure \([(g, \theta)]\) there exists a unique gauge \((g, \theta)\) – henceforth called the **natural gauge** – so that \(K_g - \delta_g \theta \equiv 1\).

**Lemma.** Let \([(g, \theta)]\) be a positive Weyl structure with natural gauge \((g, \theta)\) and let \(\pi : UM \to M\) denote the unit tangent bundle of \(g\) with coframing \((\chi, \eta, \nu)\). Then the forms

\[
\hat{\chi} := \pi^*(\star_g \theta) - \nu, \quad \hat{\eta} := -\eta, \quad \hat{\nu} := -\chi
\]

satisfy the structure equations of a Finsler metric with \(K \equiv 1\).

**Paraphrasing:** Ignoring global issues, the path geometry of a positive Weyl structure (i.e. whose paths are the geodesics of the associated Weyl connection) is dual to the path geometry of a Finsler metric with \(K \equiv 1\).
Dynamical aspects of $K \equiv 1$ Finsler metrics

**Theorem (Bryant, Foulon, Ivanov, Matveev, Ziller, 2017).** Let $F$ be a $K \equiv 1$ Finsler metric on $S^2$. Then there exists a shortest closed geodesic of length $2\pi \ell \in (\pi, 2\pi]$ and the following holds:

- **If** $\ell = 1$, all geodesics are closed and have the same length $2\pi$,
- **If** $\ell$ is irrational, there exist two closed geodesics with the same image, and all other geodesics are not closed. The length of the second closed geodesic is $2\pi \ell / (2\ell - 1)$. Moreover, the metric admits a Killing vector field.
- **If** $\ell = p/q \in (\frac{1}{2}, 1)$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$, and in this case all unit-speed geodesics have a common period $2\pi p$. Furthermore, there exists at most two closed geodesics with length less than $2\pi p$. A second one exists only if $2p - q > 1$, and its length is $2\pi p / (2p - q) \in (2\pi, 2p\pi)$.

In particular, if all geodesics of a Finsler metric on $S^2$ are closed, then its geodesic flow is periodic with period $2\pi p$ for some integer $p$.

They also show that the case when $F$ admits a Killing field can be deformed (via a Zermelo deformation) to the case $\ell = 1$. 
A duality result

A Weyl structure \([(g, \theta)]\) is called **Besse** if the associated Weyl connection has the property that all of its maximal geodesics are closed.

**Theorem (Lange–M., 2019).** There is a one-to-one correspondence between Finsler metrics on \(S^2\) with \(K \equiv 1\) and all geodesics closed on the one hand, and positive Besse–Weyl structures on weighted projective spaces \(\mathbb{CP}(a_1, a_2)\) with \(c := \gcd(a_1, a_2) \in \{1, 2\}, a_1 \geq a_2, 2|(a_1 + a_2)\) and \(c^3|a_1a_2\) on the other hand.

More precisely,

1. such a Finsler metric with shortest closed geodesic of length \(2\pi \ell \in (\pi, 2\pi]\), \(\ell = p/q \in (1/2, 1], \gcd(p, q) = 1\), gives rise to a positive Besse–Weyl structure on \(\mathbb{CP}(a_1, a_2)\) with \(a_1 = q\) and \(a_2 = 2p - q\), and

2. a positive Besse–Weyl structure on such a \(\mathbb{CP}(a_1, a_2)\) gives rise to such a Finsler metric on \(S^2\) with shortest closed geodesic of length \(2\pi \left(\frac{a_1 + a_2}{2a_1}\right) \in (\pi, 2\pi]\),

and these assignments are inverse to each other. Moreover, two such Finsler metrics are isometric if and only if the corresponding Besse–Weyl structures coincide up to a diffeomorphism.
Weighted projective space

**Projective space** $\mathbb{CP}^1$ is $\mathbb{C}^2 \setminus \{0\}$ modulo the action

$$\lambda \cdot (z, w) = (\lambda z, \lambda w), \quad \lambda \in \mathbb{C}^*$$

**Weighted projective space** $\mathbb{CP}(a_1, a_2)$ for weights $(a_1, a_2) \in \mathbb{N}^2$ is $\mathbb{C}^2 \setminus \{0\}$ modulo the action

$$\lambda \cdot (z, w) = (\lambda^{a_1} z, \lambda^{a_2} w), \quad \lambda \in \mathbb{C}^*$$

$\mathbb{CP}(1, 1) = \mathbb{CP}^1$, weighted projective space is in general an **orbifold**

There exists a natural generalisation $g_{FS}$ of the **Fubini–Study metric** to $\mathbb{CP}(a_1, a_2)$

$g_{FS}$ is a Besse orbifold metric of strictly positive Gauss curvature ($K_{g_{FS}} \neq \text{const}$).

Try to deform $g_{FS}$ among the class of positive Besse-Weyl structures to construct new examples of $K \equiv 1$ Finsler structures.
Isometric embeddings

\[ 4g_{FS} = \left( \frac{a_1 + a_2}{2} + \frac{a_1 - a_2}{2} \cos(r) \right)^2 \, dr^2 + \sin^2(r) \, d\phi^2, \quad (r, \phi) \in (0, \pi) \times S^1 \]
**Twistor space**

**Twistor bundle** $J^+ \to M$

$J^+_p := \{ \text{linear complex structures } J \text{ on } T_p M : (v, Jv) \text{ is pos. oriented } \forall v \neq 0 \}$

Bundle with fibre

$\frac{\text{GL}^+(2, \mathbb{R})/\text{GL}(1, \mathbb{C})}{\simeq} \mathcal{D} := \{ z \in \mathbb{C} : |z| < 1 \}$

Conformal structure $\leftrightarrow$ orientation compatible complex structure

$J_p : T_p M \to T_p M, \quad J_p = \text{counterclockwise rotation by } \pi/2$

Conformal structure defines section $[g] : M \to J_+.$

**Proposition (O’Brian & Rawnsley, Dubvois-Violette).** Torsion-free $\nabla$ on $TM$ equips $J^+$ with an integrable almost complex structure $J_p$ which does only depend on the projective equivalence class of $\nabla$.

At $j \in J^+$ lift $j$ horizontally and use complex structure on the fibre vertically.
Holomorphic curves

**Proposition (M., 2014).** The Weyl connection \((g, \theta) \nabla\) belongs to \(p\) iff 
\([g] : M \to (J^+, J_p)\) is a holomorphic curve.

Same statement holds for orbifolds.

**Proposition (M., 2014).** For the projective structure on \(S^2\) whose geodesics are the great circles, we have \(J^+ \hookrightarrow \mathbb{CP}^2\)

**Proposition (Lange–M., 2019).** For the projective structure arising from the Fubini–Study metric \(g_{FS}\) on \(\mathbb{CP}(a_1, a_2)\), we have \(J^+ \hookrightarrow \mathbb{CP}(a_1, (a_1 + a_2)/2, a_2)\). Furthermore, the holomorphic curve

\([g_{FS}] : \mathbb{CP}(a_1, a_2) \to \mathbb{CP}(a_1, (a_1 + a_2)/2, a_2)\)

corresponds to the **Veronese embedding**

\([z, w] \mapsto [z^2, zw, w^2]\).

Suitable deformations of the Veronese embedding yield positive Besse–Weyl structure on \(\mathbb{CP}(a_1, a_2)\) and hence new examples of Finsler 2-spheres with \(K \equiv 1\).