Discrete mechanics on unitary octonions

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December 16, 2020

Background

The subject of discrete Lagrangian mechanics concerns the study of certain discrete dynamical systems on manifolds.

-The main tool in the theory of geometric integrators is the discrete Lagrangian and Hamiltonian formalism on $Q \times Q$.

-This Cartesian products plays the role of a discretized version of standard velocity phase space TQ.

-Applying a natural discrete variational principle, we get a second order recursion operator

$$\xi : Q \times Q \to Q \times Q, \quad (x, y) \longmapsto (y, z).$$

- **J.E. Marsden, M. West**, *Discrete mechanics and variational integrators*, Acta Numerica (2001), 357–514.

Background

-Moser and Veselov consider the Lagrangian and Hamiltonian formalisms for discrete mechanics on a Lie group.

- **Moser, J., and Veselov, A.P.**, *Discrete versions of some classical integrable systems and factorization of matrix polynomials*, *Comm. Math. Phys.* **139** (1991), 217-243.

The Lagrangian function L is defined on a Lie group G, and the dynamical system is given by a diffeomorphism from G to itself. The corresponding Hamiltonian system is the mapping from the dual Lie algebra $\mathfrak{g}^*$ to itself for which L is the generating function.

-Weinstein observed that these systems could be understood as a special case of a more general theory, describing discrete Lagrangian mechanics on arbitrary Lie groupoids.

- **A. Weinstein,** *Lagrangian Mechanics and groupoids*, Fields Inst. Comm. **7** (1996), 207–231.

The aim of the talk

Discrete Lagrangian mechanics on Lie groups and groupoids has been developed in many papers.

- **J. C. Marrero, D. Martín de Diego, E. Martínez,** *Discrete Lagrangian and Hamiltonian Mechanics on Lie groupoids*, J. Nonlinear Sci. **18** (2008), 221–276..
- **J.E. Marsden and T.S. Ratiu,** *Introduction to Mechanics and Symmetry*, SpringerVerlag, (1994). Second Edition, 1999.
- **A. Weinstein,** *Lagrangian Mechanics and groupoids*, Fields Inst. Comm. **7** (1996), 207–231..

Nevertheless, the generalization of the discrete mechanics to non-associative objects is still lacking, and the goal is to fill this gap by presenting a systematic approach for the construction of discrete Lagrangian formalism on non-associative objects, smooth loops.

Non-associative generalization of the concept of a group

The theory of smooth quasi-groups and loops has already started to find interesting applications in geometry and physics.

- **A. I. Mal'cev**, *Analytic loops*, Mat. Sb. N.S. (in Russian), **36**(78) (1955), 569–576.

A remarkable development of smooth quasigroups and loops theory was presented by Lev V. Sabinin, where the large bibliography on the subject is given.

- **L. V. Sabinin**, *Smooth Quasigroups and Loops*, Kluwer Academic Press, 1999.

Overview

1. Non-associative objects
2. Octonions
3. Discrete Lagrangian mechanics on Lie group
4. Discrete Lagrangian mechanics on smooth loops
5. Tangent and cotangent bundle of a smooth loop
6. Discrete Lagrangian mechanics on unitary octonions

Non-associative objects

- A quasigroup is an algebraic structure $< G, \cdot >$ with a binary operation such that right translation $r_g : x \mapsto xg$ and left translation $l_g : x \mapsto gx$ are permutations of G, equivalently, in which the equations $ya = b$ and $ax = b$ are solvable uniquely for x and y respectively.
- A Loop is a quasigroup with a two-sided identity element, e such that $ex = xe = x$.
- An Inverse loop is a loop equipped with an inversion map $\iota : G \to G$, which we denote by $\iota(a) = a^{-1}$, such that

$$\iota(e) = e\,, \quad \iota^2 = \mathrm{Id}_{G,} \quad \text{and} \quad \iota \circ l_g = r_{g^{-1}} \circ \iota\,.$$

Smooth loops

- A smooth loop G is a smooth manifold equipped with a smooth multiplication, $m : G \times G \to G$, $(g, h) \longmapsto m(g, h) = gh$, such that:
  1. The left and right translations are diffeomorphisms.
  2. There is an identity element $e \in G$ such that $eg = ge = g$ for every $g \in G$.
- G is a smooth inverse loop if there is a smooth inverse $\iota : G \to G$.

Let $\mathfrak{g} = \mathrm{T}_e G$. If G is a smooth loop with the smooth inverse $\iota : G \to G$, then

$$\iota_*(X) = -X, \quad \text{for} \quad X \in \mathfrak{g}\,.$$

Smooth prolongations

We can left-translate (resp. right-translate) the value of $X_e \in \mathfrak{g}$ by the tangent of the left (resp. right) translation by $g \in G$.

- Although they are not invariant vector fields anymore due to the lack of associativity, we still are able to define the left (resp. right) prolongations of X_e to vector fields $\overleftarrow{X}$ (resp. $\overrightarrow{X}$) on G using the tangent maps at $g \in G$ to the left translation l_g and right translation r_g:

$$\overleftarrow{X}_g = D_e(l_g)(X_e), \quad \overrightarrow{X}_g = D_e(r_g)(X_e).$$

Here, D denotes the derivative. This prolongations are smooth vector fields.

- For the inverse smooth loop, $\iota_*(\overleftarrow{X}) = -\overrightarrow{X} \circ \iota$.

Skew algebra

Due to non-associativity

- we cannot infer that there is $[X, Y] \in \mathfrak{g}$ such that $[\overleftarrow{X}, \overleftarrow{Y}] = \overleftarrow{[X, Y]}$ nor $[\overrightarrow{X}, \overrightarrow{Y}] = \overrightarrow{[X, Y]}$.
- Moreover, in general, we do not have $[\overleftarrow{X}, \overrightarrow{Y}] = 0$.

However, the tangent space at the identity $\mathrm{T}_e G \cong \mathfrak{g}$ inherits a skew-symmetric bilinear product $[\cdot, \cdot]_l$ from the Lie product of the left prolongations of vector fields over the loop, that is $[X, Y]_l = [\overleftarrow{X}, \overleftarrow{Y}]_e$, but

- the Jacobi identity does not hold due to the non-associativity.

Mimicking the Lie theory, one can define for any smooth loop, a 'Lie functor' associating with the loop a skew algebra, i.e. a real vector space with a bilinear skew operation.

Moufang loops

Smooth Moufang loops were also considered by Mal'cev, as particular cases of loops.

- A loop is called a Moufang loop if it satisfies any of the three following equivalent conditions

$$((ax)a)y = a(x(ay)), \quad ((xa)y)a = x(a(ya)), \quad (ax)(ya) = (a(xy))a.$$

- The tangent algebra of a smooth Moufang loop is a Mal'cev algebra which is a skew algebra satisfying

$$[[X,Y],[X,Z]] = [[[X,Y],Z],X] + [[[Y,Z],X],X] + [[[Z,X],X],Y],$$

for every X, Y, Z. There is a sort of Lie's Third Theorem for smooth Moufang loops and Mal'cev algebras.

Motivating example

As a working example we will develop the discrete Lagrangian formalism on unitary octonions $\mathbb{O}_1$, understood as an inverse loop in the algebra of octonions $\mathbb{O}$, or a subloop in the loop $\mathbb{O}^\times$ of invertible octonions which as a manifold is the seven-sphere.

- It is well known that S^0, S^1, S^3 and S^7 are only spheres which are parallelizable and they correspond to elements of unit norm in the normed division algebras of the real numbers, complex numbers, quaternions, and octonions. The first three spheres are Lie groups ($S^0 = \mathbb{Z}_2$, $S^1 = U(1)$, $S^3 = SU(2)$), but S^7 is the only parallelizable sphere which is not a Lie group since it is not associative.

Octonions

The octonions $\mathbb{O}$ are the noncommutative non-associative algebra which is one of the four division algebras that exist over the real numbers.

Every octonion can be expressed in terms of a natural basis $\{e_0, e_1, \cdot, \cdot, \cdot, e_7\}$ where $e_0 = 1$ represents the identity element and the imaginary octonion units e_i, $\{i = 1, ..., 7\}$ satisfy the multiplication rule $e_i e_j = \delta_i^j + f_{ijk} e_k$, where δ_i^j is the Kronecker's delta and f_{ijk}'s are completely anti-symmetric structure constants which read as

$$f_{123} = f_{147} = f_{165} = f_{246} = f_{257} = f_{354} = f_{367} = 1.$$

The multiplication is subject to the relations

$$\forall i \neq 0 \quad e_i^2 = -1\,, \qquad e_i e_j = -e_j e_i, \quad \text{for} \quad i \neq j \neq 0.$$

The octonion multiplication rules are encoded in a triangular diagram called the Fano plane.

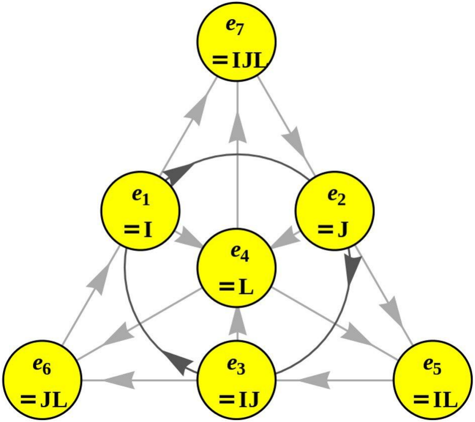

Figure: A mnemonic for the products of the unit octonions using the Fano plane

Properties of octonions

- Conjugation is an involution of $\mathbb{O}$ satisfying $(gh)^* = h^* g^*$.
- The inner product on $\mathbb{O}$ is inherited from $\mathbb{R}^8$ and can be rewritten

$$\langle g, h \rangle = \frac{(gh^* + hg^*)}{2} = \frac{(h^* g + g^* h)}{2} \in \mathbb{R}\,.$$

- The norm of an octonion is just $\|g\|^2 = gg^*$ which satisfies the defining property of a normed division algebra, namely $\|gh\| = \|g\|\|h\|$.
- The scalar product is invariant with respect to the multiplication: $\langle ag, ah \rangle = \langle g, h \rangle$ for $a \neq 0$.

Octonions as a smooth Moufang loop

- Every nonzero octonion $g \in \mathbb{O}$ has an inverse $g^{-1} = \frac{g^*}{\|g\|^2}$, such that

$$gg^{-1} = g^{-1}g = 1,$$

which makes the set of invertible octonions an inverse loop with respect to the octonion multiplication.

- We remark that the inverse is a genuine one, i.e.,

$$g(g^{-1}h) = g^{-1}(gh) = h, \quad \forall g, h \in \mathbb{O},$$

which is stronger than the standard above property for non-associative algebra.

Theorem

Invertible octonions $\mathbb{O}^\times$ form a smooth inverse Moufang loop under the octonion multiplication.

Discrete Lagrangian mechanics on Lie groups

A discrete Lagrangian system consists of a Lie group G and a smooth, real-valued function L on G.

- We define a function $(g, h) \to L(g) + L(h)$ of elements $g, h \in G$.
- A solution of the Lagrange equations for the Lagrangian function L is a sequence $g_0, g_1, g_2, \ldots$ of elements G such that (g_i, g_{i+1}) are the stationary points of the function $L(g_i) + L(g_{i+1})$ for every i.

Variational principles

Discrete Lagrangian systems on Lie groups can be based on variational principles.

- The variational principle for a Lie group G with Lie algebra $\mathfrak{g}$ is based on a set of sequences

$$\mathcal{C}_g^N = \{(g_1, g_2, ..., g_N) \in G^N \mid g_1 g_2 \cdots g_N = g \in G\}.$$

- Take a tangent vector at $(g_1, g_2 ..., g_N)$ which can be understood as the tangent vector of a curve $c(t) \in \mathcal{C}_g^N$ passing through $(g_1, g_2 ..., g_N)$ at $t = 0$.

Lemma

In a Lie group G we have $g_i g_{i+1} = g_i' g_{i+1}'$ if and only if there is $h \in G$ such that $g_i' = g_i h$ and $g_{i+1}' = h^{-1} g_{i+1}$.

- By Lemma the curve $c(t)$ is necessarily of the form

$$\begin{aligned} c(t) &= (g_1\gamma_1(t), (\gamma_1(t))^{-1}g_2\gamma_2(t), ..., \\ &\quad (\gamma_{N-2}(t))^{-1}g_{N-1}\gamma_{N-1}(t), (\gamma_{N-1}(t))^{-1}g_N), \end{aligned}$$

such that $\gamma_i : t \in (-\epsilon, \epsilon) \subseteq \mathbb{R} \to G$ are the integral curves of the left invariant vector field corresponding to $X_i \in T_eG$ that passes through the identity, that is $\gamma_i(0) = e$.

- Therefore the tangent space of $\mathcal{C}_g^N$ at $(g_1, \ldots, g_N)$ can be identified with

$$\mathrm{T}_{g_1,\ldots,g_N}\mathcal{C}_g^N = \{(X_1, X_2, ..., X_{N-1}) \in \mathfrak{g}^{N-1} \mid X_i \in T_eG \cong \mathfrak{g}\}.$$

- The curve c is called a variation of $(g_1, g_2..., g_N)$ and $(X_1, X_2..., X_{N-1})$ is called infinitesimal variational of $(g_1, g_2..., g_N)$.

Discrete Euler-Lagrange equations

- Define the discrete action sum $\mathbb{S}L : \mathcal{C}_g^N \to \mathbb{R}$ associated to the Lagrangian L as

$$\mathbb{S}L(g_1, g_2, ..., g_N) = \sum_{k=1}^{N} L(g_k).$$

- According to the Hamilton's principle of critical action, the sequence $(g_1, g_2..., g_N)$ is a solution of the Lagrangian system if and only if $(g_1, g_2..., g_N)$ is a critical point of $\mathbb{S}L$. Therefore, we calculate

$$\frac{d}{dt}\,|_{t=0}\, \mathbb{S}L(c(t)) = \sum_{k=1}^{N-1} \left[\overleftarrow{X}_k(g_k)(L) - \overrightarrow{X}_k(g_{k+1})(L)\right] = 0,$$

where $X_k \in \mathfrak{g}$.

These equations are called to be discrete Euler-Lagrange equations.

In the category of smooth loops, because of the lack of associativity there is not clear variations like what we have in Lie groups. But still we can define the discrete Euler-Lagrange equations using the smooth prolongation of vector fields as follows.

Definition: The discrete Euler-Lagrange equations for a discrete Lagrangian system on a smooth loop G with Lagrangian $L : G \to \mathbb{R}$ is given by equations

$$\overleftarrow{X}(L)(g_i) - \overrightarrow{X}(L)(g_{i+1}) = 0$$

for every $X \in T_{\mathrm{e}}G$, where $\overleftarrow{X}$ and $\overrightarrow{X}$ are the left and right prolongation, respectively. A sequence $g_1, g_2, ...$ of elements G is a solution of the Euler-Lagrange equations if g_i and g_{i+1} satisfy these equations for $i = 1, 2, \ldots$.

Discrete Legendre transformation

We have the discrete Legendre transformation for smooth loops similar to what we have for Lie groups.

- Given a Lagrangian $L : G \to \mathbb{R}$ on smooth loop G with the skew-algebra $\mathfrak{g}$, two discrete Legendre transformations $\mathbb{F}^{+}L = l_g^* \circ dL : G \to \mathfrak{g}^*$ and $\mathbb{F}^{-}L = r_g^* \circ dL : G \to \mathfrak{g}^*$, where $dL : G \to \mathrm{T}^*G$, are as follows

$$\mathbb{F}^{+}L(g)(X) = \overleftarrow{X}(L)(g), \quad \mathbb{F}^{-}L(g)(X) = \overrightarrow{X}(L)(g),$$

  for $X \in \mathfrak{g}$. Clearly, l^* and r^* are the pull backs of left and right translations.

Discrete flow

- Let $\gamma_L : G \to G$ be a smooth map on a smooth loop G for which the couples $(g, \gamma(g))$ are solutions of Euler-Lagrange equations for L. The map $\gamma_L : G \to G$ is called a discrete flow or discrete Lagrangian evolution operator for L.

Proposition

$\gamma_L : G \to G$ is the discrete flow for the Lagrangian $L : G \to \mathbb{R}$ if and only if

$$\mathbb{F}^- L \circ \gamma_L = \mathbb{F}^+ L \,.$$

Theorem

For an inverse smooth loop G the Legendre map $\mathbb{F}^+ L$ is regular at g if and only if $\mathbb{F}^- L$ is regular at g^{-1}. Moreover, $\mathbb{F}^+ L$ is a diffeomorphism if and only if $\mathbb{F}^- L$ is a diffeomorphism.

Regular Lagrangian

Definition: A discrete Lagrangian $L : G \to \mathbb{R}$ on smooth loop G is said to be regular if and only if the Legendre transformation $\mathbb{F}^+L$ is a local diffeomorphism. If $\mathbb{F}^+L$ is global diffeomorphism, L is called to be hyperregular.

Theorem

For an inverse smooth loop G the following are equivalent:

- A discrete Lagrangian $L : G \to \mathbb{R}$ on smooth loop G is to be regular;
- $\mathbb{F}^-L$ is a local diffeomorphism;

Moreover, L is hyperregular if and only if $\mathbb{F}^-L$ is a global diffeomorphism. In this case the discrete Lagrangian evolution operator is a diffeomorphism.

Tangent bundle of smooth loops

Theorem

The tangent bundle TG of a smooth loop G is a smooth loop under the multiplication

$$D_{(g,h)}m(X_g, Y_h) = D_g(r_h)(X_g) + D_h(l_g)(Y_h),$$

for $X_g \in T_g G$ and $Y_h \in T_h G$.

Cotangent bundle of smooth loops

In the category of Lie group, the cotangent bundle of a Lie group is a symplectic groupoid over the dual of the tangent algebra.

- The cotangent bundle of T^*G of the smooth loop G is equipped with a canonical symplectic structure but the lack of associativity is an obstacle for defining a natural loop structure on T^*G analogous to the Lie group.

Theorem

In other words in general there is no natural partial multiplication ('loopoid structure') on T^*G.

Projections on T^*G

For a smooth loop G with the skew-algebra $\mathfrak{g}$ and the dual $\mathfrak{g}^*$:

- There are two projections $\alpha, \beta : T^*G \to \mathfrak{g}^*$ such that

$$\begin{aligned} \langle \beta(\mu_g), X \rangle &= \langle \mu_g, D_e(l_g)(X) \rangle \,, \text{ for } \mu_g \in T^*_g G \text{ and } X \in \mathfrak{g}, \\ \langle \alpha(\nu_h), Y \rangle &= \langle \nu_h, D_e(r_h)(Y) \rangle \,, \text{ for } \nu_h \in T^*_h G \text{ and } Y \in \mathfrak{g} \,. \end{aligned}$$

  In other words,

$$\langle \beta(\mu_g), X \rangle = \left\langle \mu_g, \overleftarrow{X}(g) \right\rangle \,, \qquad \langle \alpha(\nu_h), X \rangle = \left\langle \nu_h, \overrightarrow{X}(h) \right\rangle \,.$$

Setting aside the 'loopoid structure', for any function $L : G \to \mathbb{R}$ on manifold G the submanifolds $dL(G) \subset T^*G$ is a Lagrangian submanifold of the cotangent bundle.

Equivalent definition of Discrete Euler-Lagrange dynamics

Definition: Let G be a smooth loop and L a discrete Lagrangian function on it. A sequence $\mu_1, ..., \mu_n \in \mathsf{T}^*G$ satisfies the discrete Lagrangian dynamics if $\mu_1, ..., \mu_n \in dL(G)$ and they are composable sequence in T^*G, that is

$$\beta(\mu_k) = \alpha(\mu_{k+1}), \quad k = 1, ..., n-1\,.$$

Theorem

Let G be a smooth loop and $L : G \to \mathbb{R}$ a discrete Lagrangian. Then a sequence $\mu_1, ..., \mu_n \in \mathsf{T}^*G$ satisfies the discrete Lagrangian dynamics of $dL(G) \subset \mathsf{T}^*G$ if and only if

$$\mu_k = dL(g_k) \quad \text{for some} \quad g_k \in G, \quad k = 1, \ldots, n,$$

and the discrete Euler -Lagrangian equations
$\overleftarrow{X}(L)(g_k) = \overrightarrow{X}(L)(g_{k+1})$ are satisfied, $k = 1, \ldots, n-1$.

Unitary octonions

The manifold of unitary octonions $\mathbb{O}_1 = \{a \in \mathbb{O},\ \|a\| = 1\}$ is closed under the octonion multiplication and therefore is an inverse smooth loop under the octonion multiplication. Consider the tangent space

$$\mathbf{o}_1 = \mathrm{T}_{e_0}\mathbb{O}_1 = span\{e_1, ..., e_7\}.$$

to $\mathbb{O}_1$ inside the vector space $\mathbb{O}$.

- The tangent bundle $\mathrm{T}\mathbb{O}_1$ is given by the left (or right) prolongation of imaginary octonions, that is $\mathrm{T}\mathbb{O}_1 = span\{\overleftarrow{e}_1, ..., \overleftarrow{e}_7\}$, where

$$\overleftarrow{e}_i(a) = D_{e_0}(l_a)(e_i) = ae_i \in \mathrm{T}_a\mathbb{O}_1, \quad a \in \mathbb{O}_1.$$

- Then $[\overleftarrow{e_i}, \overleftarrow{e_j}](e_0) = 2e_ie_j$ implies that $(\mathbf{o}_1, [e_i, e_j] = 2e_ie_j)$ is the skew-algebra (Mal'cev algebra) corresponding to the smooth loop $\mathbb{O}_1$.

Discrete Euler-Lagrange equations on $\mathbb{O}_1$

Discrete Euler-Lagrange equations

$$\overleftarrow{X}(L)(g_i) = \overrightarrow{X}(L)(g_{i+1}), \quad X \in \mathrm{T}_e G\,.$$

Let $L : \mathbb{O}_1 \to \mathbb{R}$ be a Lagrangian function, then the discrete Euler-Lagrange equations read as recurrence equation

$$\overleftarrow{e}_i(L)(a_n) = \overrightarrow{e}_i(L)(a_{n+1})\,,$$

where $\overleftarrow{e}_i(a) = D_{e_0}(l_a)(e_i) = ae_i$ and $\overrightarrow{e}_i(a) = D_{e_0}(r_a)(e_i) = e_i a$ are the left and the right prolongation by the element $a \in \mathbb{O}_1$. A solution for those equations is a sequence of elements $\mathbb{O}_1$.

Note: We interpret the tangent and cotangent vectors to $\mathbb{O}_1$ as octonions.

Linear Lagrangian

Take the Lagrangian as a linear function $L = e^1 = \langle e_1, \cdot \rangle$, then

$$\overleftarrow{e}_i(L)(a_n) = (a_n e_i)(L)(a_n) = \langle e_1, a_n e_i \rangle .$$

The right-hand side of the above relation is obtained by taking the integral curve $\gamma(t) = a_n + t a_n e_i$ for the tangent vector $a_n e_i$ and then we have

$$\frac{d}{dt}|_{t=0} L(a_n + t a_n e_i) = \frac{d}{dt}|_{t=0} \langle e_1, a_n + t a_n e_i \rangle .$$

Therefore, by the definition the Euler-Lagrange equations are

$$\langle e_1, a_n e_i - e_i a_{n+1} \rangle = 0, \quad \text{for} \quad i = 1, ..., 7.$$

Every element $a_n \in \mathbb{O}_1$ can be written as $a_n = \alpha_n^0 + \alpha_n^k e_k$ such that $\sum_{s=0}^{7} |\alpha_n^s|^2 = 1$, so the above equations turn to

How to find the solutions?

$$\sum_{k=1}^{7}\left\langle e_1, \alpha_n^0 e_i - \alpha_{n+1}^0 e_i + (\alpha_n^k + \alpha_{n+1}^k) e_k e_i \right\rangle = 0, \quad \text{for} \quad i = 1, ..., 7.$$

- Now, if $i = 1$, since $\langle e_1, e_1 \rangle = 1$ and $\langle e_1, e_k e_1 \rangle = 0$ for $k \neq 0$, we get $\alpha_n^0 - \alpha_{n+1}^0 = 0$.
- If $i > 1$, the two first expressions are zero because $\langle e_1, e_i \rangle = 0$ for $i \neq 1$ and thus we left by the third expression, that is

$$\sum_{k=1}^{7}\left\langle e_1, (\alpha_n^k + \alpha_{n+1}^k) e_k e_i \right\rangle = 0, \quad \text{for} \quad i = 2, ..., 7\,.$$

But for each k, there is some i such that $e_k e_i = \pm e_1$ and all $i's$ are used.

Consequently, we get the Euler-Lagrange equations

$$\alpha_n^0 - \alpha_{n+1}^0 = 0, \quad \alpha_n^k + \alpha_{n+1}^k = 0, \quad k = 1, ..., 7.$$

It is obvious from the equations that the solution of Euler-Lagrange equations for the linear Lagrangian L are just the conjugate pairs in $\mathbb{O}_1$.

Regularity of Lagrangian

Next step is to check whether the Lagrangian L is regular or hyperregular. So, we would need to find the Legendre maps associated with L.

- Consider the tangent skew-algebra $\mathbf{o}_1$ and the its dual $\mathbf{o}_1^*$ with the basis $\{e^1, ..., e^7\}$. The corresponding Legendre maps $\mathbb{F}^+L, \mathbb{F}^-L : \mathbb{O}_1 \to \mathbf{o}_1^*$ are

$$\mathbb{F}^+L(a) = \sum_{i=1}^{7} \overleftarrow{e}_i(a)(L)e^i, \quad \mathbb{F}^-L(a) = \sum_{i=1}^{7} \overrightarrow{e}_i(a)(L)e^i, \quad a \in \mathbb{O}_1.$$

Let us remark that there is no hyperregular Lagrangian on unit octonions $\mathbb{O}_1$, because the Legendre maps are $\mathbb{F}^+L, \mathbb{F}^-L : S_7 \to \mathbb{R}^7$ which cannot be diffeomorphisms. Thus we can only find Lagrangians which are (locally) regular.

Regularity of linear Lagrangian

- For the linear Lagrangian $L = \langle e_1, \cdot \rangle = e^1$, we have $\overrightarrow{e}_i(a)(e^1) = \langle e_1, e_i a \rangle$ and corresponding Legendre map

$$\mathbb{F}^- L(a) = \sum_{i=1}^{7} \overrightarrow{e}_i(e^1)(a) e^i = \sum_{i=1}^{7} \langle e_1, e_i a \rangle \, e^i, \quad a \in \mathbb{O}_1.$$

- The Lagrangian $L = e^1$ is not regular at e_0 because

$$\mathbb{F}^- L(e_0) = \sum_{i=1}^{7} \langle e_1, e_i \rangle \, e^i = e^1$$

and

$$D_{e_0}(\mathbb{F}^- L)(e_1) = \frac{d}{dt}_{|_{t=0}} \mathbb{F}^- L(e_0 + t e_1) = \sum_{i=1}^{7} \langle e_1, e_i e_1 \rangle \, e^1 = 0.$$

Lagrangian as total kinetic energy

Take the Lagrangian $L = \sum_{k=1}^{7} m_k \frac{(e^k)^2}{2}$, where $m_k > 0$, as the 'total kinetic energy' of the system. Then

$$\overrightarrow{e}_i(a)(L) = \sum_{k=1}^{7} m_k e^k(a) \overrightarrow{e}_i(a)(e^k) = \sum_{k=1}^{7} \langle m_k e_k, a \rangle \langle e_k, e_i a \rangle ,$$

and

$$\overleftarrow{e}_i(a)(L) = \sum_{k=1}^{7} m_k e^k(a) \overleftarrow{e}_i(a)(e^k) = \sum_{k=1}^{7} \langle m_k e_k, a \rangle \langle e_k, a e_i \rangle ,$$

The discrete Euler-Lagrange equations is

$$\sum_{k=1}^{7} \langle m_k e_k, a_n \rangle \langle e_k, e_i a_n \rangle = \sum_{k=1}^{7} \langle m_k e_k, a_{n+1} \rangle \langle e_k, a_{n+1} e_i \rangle , \quad i = 1, \ldots, 7 .$$

If we write $a_n = \sum_{s=0}^{7} \alpha_n^s e_s$ and $a_{n+1} = \sum_{s=0}^{7} \alpha_{n+1}^s e_s$, then this reduces to the quadratic recurrence equation

$$\alpha_n^i \alpha_n^0 = \alpha_{n+1}^i \alpha_{n+1}^0, \quad i = 1, ..., 7$$

which does not depend on m_k.

Since $\sum_{i=0}^{7} \left(\alpha_n^i\right)^2 = \alpha_n^0 + \sum_{i=1}^{7} \left(\alpha_n^i\right)^2 = 1$, we have

$$\sum_{i=1}^{7} \left(\alpha_n^i \alpha_n^0\right)^2 = (\alpha_n^0)^2 \left(1 - (\alpha_n^0)^2\right) = (\alpha_{n+1}^0)^2 \left(1 - (\alpha_{n+1}^0)^2\right)$$

Solution of Euler-Lagrange equations w.r.t L

$$\alpha_n^i \alpha_n^0 = \alpha_{n+1}^i \alpha_{n+1}^0, \quad i = 1, ..., 7$$

If α_n^0 and α_{n+1}^0 are close to 1, thus a_n and a_{n+1} are close to e_0, then $\alpha_n^0 = \alpha_{n+1}^0$, since the function $f(x) = x^2(1 - x^2)$ is monotonic on the interval $[1/\sqrt{2}, 1]$, so we get only trivial solutions $a_n = a_{n+1}$.

Theorem

The discrete Euler-Lagrange equation for the Lagrangian L as the total kinetic energy on the smooth loop $\mathbb{O}_1$ admits in a neighbourhood of e_0 only trivial solutions.

Note

However that there are nontrivial solutions lying outside the neighbourhood of e_0. For instance,

- $(a_n, a_{n+1}) = \big((0, a'_n), (0, a'_{n+1})\big)$, where a'_n and a'_{n+1} represent arbitrary imaginary and unitary octonions.

We can also take $A = \alpha_n^0 \neq \alpha_{n+1}^0 = B$, $|A| < |B| < 1$, which are different solutions of Euler Lagrange equations. Note that in this case B is not close to 1, as $|A| < 1/\sqrt{2}$.

- Then, for any $a_n = (A, a'_n)$, where a'_n represents an imaginary octonion of length $\sqrt{1 - A^2}$, the pair $((A, a'_n), (B, ua'_n))$, with $u^2 = (1 - B^2)/(1 - A^2)$, is a solution of the Euler-Lagrange equation.

Regularity of Lagrangian L

$$\mathbb{F}^{-}L(a) = \sum_{i,k=1}^{7} \langle m_k e_k, a \rangle \langle e_k, e_i a \rangle e^i.$$

$$\mathbb{F}^{+}L(a) = \sum_{i,k=1}^{7} \langle m_k e_k, a \rangle \langle e_k, a e_i \rangle e^i.$$

- $\mathbb{F}^{-}L(e_s) = 0$ for all $s \in \{0, ..., 7\}$, hence, changing the base of imaginary octonions, we infer that $\mathbb{F}^{-}L(a) = 0$ for any imaginary octonion, that supports once more the fact that the Lagrangian is not hyperregular.

Result

$$D_{e_0}\mathbb{F}^{-}L(e_s) = \sum_{i,k=1}^{7} \langle m_k e_k, e_s\rangle \langle e_k, e_i\rangle e^i = m_s e^s = D_{e_0}\mathbb{F}^{+}L(e_s)\,.$$

- Under identification $T_{e_0}\mathbb{O}_1 = \mathbf{o}_1 = \mathbf{o}_1^*$ the differential of $\mathbb{F}^{-}L : \mathbb{O}_1 \to \mathbf{o}_1$ at e_0, and similarly for $\mathbb{F}^{+}L : \mathbb{O}_1 \to \mathbf{o}_1$, can be identified with the diagonal automorphism on $\mathbf{o}_1$ for which $e_s \mapsto m_s e_s$. In particular $\mathbb{F}^{-}L$ and $\mathbb{F}^{+}L$ are regular in a neighbourhood of e_0.

Refrence and Problem

- **J. Grabowski and Z. Ravanpak**, *Discrete mechanics on unitary octonions*, *arXiv:2008.10971 [math-ph]* (2020).

Since the discrete mechanics has its version on Lie groupoids, a natural question is to find the non-associative generalizations of Lie groupoids and to construct discrete mechanics on them. The first ideas of such objects, smooth loopoids, can be found in

- **J. Grabowski**, *An introduction to loopoids*, *Comment. Math. Univ. Carolin.* **57** *(2016), 515-526*.

- **M. Kinyon**, *The coquecigrue of a Leibniz algebra*, *preprint, 2003*.


THANK YOU!