Piecewise straightening and Lipschitz simplicial volume

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Abstract

We study the Lipschitz simplicial volume, which is a metric version of the simplicial volume. We introduce the piecewise straightening procedure for singular chains, which allows us to generalize the proportionality principle and the product inequality to the case of complete Riemannian manifolds of finite volume with sectional curvature bounded from above. In particular we obtain yet another proof of the proportionality principle in the compact case by a direct approximation of the smearing map.

1 Introduction

The simplicial volume is a homotopy invariant of manifolds defined for a closed manifold $M$ as

$$\|M\| := \inf \{|c|_1 : c \text{ is a fundamental cycle with } \mathbb{R} \text{ coefficients}\},$$

where $| \cdot |_1$ is an $\ell^1$-norm on $C_*(M, \mathbb{R})$ (which we will denote for simplicity $C_*(M)$) with respect to the basis consisting of singular simplices. Although the definition is relatively straightforward, it has many applications. Most of them are mentioned in the work of Gromov [3], one of the most important is the use to the degree theorems. In general, by the degree theorem we understand a bound on the degree of a continuous map $f : M \to N$ between two $n$-dimensional Riemannian manifolds

$$\deg(f) \leq \frac{\text{const}_n}{\text{vol}(M)} \cdot \text{vol}(N).$$

Such a theorem may obviously require additional assumptions. The reason why the simplicial volume is suitable for establishing such theorems is its functoriality, i.e. if $f : M \to N$ is a map between two closed manifolds then

$$\|N\| \leq \deg(f) \cdot \|M\|.$$

One obtains easily that if $\|N\| \neq 0$, then

$$\deg(f) < \frac{\|M\|}{\|N\|}.$$

Under some curvature assumptions, Gromov proved in [3] that for a given Riemannian manifold $M$ we have $\|M\| \leq \text{const}_n \cdot \text{vol}(M)$ and $\|M\| \geq \text{const}_n \cdot \text{vol}(M)$, which imply the degree theorem if the curvature assumptions are satisfied.

In most cases simplicial volume is very difficult to compute exactly. However, it has a few properties which can be used to approximate it or at least decide if it is zero or not. Two of them which we are interested in are the product inequality and the proportionality principle.

Theorem 1.1 ([3]). Let $M$ and $N$ be two compact manifolds. Then the following inequality holds

$$\|M\| \cdot \|N\| \leq \|M \times N\|.$$

Theorem 1.2 ([3], [10]). Let $M$ and $N$ be two compact Riemannian manifolds. Assume also that their universal covers are isometric. Then

$$\frac{\|M\|}{\text{vol}(M)} = \frac{\|N\|}{\text{vol}(N)}.$$
A natural question to ask is if these properties generalise somehow to the non-compact case. In order to have a fundamental class, one needs to consider $\ell^1$ norm on locally finite singular chains instead of just (finite) singular chains. In this case simplicial volume obviously does not have to be finite. Unfortunately, neither of the above properties holds in such generality. The product inequality does not hold because of another result of Gromov from [3] that the simplicial volume of a product of at least 3 open manifolds is 0, while there are examples of products of two such manifolds with nonzero simplicial volume [9]. The proportionality principle fails because of a similar reason. Take a product of three non-compact, locally symmetric space of finite volume. Its simplicial volume vanishes, but on the other hand there always exists a compact locally symmetric space with isometric universal cover [1] and the simplicial volume of closed locally symmetric spaces of non-compact type is known to be nonzero [6].

The solution to the problems above, also proposed by Gromov in [3], is to consider a geometric variant of simplicial volume by taking only Lipschitz chains. This way one obtains the Lipschitz simplicial volume

$$\|M\|_{Lip} := \inf \{ |\alpha|_1 : \alpha \in C^f_*(M) \text{ is a fundamental cycle with } \mathbb{R} \text{ coefficients, } Lip(\alpha) < \infty \}.$$

In the case of closed manifolds the classical and the Lipschitz simplicial volumes coincide. Löh and Sauer studied the above invariant in [9] and proved that it may be a proper generalisation of the simplicial volume to the case of complete Riemannian manifolds of finite volume, not necessarily compact. In particular, in the presence of non-positive curvature they proved the proportionality principle and the product inequality. The main result of this article is a generalisation of their proofs to the case of manifolds with curvature bounded from above.

**Theorem 1.3** (Product inequality). Let $M$ and $N$ be two complete, Riemannian manifolds with sectional curvatures bounded from above. Then the following inequality holds

$$\|M\|_{Lip} \cdot \|N\|_{Lip} \leq \|M \times N\|_{Lip}.$$

**Theorem 1.4** (Proportionality principle). Let $M$ and $N$ be two complete Riemannian manifolds of finite volume with sectional curvatures bounded from above. Assume also that their universal covers are isometric. Then

$$\frac{\|M\|_{Lip}}{\text{vol}(M)} = \frac{\|N\|_{Lip}}{\text{vol}(N)}.$$

In the work of Löh and Sauer, non-positive curvature assumption is needed to introduce the procedure of straightening the simplices. Namely, given a singular chain, one can homotopy it to the chain consisting of straight simplices by using the fact that in simply connected, non-positively curved Riemannian manifolds geodesics are unique. We develop the straightening procedure and apply it piecewise, which allows us to use it in the case of manifolds with curvature bounded from above.

Since for closed manifolds sectional curvature is always bounded and the Lipschitz simplicial volume equals the classical one, we obtain yet another proof of Theorem 1.2. We follow Thurston’s approach from [11] (used in [10]), however, we obtain the proof without any use of bounded cohomology by approximating directly the smearing map.

The proportionality principle provides direct connection between Lipschitz simplicial volume and volume, therefore one obtains immediately

**Corollary 1.5.** Let $f : M \to N$ be a proper Lipschitz map between two complete Riemannian manifolds of finite volume with sectional curvatures bounded from above, which in addition have isometric universal covers. Assume moreover that $\|N\|_{Lip} \neq 0$. Then

$$\text{deg}(f) \leq \frac{\text{vol}(M)}{\text{vol}(N)}.$$

Löh and Sauer combined this fact for non-positively curved manifolds with the facts that Lipschitz simplicial volume is strictly positive for locally symmetric spaces of non-compact type of finite volume [1, 6, 9], that there are only finitely many symmetric spaces (with the standard metric) in each dimension and that $\|N\| \leq C_n \text{vol}(N)$ if $\text{Ricci}(N) \geq -(n-1)$ and $\text{sec}(N) \leq 1$ [3, 9] to prove the following theorem.
Theorem 1.6 (Degree theorem, [2, 6, 9]). For every \( n \in \mathbb{N} \) there is a constant \( C_n > 0 \) with the following property: Let \( M \) be an \( n \)-dimensional locally symmetric space of non-compact type with finite volume. Let \( N \) be an \( n \)-dimensional complete Riemannian manifold of finite volume with \( \text{Ricci}(N) \geq -(n-1) \) and \( \text{sec}(N) \leq 1 \), and let \( f : N \to M \) be a proper Lipschitz map. Then

\[
\deg(f) \leq C_n \cdot \frac{\text{vol}(N)}{\text{vol}(M)}.
\]

Possible generalisation of the above theorem depends on the further results on non-vanishing of the Lipschitz simplicial volume. At the moment most results in this direction are based on the proportionality principle indicated above, non-vanishing of the simplicial volume for negatively curved spaces [11], locally symmetric spaces of non-compact type [6] and their products and connected sums [3].

Organization of this work

In Section 2 we recall the basic facts about straight simplices and develop the piecewise straightening procedure for singular chains. Section 3 is devoted to the proofs of theorems 1.2, 1.3 and 1.4.

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2 Piecewise straightening of simplices

In Section 2.1 we recall basic notions concerning geodesic simplices and joins and prove that under some curvature and diameter conditions a geodesic join of Lipschitz maps is also Lipschitz. In Section 2.2 we define a piecewise straightening procedure for locally finite Lipschitz chains. Finally in Section 2.3 we define variants of piecewise barycentric homology, which will be used to prove Theorem 1.4.

2.1 Straight simplices and homotopies

Let \( M \) be an \( n \)-dimensional, complete, simply connected Riemannian manifold of finite volume and sectional curvature bounded from above by \( 0 < K < \infty \). Recall that \( M \) has therefore injectivity radius \( \rho \geq \frac{1}{\sqrt{K}} \). Before we develop the straightening procedure, we need some technical facts about geodesic simplices in \( M \). If \( x, y \in M \) are two points such that \( d_M(x, y) < \rho \), denote by \([x, y]\) the unique shortest geodesic joining them. Following [9], we can define the geodesic join of two maps \( f, g : X \to M \).

Definition 2.1. Let \( f, g : X \to M \) be two maps such that \( \text{diam}(\text{im}(f) \cup \text{im}(g)) < \rho \). Then there exists an unique homotopy \([f, g] : X \times [0, 1] \to M \) defined by \((x, t) \mapsto [f(x), g(x)](t)\).

Consequently we can define geodesic simplices. We identify the standard simplex \( \Delta^k \) with the subset \( \{(z_0, \ldots, z_k) \in \mathbb{R}^{k+1}_{\geq 0} : \sum_{i=0}^k z_i = 1\} \) equipped with the Euclidean metric and \( \Delta^{k-1} \) with the subset \( \{(z_0, \ldots, z_k) \in \Delta^k : z_k = 0\} \).

Definition 2.2. The geodesic simplex \([x_0, \ldots, x_k] : \Delta^k \to M \) with vertices \( x_0, \ldots, x_k \) such that \( \max_{i,j=0,\ldots,k} d_M(x_i, x_j) < \frac{\rho}{\sqrt{k+1}} \) is defined inductively by

- \([x_0](\Delta^0) = \{x_0\} \subset M\);
- \([x_0, \ldots, x_k]((1-t)s + t(0, \ldots, 0, 1)) = [[x_0, \ldots, x_{k-1}](s), x_k](t) \) for \( s \in \Delta^{k-1} \).

To prove that the definition is correct we need to check that \( \text{diam}([x_0, \ldots, x_k] \cup \{x_k\}) < \rho \). We will use the following lemma
Lemma 2.3. \( \text{diam}([x_0, \ldots, x_{k-1}]) \leq (2^k - 2)D, \) where \( D = \max_{i,j=0,\ldots,k-1} d_M(x_i, x_j). \)

Proof. We prove the statement by induction. For \( k = 2 \) the statement is obvious. For \( k > 2 \) let \( x, y \in [x_0, \ldots, x_{k-1}] \). We have

\[
\begin{align*}
d_M(x, y) & \leq d_M(x, x_{k-1}) + d_M(x_{k-1}, y) \\
& \leq 2(d_M([x_0, \ldots, x_{k-2}], x_{k-1}) + \text{diam}([x_0, \ldots, x_{k-2}])) \\
& \leq 2(D + (2^{k-1} - 2)D) \leq 2(2^{k-1}D - D) = (2^k - 2)D.
\end{align*}
\]

Now we can calculate

\[
\begin{align*}
\text{diam}([x_0, \ldots, x_{k-1}] \cup \{x_k\}) & \leq \text{diam}([x_0, \ldots, x_{k-1}]) + d_M([x_0, \ldots, x_{k-1}], x_k) \\
& < (2^k - 2)rac{\rho}{2^k - 1} + \frac{\rho}{2^k - 1} < \rho.
\end{align*}
\]

To proceed, we need the positive curvature analogue of Proposition 2.1 in [9].

Proposition 2.4. Let \( M \) be a complete, simply connected Riemannian manifold with sectional curvature bounded from above by \( 0 < K < \infty \) and let \( X \) be a compact, smooth manifold (possibly with boundary). Let \( f, g : X \to M \) be two Lipschitz maps such that \( \text{diam}(\text{im}(f) \cup \text{im}(g)) < C_K \), where \( C_K \) is a constant depending only on \( K \). Then \([f,g] \) has Lipschitz constant depending only on \( K \) and the Lipschitz constants for \( f \) and \( g \). Moreover, \([f,g] \) is smooth if \( f \) and \( g \) are.

We need two technical lemmas concerning Riemannian geometry. First is the technical result proved in [9](Proposition 2.6).

Lemma 2.5. Let \( M \) be a complete, simply connected Riemannian manifold whose sectional curvature is bounded from above by \( 0 < K < \infty \). Then every geodesic simplex \( \sigma \) of diameter less than \( \frac{\pi}{2\sqrt{K}} \) is smooth. Further, there is a constant \( L > 0 \) such that every geodesic \( k \)-simplex \( \sigma \) of diameter less than \( \frac{\pi}{2\sqrt{K}} \) satisfies \( ||T_\sigma \sigma|| < L \) for every \( x \in \Delta^k \).

Lemma 2.6. Let \( M \) be a complete, simply connected Riemannian manifold whose sectional curvature is bounded from above by \( 0 < K < \infty \). Consider the geodesic triangle in \( M \) with vertices \( x,y,z \) such that \( \text{diam}([x,y,z]) < C_K = \frac{\pi}{24\sqrt{K}} \). Then there exists a constant \( D_K \), depending only on the curvature bound \( K \), such that for any \( t \in [0,1] \)

\[
d_M([x,z](t), [y,z](t)) \leq D_K d_M(x,y)
\]

Proof. Consider the extension (in any direction) of \([x,y]\) to a geodesic of length \( \frac{\pi}{2\sqrt{K}} \) and denote the endpoints of such geodesic by \( x_0, y_0 \). Now consider the geodesic triangle \([x_0, y_0, z]\). Note that

\[
d_M(x_0, y) \leq d_M(x_0, x) + d_M(x, z) < \frac{\pi}{24\sqrt{K}} + \frac{\pi}{24\sqrt{K}} = \frac{\pi}{12\sqrt{K}}.
\]

similarly, \( d(y_0, z) < \frac{\pi}{12\sqrt{K}} \), hence, by Lemma 2.3 we have \( \text{diam}([x_0, y_0, z]) < \frac{6\pi}{12\sqrt{K}} = \frac{\pi}{2\sqrt{K}} \). We can therefore use Lemma 2.5 to conclude that the diffeomorphic simplex map \( \sigma : \Delta \to M \) from the standard 2-simplex onto \([x_0, y_0, z]\) is Lipschitz with constant \( L \) depending on the size of the simplex (which is bounded). Hence

\[
d_M([x,z](t), [y,z](t)) \leq L\cdot d_\Delta(\sigma^{-1}([x,z](t)), \sigma^{-1}([y,z](t))) \leq L\cdot d_\Delta(\sigma^{-1}(x), \sigma^{-1}(y)) = L\sqrt{2}\frac{d(x,y)}{\pi/24\sqrt{K}}
\]

so one can take \( D_K = \frac{24L\sqrt{2K}}{\pi} \).

Proof of Proposition 2.4. Put \( C_K = \frac{\pi}{24\sqrt{K}} \). To prove smoothness in the case \( f \) and \( g \) are smooth, one can rewrite \([f,g] \) as

\[
[f,g](x,t) = \exp_{f(x)}(t \cdot \exp_{f(x)}^{-1}(g(x)))
\]
where we use the fact that $\exp : TM_\rho \to M \times M$, $\exp(x, t) \mapsto (x, \exp_x(t))$ is a diffeomorphism onto its image, where $TM_\rho = \{(x, t) \in TM : d(x, \exp_x(t)) < \rho\}$ and $\rho \geq \frac{2}{\sqrt{\kappa}}$ is injectivity radius of $M$.

Now, let $(x, t), (x', t') \in X \times [0, 1]$. We have

$$d_M([f, g](x, t), [f, g](x', t')) \leq d_M([f(x), g(x)](t), [f(x), g(x)](t')) + d_M([f(x), g(x)](t'), [f(x'), g(x')])(t'))$$

The first term can be easily estimated as follows

$$d_M([f(x), g(x)](t), [f(x), g(x)](t')) \leq |t - t'| \cdot d_M(f(x), g(x)) \leq |t - t'| \cdot \text{diam}(im(f) \cup im(g)).$$

Recall that by assumption $\text{diam}(im(f) \cup im(g)) < C_K = \frac{\pi}{24\sqrt{\kappa}}$. The second term can be estimated using Lemma 2.6

$$d_M([f(x), g(x)](t'), [f(x'), g(x')](t')) \leq d_M([f(x), g(x)](t'), [f(x), g(x)](t')) + d_M([f(x), g(x)](t'), [f(x'), g(x')])(t'))$$

Finally, we obtain

$$d_M([f, g](x, t), [f, g](x', t')) \leq |t - t'| C_K + D_K(Lip(f) + Lip(g))d_X(x, x')$$

$$\leq (C_K + D_K(Lip(f) + Lip(g)))d_X \times [0, 1](x, t), (x', t')).$$

2.2 Piecewise straightening procedure

The straightening procedure on non-positively curved manifolds is well known and applied successfully to many problems. Roughly speaking, given a complete, simply connected Riemannian manifold $M$ with non-positive curvature and a singular simplex $\sigma : \Delta^k \to M$ with vertices $x_0, \ldots, x_k$ the straightening of this simplex is the geodesic simplex $[x_0, \ldots, x_k]$. Because geodesics on $M$ are unique, there exists a (unique) geodesic homotopy between $\sigma$ and $[x_0, \ldots, x_k]$ which is defined as the geodesic join $[\sigma, [x_0, \ldots, x_k]]$. We can apply the same procedure to the singular simplex $\sigma$ on non necessarily simply-connected Riemannian manifold $M$ by taking its lift to the universal cover $\tilde{\sigma} : \Delta^k \to \tilde{M}$, applying straightening there and pushing down the homotopy. It can be shown that it does not depend on the choice of the lift, therefore it can be extended to the straightening on singular chains and induces an isomorphism on homology. The same applies to locally finite Lipschitz chains and homology. The straightening procedure has also the advantage that it does not increase $l_1$-norm on chains, therefore the above isomorphism turns out to be isometric and the simplicial volume can be computed by considering only straight simplices. This fact, together with a careful control of the set of vertices, is a key to prove e.g. proportionality principle for Lipschitz simplicial volume and inequalities for products of manifolds, assuming all the manifolds have non-positive curvature.

The fact which obviously fails if we consider possibly positive curvature is the existence of unique geodesics on the simply connected manifolds. However, they exist locally, therefore procedure can be also extended, but not globally. In particular, we need a ‘skeleton’ for the procedure to continue.

Let $M$ be a complete, n-dimensional Riemannian manifold with curvature bounded from above by $0 < K < \infty$ and let $\bar{M}$ be its universal cover. Choose a locally finite family $(F_j)_{j \in J}$ of pairwise disjoint Borel subsets of $\bar{M}$ together with points $z_j \in F_j$ for $j \in J$ such that

- $\bigcup_{j \in J} F_j = \bar{M}$;
- the families $(F_j)_{j \in J}$ and $(z_j)_{j \in J}$ are $\pi_1(M)$-invariant;
- $\text{diam}(F_j) < E_{n, K} = \frac{C_K}{2^{(4/\kappa - 2)}}$ for every $j \in J$, where $C_K$ is constant from Proposition 2.4.
A family with properties described above always exists. To see this choose a triangulation of \( M \) (which exists because \( M \) is Riemannian) and divide every triangle into a locally finite family of disjoint Borel sets with sufficiently small diameters and such that restrictions of the covering \( \tilde{M} \to M \) to these sets are trivial. Then lift the sum of all these families to \( \tilde{M} \).

**Definition 2.7.** For \( m \in \mathbb{N} \) an \((m-)\)piecewise straight simplex \( \sigma \) on \( M \) (with respect to \((F_j)_{j \in J}\)) is a singular simplex \( \sigma : \Delta \to M \) with the property that (any) lift of any simplex in \( S^m(\sigma) \) to \( \tilde{M} \) is geodesic with vertices in \((\tilde{z}_j)_{j \in J} \), where \( S^m(\sigma) \) is \( m \) times iterated barycentric subdivision operator \( S \).

We say that a (locally finite) chain \( c = \sum_{i \in I} a_i \sigma_i \in C_*(M) \) is piecewise straight (with respect to \((F_j)_{j \in J}\)) if there exists \( m \in \mathbb{N} \) such that every \( \sigma_i, i \in I \), is \( m \)-piecewise straight.

Let \( \sigma : \Delta^k \to M \) for \( k \leq n \) be a singular simplex with Lipschitz constant \( L \). We define the straightening of \( \sigma \) (with respect to \((F_j)_{j \in J}\)) as follows. Choose any lift \( \tilde{\sigma} : \Delta^k \to \tilde{M} \) of \( \sigma \) to \( \tilde{M} \) such that each simplex in \( \tilde{S}(\tilde{\sigma}) = \sum_i \tilde{\sigma}_i \) has diameter less than \( E_{n,K} \). Such \( \tilde{\sigma} \) exists because diameter of subdivided simplices tends to 0 ([4], Corollary 9.4.9). Moreover, we can choose \( \tilde{\sigma} \) depending only on \( n, K \) and the Lipschitz constant of \( \sigma \). Now for any \( \tilde{\sigma} \), and its vertices \( \tilde{z}_i, \tilde{z}_{i,0}, \ldots, \tilde{z}_{i,k} \) consider the sequence of points \( z_i, 0, \ldots, z_{i,k} \) such that \( \tilde{z}_{i,l} \in F_i, l \leq \ldots \leq k \). Note that the straightening defined as above does not depend on the lift of \( \sigma \).

Hence

\[
\max_{l,m=0,\ldots,k} d_{\tilde{M}}(z_{i,l}, z_{i,m}) \leq d_{\tilde{M}}(z_{i,l}, x_{i,l}) + \text{diam}(\tilde{\sigma}_j) + d_{\tilde{M}}(z_{i,m}, x_{i,m}) < 3E_{n,K}.
\]

The geodesic simplex \([z_{i,0}, \ldots, z_{i,k}]\) exists because

\[
3E_{n,K} = \frac{3C_K}{2(3 \cdot 2^n - 2)} < \frac{2\pi}{48\sqrt{K(3 \cdot 2^n - 2)}} < \frac{2\pi}{\sqrt{K(3 \cdot 2^n - 2)}} < \frac{2\pi}{\sqrt{K(2^n - 1)}} \leq \frac{\rho}{2^n - 1}.
\]

By Lemma 2.3 we have also

\[
\text{diam}([z_{i,0}, \ldots, z_{i,k}]) < 3(2^{k+1} - 2)E_{n,K}.
\]

It follows from Proposition 2.4 that there exists a Lipschitz homotopy \( H_j : \Delta \times I \to \tilde{M} \) between these simplices, with Lipschitz constant depending only on \( \text{Lip}(\sigma) \), \( m \) and \( K \). Define

\[
\text{str}_m(\sigma) = p_{\tilde{M}}((S^m(\sigma))^{-1} \sum_{i \in I} [z_{i,0}, \ldots, z_{i,k}]).
\]

It is indeed well defined: because the family \((F_j)_{j \in J}\) is \( \pi_1(M) \)-invariant, the construction does not depend on the lift of \( \sigma \). Moreover, the above construction is consistent with taking boundaries in the sense that if \( \iota : \Delta^{k-1} \to \Delta^k \) is the canonical injection onto one of the faces then \( H_j \circ (\tilde{\sigma}_j \circ \iota, \cdot, -) \) depends only on the singular \((k-1)\)-simplex \( \tilde{\sigma}_j \circ \iota \), not on \( i \). Therefore \( \sum_i [z_{i,0}, \ldots, z_{i,n}] \) lies in the image of \( S^m(\sigma) \) and we can choose the preimage in an obvious way.

Let \( c = \sum_{i \in I} a_i \sigma_i \) be a locally finite Lipschitz chain with Lipschitz constant \( L \). We see that we can choose \( m \in \mathbb{N} \), depending only on \( n, L \) and \( K \), such that \( \text{str}_m(\sigma_i) \) is defined for every \( i \), so we can define \( \text{str}_m(c) \) simply as \( \sum_i a_i \text{str}_m(\sigma_i) \). Note that the straightening defined as above does not define a chain operator \( C_{s \leq n}^{f,Lip}(\tilde{M}) \to C_{s \leq n}^{f,Lip}(M) \), where \( C_{s \leq n}^{f,Lip}(M) \) are locally finite Lipschitz chains, because we cannot choose \( m \) uniformly. However, it is sufficient for a proof of the following fact.

**Proposition 2.8.** Every homology class \( \xi \) in \( H_{s \leq n}^{f,Lip}(M) \) can be represented by a piecewise straight chain with vertices in \((z_j)_{j \in J} \). Moreover, \( L^1 \) semi-norm on \( H_{s \leq n}^{f,Lip}(M) \) can be computed on piecewise straight chains.
Moreover, by construction boundaries of each simplex, hence we can subdivide $\bigcup\Delta_i \times I \to M$ joining $c$ and $str_m(c)$, which exists by the above construction. It is also Lipschitz by construction and Proposition 2.4, and commutes with taking $\overline{\partial}c$. Hence $[c - \partial \partial c] = [str_m(c)]$. Moreover, by construction $|c||str_m(c)|$, hence the second statement follows.

**Remark 2.9.** We showed in fact the result above for $* \leq n$. However, for $* > n$ groups $H^f_{\lf, \lip}(M)$ vanish ([9], Theorem 3.3). Moreover, we could simply modify the constants used in the straightening to work for $* \leq N$ for $N$ arbitrarily large. In further work we will without loss of generality assume that all chains and homology classes are of dimension $* \leq n$.

**Remark 2.10.** It is obvious that the straightening procedure depends on the choice of sets $(F_j)_{j \in J}$ and $m \in \mathbb{N}$, which depends on particular chain which we would like to straighten. However, in most cases these details are of secondary interest, therefore we will just say shortly about applying (piecewise) straightening procedure meaning applying it with respect to any suitable family $(F_j)_{j \in J}$ and any $m \in \mathbb{N}$ for which the procedure is defined.

**Remark 2.11.** Although the above procedure does not define a chain operator $C^f_{* \leq n}(M) \to C^f_{* \leq n}(M)$, it is easy to see that for every $L < \infty$ there exists $m$ such that the operator $str_m : C^f_{* \leq n}(M) \to C^f_{* \leq n}(M)$ is well defined, where $C^f_{* \leq L}(M)$ is a chain complex of locally finite chains $c$ with $\text{Lip}(c) < L$. Moreover, it is chain homotopic to the identity and induces an isometry on homology. The proof is essentially the same as for Proposition 2.8, one needs only to show that an obvious chain homotopy $H : C^f_{* \leq L}(M) \to C^f_{* \leq L}(M)$ can be improved to have its image in $C^f_{* \leq L}(M)$. This can be done by subdividing barycentrically each simplex in the image of $H$ to satisfy the restrictions on Lipschitz constant and then add chain homotopies between identity and certain iterated barycentric subdivisions (see proof of Lemma 2.13). Because $C^f_{* \leq L}(M) = \lim_{L \to \infty} C^f_{* \leq L}(M)$ and taking homology commutes with direct limits, we get an alternative proof of Proposition 2.8.

### 2.3 $C^1$-barycentric homology and barycentric measure homology

The straightening procedure described in the previous section is sufficient for some applications, though we need some more extensive machinery. One of the key properties of the classical straightening procedure for non-positively curved manifolds is that the straightened chains are smooth, because they consist of geodesic simplices. It is important e.g. in the proof of the proportionality principle in non-positively curved case, which depends on measure homology with $C^1$-topology, with additional assumption that support of each ‘chain’ is contained in $L$-Lipschitz simplices for some $L < \infty$. Smoothness is strictly technical, but necessary, because it allows to recognise fundamental cycle by integrating the volume form. However, the piecewise straight simplices which we use are only piecewise smooth. The aim of this section is to define piecewise $C^1$ simplices and chains and provide some reasonable topology on them in order to define corresponding smooth and measure homology theories.

We begin with the definition of $C^1$-barycentric chains and homology. Recall that a map $f : \Delta \to M$ is smooth if it can be extended to a smooth map $f' : U \to M$, where $U \subset \mathbb{R}^n$ is some open neighbourhood of $\Delta \subset \mathbb{R}^n$.

**Definition 2.12.** Let $C^b_{*}(M)$ be a chain complex consisting of singular chains $c$ on $M$ for which there exists $m(c) < \infty$ (depending on chain) such that $S^{m(c)}$ is a $C^1$ chain. We call this complex a $C^1$-barycentric complex. Because the boundary of a $C^1$-barycentric chain is $C^1$ barycentric, it is a subcomplex of $C_*(M)$. We call the corresponding homology theory a $C^1$-barycentric homology $H^b_*(M)$.

We consider also $C^1$-barycentric locally finite Lipschitz chains $C^b_{\lf, \lip}(M)$ and corresponding $C^1$-barycentric locally finite Lipschitz homology $H^b_{\lf, \lip}(M)$. 

7
Obviously every piecewise straight chain is $C^1$-barycentric by Lemma 2.5. To show that these homology theories are isometric to the corresponding non-smooth ones, we need the following lemma.

**Lemma 2.13.** Let $\sigma, \sigma' : \Delta \to M$ be two $k$-dimensional singular simplices such that there exists $m \in \mathbb{N}$ such that both $S^{(m)}(\sigma)$ and $S^{(m)}(\sigma')$ are $C^1$ chains. Let also $H : \Delta \times I \to M$ be a homotopy between $\sigma$ and $\sigma'$ such that it is differentiable on $\Delta' \times I$ for every $\Delta' \in S^{(m)}(\Delta)$. Then there exists a singular chain $D \in C_{k+1}(M)$ such that $S^{(m)}(D)$ is $C^1$ chain, $\partial D = \sigma - \sigma + \sum_{\Delta' \in \partial \Delta} A_{\Delta'}$, where $A_{\Delta''}$ for $\Delta'' \in \partial \Delta$ depends only on $H|_{\Delta'' \times 1}$. Moreover, $|D|_1 \leq N(k, m) < \infty$ is bounded by a constant depending uniformly on $m$ and $k$.

**Proof.** We recall the definition of chain homotopy $T : C_{s}(M) \to C_{s+1}(M)$ between identity and barycentric subdivision operator $S$, following [5], Section 2.1. Let $\lambda : \Delta \to M$ be a singular simplex and let $b : C_{s}(\Delta) \to C_{s+1}(M)$ be a ‘cone operator’ assigning to a simplex $\lambda' : \Delta' \to \Delta$ a simplex $b(\lambda')$ where $b$ is a barycentre of $\Delta$. It is easy to see that if $\lambda$ is differentiable on $m$-th barycentric subdivision, then for any face $\lambda''$ of $\lambda$, $b(\lambda'')$ is also. The chain homotopy operator $T : C_{s}(M) \to C_{s+1}(M)$ such that $\partial T + T \partial = \text{Id} - S$ is then defined inductively as:

- $T(0) = 0$
- $T(\lambda) = b_\lambda (\text{Id}_\Delta - T(\partial \lambda))$

By construction, $|T(\lambda)|_1$ depends only on the dimension of $\lambda$. Moreover, if $S^{(m)}(\lambda)$ is smooth, then $S^{(m)}(T(\lambda))$ is so. The homotopy between identity and the iterated subdivision operator $S^m$ is defined as $R_m = \sum_{0 \leq i < m} TS^{(i)}$. It is clear that $|R_m(\lambda)|_1 \leq m$ and the dimension of $\lambda$ and $S^{(m)}(R_m(\lambda))$ is smooth if $S^{(m)}(\lambda)$ is.

Now let $k = \dim(\Delta)$. For any map $h : \Delta \times I \to M$ let $D_h \in C_{k+1}(M)$ be a canonical subdivision of $h$ into $k + 1$ singular simplices, described also in [5], Section 2.1. Note that

$$\partial D_h = h|_{\Delta \times \{1\}} - h|_{\Delta \times \{0\}} + \sum_{\Delta' \in \partial \Delta} A_{\Delta'},$$

where $A_{\Delta'}$ depends only on $h|_{\Delta' \times I}$ for $\Delta' \in \partial \Delta$. Define $D$ as

$$D = -R_m(\sigma) + \sum_{\Delta' \in S^{(m)}(\Delta)} D_{H|_{\Delta'}} + R_m(\sigma')$$

The interpretation of $D$ is following: first subdivide $\sigma$ barycentrically $m$ times, then subdivide canonically each cylinder $\Delta' \times I$ for $\Delta' \in S^{(m)}(\Delta)$ and finally glue $\sigma'$ from its $m$-th barycentric subdivision. $S^{(m)}(D)$ is smooth because $m$-th barycentric subdivision of each term in the definition of $D$ is smooth. It is also clear that

$$|D|_1 \leq 2|R_m|_1 + (k + 1)((k + 1)!)^m,$$

which depends only on $k$ and $m$. Moreover we have

$$\partial D = S^{(m)}(\sigma) - \sigma + R_m \partial(\sigma) + \sum_{\Delta' \in S^{(m)}(\Delta)} (\sigma'|_{\Delta'} - \sigma|_{\Delta'}) + \sum_{\Delta'' \in \partial \Delta} A_{\Delta''} - R_m \partial(\sigma') - S^{(m)}(\sigma') + \sigma'$$

$$= \sigma' - \sigma + \sum_{\Delta'' \in \partial \Delta} A_{\Delta''} + R_m \partial(\sigma - \sigma'),$$

where both $R_m \partial(\sigma - \sigma')$ and $A_{\Delta''}$ depend only on $H|_{\Delta'' \times I}$ for $\Delta'' \in \partial \Delta$.

**Corollary 2.14.** Let $c \in C_{s+1}^{b, l, Lip}(M)$ be a $C^1$-barycentric locally finite Lipschitz cycle and let $m \in \mathbb{N}$ be such that $\text{str}_m(c)$ is defined. Then $c$ and $\text{str}_m(c)$ are homologous in $C_{s+1}^{b, l, Lip}(M)$.

**Proof.** It is easy to see that a homotopy $H$ constructed in the proof of Proposition 2.8 satisfies the assumptions of Lemma 2.13 for each simplex. Hence it can be subdivided into a $C^1$-barycentric chain $H' \in C_{s+1}^{b, l, Lip}(M)$ such that $\partial H' = c - \text{str}_m(c)$. It has finite $l$ norm if $c$ has and is Lipschitz by Proposition 2.4, therefore $H' \in C_{s+1}^{b, l, Lip}(M)$.
Proposition 2.15. Let \( M \) be a complete Riemannian manifold with sectional curvature bounded from above by \( 0 < K < \infty \). Then the map \( I_* : H^b_{*+1}M \to H^b_1M \) induced by the inclusion of chains is an isometric isomorphism.

Proof. The map \( I_* \) is onto by Proposition 2.8. To see that it is injective consider \( c_1, c_2 \in C^b_{*+1}M \) which represent the same class in \( H^b_{*+1}M \). Then there exists a \( D \in C^b_{*+1}M \) such that \( \partial D = c_2 - c_1 \). We can apply now the straightening procedure to \( D \) to obtain a chain \( \text{str}_m(D) \subset C^b_{*+1}M \) such that \( \partial \text{str}_m(D) = \text{str}_m(c_2) - \text{str}_m(c_1) \). Now apply Corollary 2.14 to see that \( c_1 \) and \( c_2 \) are homologous (in \( C^b_{*+1}M \)) to \( \text{str}_m(c_1) \), \( \text{str}_m(c_2) \) respectively. It is an isometry on homology by Proposition 2.8. \( \square \)

Now we turn our attention to chains with finite \( \ell^1 \)-norm and corresponding measure homology theory.

Definition 2.16. Let \( C^b_{*+1}M \) be a chain subcomplex of \( C^b_{*+1}M \) consisting of chains which have finite \( \ell^1 \)-norm. We call the homology of this complex \( \ell^1 \)-barycentric Lipschitz homology and denote it by \( H^b_{*+1}M \).

Remark 2.17. Note that Corollary 2.14 also applies to \( C^b_{*+1}M \), so an analogue of Proposition 2.15 for \( H^b_{*+1}M \) is true.

Definition 2.18. Let \( m \in \mathbb{N} \) and \( B^mC^1(\Delta, M) \) be a set of singular simplices \( \sigma : \Delta \to M \) such that \( S^{(m)}(\sigma) \) is a smooth singular chain. We call it the set of \( m \)-piecewise smooth singular simplices. We equip it with the topology induced from the embedding

\[
B^mC^1(\Delta, M) \to \prod_{\Delta' \in S^{(m)}(\Delta)} C^1(\Delta', M)
\]

For each \( m \in \mathbb{N} \) we have embedding \( B^mC^1(\Delta, M) \to B^{m+1}C^1(\Delta, M) \) onto a closed subset. We denote the direct limit of these spaces with weak topology as \( BC^1(\Delta, M) \).

The properties of the above topology on \( B^mC^1(\Delta, M) \) which are crucial to us are the following

- \( B^mC^1(\Delta, M) \) is a locally compact Hausdorff space;
- If \( \Delta = \Delta^n \), \( n = \text{dim } M \) then for every differential form \( \omega \in \Omega^n(M) \) the map

\[
I_\omega : B^mC^1(\Delta, M) \to \mathbb{R} \, , \, f \mapsto \int_\Delta f^* \omega
\]

is continuous;
- for every \( \sigma \in B^mC^1(\Delta, M) \) the map

\[
I\text{som}^+(M) : B^mC^1(\Delta, M) \, , \, g \mapsto g\sigma
\]

is continuous.

Definition 2.19. Let \( C^b_{*+1}M \) be a chain complex of measures on \( BC^1(\Delta^*, M) \) such that

1. for every measure \( \mu \in C^b_{*+1}M \) there exists \( m < \infty \) such that it is a Borel measure with finite variation supported on \( B^mC^1(\Delta^*, M) \);
2. every measure has Lipschitz determination, i.e. there exists \( L < \infty \) such that it is supported on simplices with Lipschitz constant \( L \).

The boundary operators are the push-forwards of measures by the boundary maps \( \partial : C^b_{*+1}M \to C^b_{*+1}M \). The obtained homology theory is called \( C^1 \)-barycentric measure homology with Lipschitz determination \( H^b_{*+1}M \).
Remark 2.20. The space $BC^1(\Delta, M)$ is in general not locally compact, therefore it is a problem with the definition of Borel measures. However, because every measure in $C^{b_1, Lip}(M)$ is $C^1$-barycentric, it can be treated as a Borel measure on $B^n C^1(\Delta^*, M)$ for some $m \in \mathbb{N}$, which is a locally compact space. For simplicity we will say that measure is Borel on $BC^1(\Delta, M)$ meaning that it is a push-forward of Borel measure on $B^n C^1(\Delta, M)$ for some $m \in \mathbb{N}$. Similarly, when integrating over $BC^1(\Delta, M)$, we will understand it as an integral over some $B^n C^1(\Delta, M)$ for $m$ sufficiently large.

The above homology theory is a variant of Milnor-Thurston homology. We can introduce a semi-norm $\| \cdot \|_1$ on it by taking infimum of absolute variations over all measures representing given homology class. An important consequence of the above construction is the following

**Proposition 2.21.** Let $M$ be a complete Riemannian manifold with sectional curvature bounded from above by $0 < K < \infty$. Then the homology groups $H_*^{b_1, Lip}(M)$ and $\mathcal{H}^{b_1, Lip}(M)$ are isometrically isomorphic.

**Proof.** By interpreting singular chains with finite $\ell^1$-norm as discrete measures with finite variation we have an obvious inclusion of chains $I : C_*^{b_1, Lip}(M) \rightarrow C_*^{b_1, Lip}(M)$ which commutes with differentials, hence it is a morphism of chain complexes and induces a homomorphism $I_*$ on homology. It is obviously injective, because chain homotopy joining two singular cycles can also be interpreted as a discrete measure on simplices. The surjectivity follows from applying straightening procedure. To be more precise, let $\mu \in C_k^{b_1, Lip}(M)$ be a measure cycle determined in $L$-Lipschitz simplices. Choose any family $(F_j)_{j \in J}$ of Borel subsets of $M$ with the properties indicated in the description of the piecewise straightening procedure and $m \in \mathbb{N}$ such that $str_m$ is defined for any simplex with Lipschitz constant $L$. Then after applying $str_m$ to the measure $\mu$ we obtain a cycle $\sum_{i \in I} a_i \sigma_i$, where each $\sigma_i, i \in I$ is an $m$-piecewise straight simplex and

$$a_i = \mu(\{ \sigma \in BC^1(\Delta^k, M); Lip(\sigma) \leq L, \text{ str}_m(\sigma) = \sigma_i \}).$$

The subset of $BC^1(\Delta^k, M)$ described above is Borel by the construction of sets $(F_j)_{j \in J}$, so the cycle is well defined. It is also $C^1$-barycentric by Proposition 2.4, locally finite by the local finiteness of $(F_j)_{j \in J}$ and Lipschitz by Proposition 2.4 and Lipschitz determination of $\mu$. It is also easy to see that $\mu$ and $str_m(\mu)$ are homologous in $C_*^{b_1, Lip}(M)$ by the same argument as in Corollary 2.14. The fact that $I_*$ is an isometry is a consequence of the facts that $I$ is an isometric inclusion and the straightening procedure does not increase the norm. \hfill $\square$

**Remark 2.22.** The existence of an isometric isomorphism as above for ‘finite’ $C^1$-barycentric theory $H_*^{b_1}(M)$ and $C^1$-barycentric measure homology with compact supports $\mathcal{H}_*^{b_1}(M)$ can be proved without any curvature assumptions as in [8]. However, the proof given in [8] depends heavily on bounded cohomology and cannot be easily generalised to the locally finite Lipschitz case.

### 3 Applications

#### 3.1 Product inequality

There is a classical result concerning the behaviour of simplicial volume under taking products. Namely, if $M$ and $N$ are compact manifolds of dimensions $n$ and $m$ respectively there are inequalities (see [3] for more details)

$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq \binom{n+m}{n} \|M\| \cdot \|N\|.$$  

The second inequality is obtained by simply taking a simplicial approximation of a cross product and can be easily generalised to the noncompact case. On the other hand, first inequality can be established by passing to bounded cohomology and using the duality between $\ell^1$ semi-norm on homology and $\ell^\infty$ semi-norm on cohomology. However, this approach does not generalize directly.
to the case of noncompact manifolds and Lipschitz simplicial volume (and in general is false in noncompact, non-Lipschitz case). Two main problems which arise are more subtle relation between \(\ell^1\) semi-norm on locally finite homology and \(\ell^\infty\) semi-norm on cohomology with compact supports and the existence of a good product in cohomology with compact supports. However, for the Lipschitz simplicial volume the inequality was proved in the case of complete, non-positively curved Riemannian manifolds ([9], Theorem 1.7). Using piecewise straightening procedure, we are able to generalize it slightly and obtain Theorem 1.3.

The proof is a modification of the proof from [9] with one proposition generalised to the case of bounded positive curvature. We introduce necessary notions and facts. By \(S_{k,l}^{l,Lip}(M)\) we denote the family of subsets of \(Map(\Delta^k, M)\) such that each element \(A \in S_{k,l}^{l,Lip}(M)\) is locally finite (in the sense that for any given compact subset \(K \subset M\) we have \#\{\(\sigma \in A : \sigma \cap K \neq \emptyset\) \(<\infty\)\} and consists of \(L\)-Lipschitz simplices for some \(L\), depending on \(A\). We recall the most important definitions and results from [9].

**Definition 3.1** ([9], Definition 3.11). Let \(M\) be a topological space, \(k \in \mathbb{N}\), and let \(A \subset Map(\Delta^k, M)\).

1. For a locally finite chain \(c = \sum_{i \in I} a_i \sigma_i \in C_k^A(M)\), let
   \[
   |c|_1^A = \begin{cases} |c|_1 & \text{if } supp(c) \subset A, \\ \infty & \text{otherwise}. \end{cases}
   \]
   Here, \(supp(c) := \{\sigma_i : i \in I, a_i \neq 0\}\).
2. The semi-norms on \((\text{Lipschitz})\) locally finite homology induced by \(|\cdot|_1^A\) are denoted by \(|\cdot|_1^A\).
3. If \(M\) is an oriented, connected \(n\)-manifold, then
   \[
   \|M\|^A := ||[M]||_1^A.
   \]

**Definition 3.2** ([9], Definition 3.19). Let \(M\) and \(N\) be two topological spaces, and let \(k, l \in \mathbb{N}\). A locally finite family \(A \in S_{k+l}^{l,Lip}(M \times N)\) is called \((k, l)\)-sparse if
   \[
   A_M := \{\pi_M \circ \sigma]_k, \sigma \in A\} \in S_k^{l}(M) \quad \text{and} \quad A_N := \{\pi_N \circ \iota]_l, \sigma \in A\} \in S_l^{l}(N)
   \]
   where \(\sigma]_k\) is an \(k\)-dimensional face of \(\sigma\) spanned by the last \(k\) vertices, \(\iota]_l, \sigma\) is an \(l\)-dimensional face of \(\sigma\) spanned by the first \(l\) vertices and \(\pi_M : M \times N \rightarrow M\) and \(\pi_N : M \times N \rightarrow N\) are the canonical projections.

A locally finite chain \(c \in C_{k+l}^{l,Lip}(M \times N)\) is called \((k, l)\)-sparse if its support is \((k, l)\)-sparse.

The proof of product inequality given in [9] uses non-positive curvature only to prove that for two non-positively curved manifolds the simplicial volume can be computed on sparse cycles. The following proposition is not stated as such in [9], it is, however, a meta-theorem actually proved there.

**Proposition 3.3.** Let \(M\) and \(N\) be two complete, oriented manifolds of dimensions \(m\) and \(n\) respectively such that the Lipschitz simplicial volume of \(M \times N\) can be computed via \((m, n)\)-sparse fundamental cycles, i.e.
   \[
   \|M \times N\|_{Lip} = \inf \{|[M \times N]|^A ; A \in S_{m+n}^{l,Lip}(M \times N), A-(m, n)\text{-sparse}\}.
   \]

Then
   \[
   \|M\|_{Lip} \cdot \|N\|_{Lip} \leq \|M \times N\|_{Lip}.
   \]

The outline of the proof is as follows. Consider cohomology with Lipschitz compact supports \(H^{*}_{cs,Lip}\), i.e. cohomology of cochain complex consisting of those singular cochains for which there exists a compact set \(K\) and constant \(L\) such that the evaluation on a simplex \(\Delta\) is 0 if \(\Delta\) has support disjoint from \(K\) and Lipschitz constant less than \(L\). For a given space \(X\) and family \(A \in S_k^{l,Lip}(X)\) an \(\ell^\infty\) semi-norm of a cohomology with Lipschitz compact support class computed on \(A\) is dual to the \(\ell^1\) semi-norm \(|\cdot|_1^A\) on Lipschitz locally finite homology ([9], Proposition 3.12). Therefore one
can compute a Lipschitz simplicial volume using a dual point of view. Moreover for two cochains with Lipschitz compact supports on $M$ and $N$ we can define their product on $M \times N$ which also has Lipschitz compact support ([9], Lemma 3.15). Finally, if $A \in S_{m+n}^{f,Lip}(M \times N)$ is $(m, n)$-sparse and $A_M$, $A_N$ are corresponding projections of $A$ to $S_m^{f,Lip}(M)$ and $S_n^{f,Lip}(N)$, then we have a product inequality ([9], Remark 3.17)

$$\|\phi \times \psi\|_A^N \leq \|\phi\|_{A_M}^M \cdot \|\psi\|_{A_N}^N$$

for $\phi \in H^{m}_{cs,Lip}(M)$ and $\psi \in H^{n}_{cs,Lip}(N)$. In particular if $A$ is $(m, n)$-sparse we obtain

$$\|M \times N\|^A = \frac{1}{\|[M \times N]^*\|^A_N} \leq \frac{1}{\|[M]\|^A_M \cdot \|[N]\|^A_N} = \|M\|^{A_M} \cdot \|N\|^{A_N} \geq \|M\|_{Lip} \cdot \|N\|_{Lip},$$

hence if the simplicial volume of $M \times N$ can be computed on sparse cycles we get the desired inequality.

To finish the proof of Theorem 1.3 we will prove the following proposition, which is a generalization of Proposition 3.20 in [9], where it was proved assuming non-positive curvature.

**Proposition 3.4.** Let $M$ and $N$ be two oriented, connected, complete Riemannian manifolds (without boundary) of dimensions $m$ and $n$ respectively with sectional curvatures bounded from above by $-K < 0$.

1. Let $k, l \in \mathbb{N}$. For any cycle $c \in C^{f,Lip}_{k+l}(M \times N)$ there is a $(k, l)$-sparse cycle $c' \in C^{f,Lip}_{k+l}(M \times N)$ satisfying

$$|c'|_1 \leq |c|_1 \quad \text{and} \quad c \sim c' \text{ in } C^{f,Lip}_{k+l}(M \times N).$$

2. In particular, the Lipschitz simplicial volume can be computed via sparse fundamental cycles, i.e.

$$\|M \times N\|_{Lip} = \inf \{\|M \times N\|^A; A \in S^{f,Lip}_{m+n}(M \times N), A \text{-(}m,n\)-sparse\}.$$

**Proof.** The second statement is a direct consequence of the first one. To prove the first one it is enough to just apply straightening procedure, but with more carefully chosen sets $(F_i)_{i \in J}$. Let $Q = 2 \max\{m, n\}$. Choose a family of Borel subsets $(F^M_j)_{i \in J^M}$ of $\widetilde{M}$ together with the points $(z_j^M)_{i \in J^M}$ with all the properties indicated in the description of the straightening procedure, but with the additional assumption that $\text{diam}(F^M_j) < \frac{E_{Q,K}}{12}$ for every $j \in J^M$. Similarly choose a family $(F^N_j)_{i \in J^N}$ of Borel subsets of $\widetilde{N}$ together with points $(z_j^N)_{i \in J^N}$ and as a base of the straightening procedure for $M \times N$ take a family $(F^M_j \times F^N_j)_{(j_1, j_2) \in J^M \times J^N}$ together with points $(z_{j_1}^M, z_{j_2}^N)_{(j_1, j_2) \in J^M \times J^N}$. This family is locally finite, $\pi_1(M \times N)$-invariant and $\text{diam}(F^M_j \times F^N_j) < \frac{E_{Q,K}}{8}$ for every $(j_1, j_2) \in \mathcal{J}^M \times \mathcal{J}^N$. Hence if $c \in C^{f,Lip}_k(M \times N)$ is any given locally finite Lipschitz chain it can be straightened with respect to that family. Note also that for any given $L < \infty$ and $p \in \mathbb{N}$ the family

$$A_{L,p} := \{\sigma \in \text{Map}(\Delta^{k+l}, M \times N); \text{Lip}(\sigma) \leq L; \sigma \text{ is } p\text{-piecewise straight simplices}\}$$

belongs to $S^{f,Lip}_{k+l}(M \times N)$ and is $(k, l)$-sparse by the construction of $(F^M_j \times F^N_j)_{(j_1, j_2) \in J^M \times J^N}$ and Lipschitz condition. To finish the proof note that $c \sim \text{str}_p(c)$ for some $p \in \mathbb{N}$, $|c|_1 \geq |\text{str}_p(c)|_1$ and $\text{str}_p(c)$ has a support in $A_{L,p}$ for some $L$, thus it is $(k, l)$-sparse. 

### 3.2 Proportionality Principle

Another result obtained in [9] for non-positively curved manifolds is the proportionality principle for the Lipschitz simplicial volume. We generalize it here and prove Theorem 1.4. As a corollary we obtain a proof of Theorem 1.2, based on Thurston’s approach [11], but slightly different from the proof given in [10].

The idea of the original proof is as follows. Using the common universal cover one can construct a “smearing map” from $C^1$ locally finite Lipschitz chain complex on $M$ into the chain complex of Borel measures on $C^1(\Delta, N)$ with finite variation and Lipschitz determination. This map does not
increase the norm and has the property that the image of locally finite real fundamental class of $M$ maps to a (measure) fundamental class of $N$ multiplied by $\frac{\text{vol}(M)}{\text{vol}(N)}$ (or more precisely a measure cycle such that every singular chain homologous to it, if it exists, is a certain multiplicity of a fundamental cycle). Moreover, the image of this map can be approximated 'isometrically' by a singular locally finite Lipschitz cycle, which finishes the proof.

The usage of $C^1$ chains and measures is strictly technical and is used to recognise the image of the smearing map. In our approach we cannot use singular locally finite Lipschitz cycle, which finishes the proof.

Proposition 3.5 ([9], Proposition 4.4). Let $M$ be a Riemannian $n$-manifold, and let $c = \sum_{k \in \mathbb{N}} a_k \sigma_k \in C_{lf}^f(M)$ be a cycle with $|c|_1 < \infty$ and $\text{Lip}(c) < \infty$.

1. Then $\langle d\text{vol}_M, \sigma_k \rangle \leq \text{Lip}(c) \text{vol}(\Delta^n)$ for every $k \in \mathbb{N}$

2. Furthermore, we have the following equivalence:

$$\sum_{k \in \mathbb{N}} a_k \cdot \langle d\text{vol}_M, \sigma_k \rangle = \text{vol}(M) \Leftrightarrow c \text{ is a fundamental cycle.}$$

Remark 3.6. The second statement of the above proposition gives an easy criterion to distinguish fundamental class. It can be also applied to a given measure cycle, but only if it is homologous to some singular cycle. The reason is that there is no obvious map $C_{b1,\text{Lip}}^b(M) \to C_{b1}^f(M)$. However, by Proposition 2.21 we obtain a map $H_{b1,\text{Lip}}^b(M) \to H_{b1}^f(M)$ by composing the inverse of the isometric isomorphism $H_{b1,\text{Lip}}^b(M) \to H_{b1}^b(M)$ and a map $H_{b1,\text{Lip}}^b(M) \to H_{b1}^f(M)$ induced by the inclusion of chains. We can therefore define fundamental cycles in $C_{b1,\text{Lip}}^b(M)$ as the cycles representing any class in the preimage of the fundamental class in $H_{b1}^f(M)$.

The following proposition is a variation of the results from [9] and the proof is completely analogous. Let $U$ be a common universal cover of $M$ and $N$ with covering maps $p_M$ and $p_N$ respectively, let $G = \text{Isom}^+(U)$ and let $\Lambda = \pi_1(N)$. Then $\Lambda$ is a lattice in $G$ ([9], Lemma 4.2). Denote by $\mu_{\Lambda \setminus G}$ the normalized Haar measure on $\Lambda \setminus G$.

Proposition 3.7 ([9], Proposition 4.9, Lemma 4.10). Let $\sigma : \Delta^i \to M$ be a $C^1$-barycentric simplex, and let $\tilde{\sigma} : \Delta^i \to U$ be a lift of $\sigma$ to $U$. Then push-forward of $\mu_{\Lambda \setminus G}$ under the map

$$\text{smear}_{\tilde{\sigma}} : \Lambda \setminus G \to BC^1(\Delta^i, N), \Lambda g \mapsto p_N \circ g \tilde{\sigma}$$

does not depend on the choice of the lift of $\sigma$ as is denoted by $\mu_{\sigma}$. Further there is a well-defined chain map

$$\text{smear}_* : C_{b1,\text{Lip}}^b(M) \to C_{b1,\text{Lip}}^b(N), \sum_\sigma a_\sigma \sigma \mapsto \sum_\sigma a_\sigma \mu_{\sigma}.$$ 

Moreover, for every fundamental cycle $c \in C_{b1,\text{Lip}}^b(M)$ we have

$$\langle d\text{vol}_N, \text{smear}_n(c) \rangle = \int_{BC^1(\Delta^n, N)} \int_{\Delta^n} \sigma^* d\text{vol}_N \text{dsmear}_n(c)(\sigma) = \text{vol}(M).$$
Proof of theorem 1.4 and 1.2. We will show that
\[
\frac{\|N\|_{Lip}}{vol(N)} \leq \frac{\|M\|_{Lip}}{vol(M)}
\]
and the opposite inequality will follow by symmetry. For \(\|M\|_{Lip} = \infty\) the inequality is obvious, so we can assume \(\|M\|_{Lip} < \infty\). By Proposition 2.15, in this case there exists a fundamental cycle in \(C_{n1}^{d1,Lip}(M)\). Let \(c = \sum a_{\sigma} \sigma \in C_{n1}^{d1,Lip}(M)\) be a fundamental cycle and consider its image under the smearing map. It follows from Propositions 3.7, 3.5 and Remark 3.6 that its image \(\text{smear}_n(c)\) represents a fundamental class in \(C_{b1,Lip}^{d1}(N)\) multiplied by \(\frac{vol(M)}{vol(N)}\). Moreover, by the construction of the smearing map
\[
|\text{smear}_n(c)| = \left| \sum a_{\sigma} \mu_{\sigma} \right| \leq \sum |a_{\sigma}||\mu_{\sigma}| = \sum |a_{\sigma}| = |c|_1.
\]
By Proposition 2.21 there exists a cycle in \(C_{n1}^{d1,Lip}(N)\) which represents the same homology class as \(\text{smear}_n(c)\) with not greater \(\ell^1\) norm. Because Proposition 2.15 implies that the Lipschitz simplicial volume of \(M\) can be computed on \(C^1\)-barycentric cycles we obtain
\[
\|N\|_{Lip} \leq \frac{vol(N)}{vol(M)} \|M\|_{Lip} \Rightarrow \frac{\|N\|_{Lip}}{vol(N)} \leq \frac{\|M\|_{Lip}}{vol(M)}.
\]
To prove Theorem 1.2 we need only to use the fact that for closed manifolds the classical and Lipschitz simplicial volumes coincide ([9], Remark 1.4).

References

[8] C. Löh Measure homology and singular homology are isometrically isomorphic, Mathematische Zeitschrift 253 (2006), no. 1, 197-218