Polish $G$-spaces similar to logic $G$-spaces of continuous structures

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Logic $S_\infty$-space

Let $L = (R_i^{n_i})_{i \in I}$ be a countable relational language and

$$X_L = \prod_{i \in I} 2^{\omega^{n_i}}$$

be the corresponding topol. space under the product topology $\tau$.

We view $X_L$ as the space of all $L$-structures on $\omega$ identifying every $x = (\ldots x_i \ldots) \in X_L$ with the structure $(\omega, R_i)_{i \in I}$ where $R_i$ is the $n_i$-ary relation defined by the characteristic function $x_i : \omega^{n_i} \to 2$.

The logic action of the group $S_\infty$ of all permutations of $\omega$ is defined on $X_L$ by the rule:

$$g \circ x = y \iff \forall i \forall \bar{s}(y_i(\bar{s})) = x_i(g^{-1}(\bar{s})).$$
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Let $(\langle X, \tau \rangle, G)$ be a Polish $G$-space with a countable basis $\{C_j\}$.

**H. Becker**: there exists a unique partition of $X$, 

$$X = \bigcup \{ Y_t : t \in T \}$$

into invariant $G_\delta$-sets $Y_t$ s. t. every orbit from $Y_t$ is dense in $Y_t$.

To construct it take $\{C_j\}$ and for any $t \in 2^\mathbb{N}$ define

$$Y_t = (\bigcap \{ GC_j : t(j) = 1 \}) \cap (\bigcap \{ X \setminus GC_j : t(j) = 0 \})$$

and take $T = \{ t \in 2^\mathbb{N} : Y_t \neq \emptyset \}$.

In the case of the logic action of $S_\infty$ on the space $X_L$ each piece consists of structures which satisfy the same $\forall$-sentences and $\exists$-sentences.
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**Vaught transforms**

Let $X$ be a Polish $G$-space, $B \subseteq X$ and $u \subseteq G$ is open.

Vaught transforms:

$$B^u = \{ x \in X : \{ g \in u : gx \in B \} \text{ is comeagre in } u \}$$

$$B^\Delta u = \{ x \in X : \{ g \in u : gx \in B \} \text{ is not meagre in } u \}.$$ 

In the case of the **logic action** of $S_\infty$ on the space $X_L$ if

$$B = \{ M \in X_L : M \models \phi(s) \} \text{ with } s \in \omega$$

then

$$B^{S_\infty} = \{ M \in X_L : M \models \forall x \phi(x) \}.$$
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If $B \in \Sigma_\alpha$, then $B^{\Delta H} \in \Sigma_\alpha$ and
if $B \in \Pi_\alpha$, then $B^{*H} \in \Pi_\alpha$.

For any open $B \subseteq X$ and any open $K < G$ we have $B^{\Delta K} = KB$, where $KB = \{gx : g \in K, x \in B\}$. 
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$B^{\Delta K} = KB$, where $KB = \{gx : g \in K, x \in B\}$. 
Let $G$ be a closed subgroup of $S_\infty$. Let $\mathcal{N}^G$ be the standard basis of the topology of $G$ consisting of cosets of pointwise stabilisers of finite subsets of $\omega$.

Let $(\langle X, \tau \rangle, G)$ be a Polish $G$-space with a countable basis $\mathcal{A}$. Along with $\tau$ we shall consider another topology on $X$.

**Nice topology:**
Let $G$ be a closed subgroup of $S_\infty$.

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**Nice topology:**
Nice topology

Definition (H. Becker) A topology $t$ on $X$ is nice for the $G$-space $(\langle X, \tau \rangle, G)$ if the following conditions are satisfied.

(A) $t$ is a Polish topology, $t$ is finer than $\tau$ and the $G$-action remains continuous with respect to $t$.

(B) There exists a basis $B$ for $t$ (called nice) such that:

1. $B$ is countable;
2. for all $B_1, B_2 \in B$, $B_1 \cap B_2 \in B$;
3. for all $B \in B$, $X \setminus B \in B$;
4. for all $B \in B$ and $u \in \mathcal{N}^G$, $B^{\Delta u}, B^{*u} \in B$;
5. for any $B \in B$ there exists an open subgroup $H < G$ such that $B$ is invariant under the corresponding $H$-action.
For any countable fragment $F$ of $L_{\omega_1\omega}$, which is closed under quantifiers, all sets

$$\text{Mod}(\phi, \bar{s}) = \{ M \in X_L : M \models \phi(\bar{s}) \} \text{ with } \bar{s} \subset \omega$$

form a nice basis defining a nice topology (denoted by $t_F$) of the $S_\infty$-space $X_L$.

Each piece of the canonical partition corresponding to $t_F$ consists of structures which satisfy the same $F$-sentences (without parameters).
Logic action

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Example and illustration

Let $G$ be a closed subgroup of $S_\infty$ and $(X, \tau)$ be a Polish $G$-space. Let $t$ be a nice topology for $\langle X, \tau \rangle, G$.

A generalized version of Lindström’s model completeness theorem:

**Theorem (B.M-I)**

For any $x_1 \in X$ if $X_1 = Gx_1$ is a $G_\delta$-subset of $X$, then both topologies $\tau$ and $t$ coincide on $X_1$.

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Existence

**Theorem**

(H.Becker) Let $G$ be a closed subgroup of $S_\infty$ and $(X, \tau)$ be a Polish $G$-space.

Let $\mathfrak{t}'$ be a topology on $X$ finer than $\tau$, such that the action remains continuous with respect to $\mathfrak{t}'$.

Then there is a nice topology $\mathfrak{t}$ for $(\langle X, \tau \rangle, G)$ such that $\mathfrak{t}$ is finer than $\mathfrak{t}'$.

**Remark:** All elements of $\mathfrak{t}$ are $\tau$-Borel.
Question:
Is it possible to extend the generalised model theory of H. Becker to actions of Polish groups (without the assumption $G \leq S_\infty$)?
Continuous structures

A countable continuous signature:

\[ L = \{ d, R_1, ..., R_k, ..., F_1, ..., F_l, ... \}. \]

**Definition**

A **metric \( L \)-structure** is a complete metric space \((M, d)\) with \(d\) bounded by 1, along with a family of uniformly continuous operations \(F_k\) on \(M\) and a family of predicates \(R_l\), i.e. uniformly continuous maps from appropriate \(M^{k_l}\) to \([0, 1]\).

It is usually assumed that to a predicate symbol \(R_i\) a continuity modulus \(\gamma_i\) is assigned so that when \(d(x_j, x_j') < \gamma_i(\varepsilon)\) with \(1 \leq j \leq k_i\) the corresponding predicate of \(M\) satisfies

\[ |R_i(x_1, ..., x_j, ..., x_{k_i}) - R_i(x_1, ..., x_j', ..., x_{k_i})| < \varepsilon. \]
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Let \((G, d)\) be a Polish group with a left invariant metric \(\leq 1\). If \((X, d)\) is its completion, then \(G \leq Iso(X)\). Let \(S\) be a countable dense subset of \(X\). Enumerate all orbits of \(G\) of finite tuples of \(S\).

For the closure of such an \(n\)-orbit \(C\) define a predicate \(R_{\overline{C}}\) on \((X, d)\) by

\[
R_{\overline{C}}(y_1, ..., y_n) = d((y_1, ..., y_n), \overline{C}) \text{ (i.e. } \inf \{d(\overline{y}, \overline{c}) : \overline{c} \in \overline{C}\}).
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It is observed by J. Melleray that \(G\) is the automorphism group of the continuous structure \(M\) of all these predicates on \(X\), with continuous moduli = \(id\).
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The space of continuous structures

Fix a relational continuous signature $L$ and a Polish space $(\mathcal{Y}, d)$. Let $S$ be a dense countable subset of $\mathcal{Y}$.

Define the space $\mathcal{Y}_L$ of continuous $L$-structures on $(\mathcal{Y}, d)$ as follows.

**Metric on the set of $L$-structures:** Enumerate all tuples of the form $(j, \bar{s})$, where $\bar{s}$ is a tuple $\in S$ of the length of the arity of $R_j$. For $L$-structures $M$ and $N$ on $\mathcal{Y}$ let

$$\delta(M, N) = \sum_{i=1}^{\infty} \{2^{-i} |R_j^M(\bar{s}) - R_j^N(\bar{s})| : i \text{ is the number of } (j, \bar{s})\}.$$ 

Logic action

- the space $\mathcal{Y}_L$ is Polish;
- the Polish group $Iso(\mathcal{Y})$ acts on $\mathcal{Y}_L$ continuously.
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**Logic action**

- the space \( Y_L \) is Polish;
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Theorem

For any Polish group $G$ there is a Polish space $(\mathcal{Y}, d)$ and a continuous relational signature $L$ such that

1. $G < \text{Iso(} \mathcal{Y} \text{)}$
2. for any Polish $G$-space $\mathcal{X}$ there is a Borel 1-1-map $\mathcal{M} : \mathcal{X} \rightarrow \mathcal{Y}_L$ such that for any $x, x' \in \mathcal{X}$ structures $\mathcal{M}(x)$ and $\mathcal{M}(x')$ are isomorphic if and only if $x$ and $x'$ are in the same $G$-orbit,

The map $\mathcal{M}$ is a Borel $G$-invariant 1-1-reduction of the $G$-orbit equivalence relation on $\mathcal{X}$ to the $\text{Iso(} \mathcal{Y} \text{)}$-orbit equivalence relation on the space $\mathcal{Y}_L$ of all $L$-structures.
A **graded subset** of $X$, denoted $\phi \subseteq X$, is a function $X \rightarrow [0, 1]$.

It is **open** (**closed**), $\phi \in \Sigma_1$ (**resp.** $\phi \in \Pi_1$), if it is upper (**lower**) semi-continuous, i.e. the set $\phi <_r$ (**resp.** $\phi >_r$) is open for all $r \in [0, 1]$ (**here** $\phi <_r = \{z \in X : \phi(z) < r\}$).

When $G$ is a Polish group, then a graded subset $H \subseteq G$ is called a **graded subgroup** if $H(1) = 0$, $\forall g \in G(H(g) = H(g^{-1}))$ and $\forall g, g' \in G(H(gg') \leq H(g) + H(g'))$.

We also define Borel classes $\Sigma_\alpha$, $\Pi_\alpha$ so that $\phi$ is $\Sigma_\alpha$ if $\phi = \inf \Phi$ for some countable $\Phi \subseteq \bigcup \{\Pi_\gamma : \gamma < \alpha\}$ and $\Pi_\alpha = \{1 - \phi : \phi \in \Sigma_\alpha\}$.
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Graded subsets of $\mathcal{Y}_L$

For $\bar{c}$ from $(\mathcal{Y}, d)$ and a linear $\delta$ with $\delta(0) = 0$ graded subgroup $H_{\delta, \bar{c}} \subseteq Iso(\mathcal{Y})$:

$$H_{\delta, \bar{c}}(g) = \delta\left(\max(d(c_1, g(c_1)), \ldots, d(c_n, g(c_n))))\right),$$

where $g \in Iso(\mathcal{Y})$.

A continuous formula is an expression built from 0, 1 and atomic formulas by applications of the following functions:

$$\frac{x}{2}, x \cdot y = \max(x - y, 0), \min(x, y), \max(x, y), |x - y|,$$

$$\neg(x) = 1 - x, x \check{+} y = \min(x + y, 1), \sup_x \text{ and } \inf_x.$$

Any continuous sentence $\phi(\bar{c})$ defines a graded subset of $\mathcal{Y}_L$ which belongs to $\Sigma_n$ for some $n$:

$$\phi(\bar{c}) \text{ takes } M \text{ to the value } \phi^M(\bar{c}).$$
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Invariant graded subsets

Assuming that continuity moduli of $L$-symbols are $\text{id}$ for any $\phi(\bar{x})$ as above we find a linear function $\delta$ such that the graded subgroup

$$H_{\delta,\bar{c}}(g) = \delta(\max(d(c_1, g(c_1)), \ldots, d(c_n, g(c_n)))),$$

where $g \in \text{Iso}(Y)$.

and the graded subset $\phi(\bar{c}) \sqsubseteq Y_L$ satisfy

$$\phi^{g(M)}(\bar{c}) \leq \phi^M(\bar{c}) + H_{\delta,\bar{c}}(g).$$

Definition

Let $X$ be a continuous $G$-space. A graded subset $\phi \sqsubseteq X$ is called invariant with respect to a graded subgroup $H \sqsubseteq G$ if for any $g \in G$ we have $\phi(g(x)) \leq \phi(x) + H(g)$. 
Invariant graded subsets

Assuming that continuity moduli of \( L \)-symbols are \( \text{id} \) for any \( \phi(\bar{x}) \) as above we find a linear function \( \delta \) such that the graded subgroup

\[
H_{\delta,\bar{c}}(g) = \delta(\max(d(c_1, g(c_1)), \ldots, d(c_n, g(c_n))))
\]

where \( g \in \text{Iso}(\mathcal{Y}) \).

and the graded subset \( \phi(\bar{c}) \subseteq \mathcal{Y}_L \) satisfy

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\phi^{g(M)}(\bar{c}) \leq \phi^{M}(\bar{c}) + H_{\delta,\bar{c}}(g).
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Definition

Let \( X \) be a continuous \( G \)-space. A graded subset \( \phi \subseteq X \) is called invariant with respect to a graded subgroup \( H \subseteq G \) if for any \( g \in G \) we have \( \phi(g(x)) \leq \phi(x) + H(g) \).
Vaught transforms 3

For any non-empty open $J \subseteq G$ let

$$\phi^{\Delta J}(x) = \inf \{ r \dot{+} s : \{ h : \phi(h(x)) < r \} \text{ is not meagre in } J_{<s} \}. $$

$$\phi^{*J}(x) = \sup \{ r \dot{-} s : \{ h : \phi(h(x)) \leq r \} \text{ is not comeagre in } J_{<s} \},$$

**Theorem**

- $\phi^{*J}(x) = 1 - (1 - \phi)^{\Delta J}(x)$ for all $x \in X$.
- $\phi^{\Delta J}(x) \leq \phi^{*J}(x)$ for all $x \in X$.
- If $\phi$ is a graded $\Sigma_\alpha$-subset, then $\phi^{\Delta J}$ is also $\Sigma_\alpha$.
- If $\phi$ is a graded $\Pi_\alpha$-subset, then $\phi^{*J}(x)$ is also $\Pi_\alpha$.
- Vaught transforms of Borel graded subsets are Borel.
For any non-empty open $J \subseteq G$ let

\[ \phi^\Delta J(x) = \inf \{ r + s : \{ h : \phi(h(x)) < r \} \text{ is not meagre in } J_{<s} \} \]

\[ \phi^* J(x) = \sup \{ r - s : \{ h : \phi(h(x)) \leq r \} \text{ is not comeagre in } J_{<s} \} \]

**Theorem**

- \[ \phi^* J(x) = 1 - (1 - \phi)^\Delta J(x) \text{ for all } x \in X. \]
- \[ \phi^\Delta J(x) \leq \phi^* J(x) \text{ for all } x \in X. \]

- If $\phi$ is a graded $\Sigma_\alpha$-subset, then $\phi^\Delta J$ is also $\Sigma_\alpha$.
  
  If $\phi$ is a graded $\Pi_\alpha$-subset, then $\phi^* J(x)$ is also $\Pi_\alpha$.

- Vaught transforms of Borel graded subsets are Borel.
Theorem

If $H$ is a graded subgroup of $G$, then both $\phi^*H(x)$ and $\phi^{\Delta H}(x)$ are $H$-invariant:

$$\phi^*H(x) - H(h) \leq \phi^*H(h(x)) \leq \phi^*H(x) + H(h) \text{ and}$$

$$\phi^{\Delta H}(x) - H(h) \leq \phi^{\Delta H}(h(x)) \leq \phi^{\Delta H}(x) + H(h).$$

Moreover if $\phi(x) \leq \phi(h(x)) + H(h)$ for all $x$ and $h$, then

$$\phi^*H(x) = \phi(x) = \phi^{\Delta H}(x).$$
We consider $G$ together with a distinguished countable family of open graded subsets $\mathcal{R}$ so that all $\rho < r$ for $\rho \in \mathcal{R}$ and $r \in \mathbb{Q}$, form a basis of the topology of $G$.

We usually assume that $\mathcal{R}$ consists of graded cosets, i.e. for such $\rho \in \mathcal{R}$ there is a graded subgroup $H \in \mathcal{R}$ and an element $g_0 \in G$ so that for any $g \in G$, $\rho(g) = H(gg_0^{-1})$. (For every Polish group $G$ there is a countable family of open graded subsets $\mathcal{R}$ as above.)

Considering a $(G, \mathcal{R})$-space $X$ we distinguish a similar family too: a countable family $\mathcal{U}$ of open graded subsets of $X$ generating the topology.
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Considering a $(G, \mathcal{R})$-space $X$ we distinguish a similar family too: a cntble family $\mathcal{U}$ of open graded sbsts of $X$ generating the topol.
Graded bases

We consider $G$ together with a distinguished countable family of open graded subsets $\mathcal{R}$ so that all $\rho < r$ for $\rho \in \mathcal{R}$ and $r \in \mathbb{Q}$, form a basis of the topology of $G$.

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Considering a $(G, \mathcal{R})$-space $X$ we distinguish a similar family too: a countable family $\mathcal{U}$ of open graded subsets of $X$ generating the topology.
Definition. A family $\mathcal{B}$ of Borel graded subsets of the $G$-space $X$ is a **nice basis** w.r.to $\mathcal{R}$ if:

- $\mathcal{B}$ is countable and generates the topol. finer than $\tau$;
- for all $\phi_1, \phi_2 \in \mathcal{B}$, the functions $\min(\phi_1, \phi_2)$, $\max(\phi_1, \phi_2)$, $|\phi_1 - \phi_2|$, $\phi_1 - \phi_2$, $\phi_1 + \phi_2$ belong to $\mathcal{B}$;
- for all $\phi \in \mathcal{B}$ and rational $r \in [0, 1]$, $r\phi$ and $1 - \phi \in \mathcal{B}$;
- for all $\phi \in \mathcal{B}$ and $\rho \in \mathcal{R}$, $\phi^*\rho$, $\phi^\Delta\rho \in \mathcal{B}$;
- for any $\phi \in \mathcal{B}$ there exists an open graded subgroup $H \in \mathcal{R}$ such that $\phi$ is invariant under the corresponding $H$-action.

A topology $\mathbf{t}$ on $X$ is **$\mathcal{R}$-nice** for the $G$-space $\langle X, \tau \rangle$ if:

(a) $\mathbf{t}$ is a Polish topology, $\mathbf{t}$ is finer than $\tau$ and the $G$-action remains continuous with respect to $\mathbf{t}$;
(b) there exists a nice basis $\mathcal{B}$ so that $\mathbf{t}$ is generated by all $\phi < q$ with $\phi \in \mathcal{B}$ and $q \in \mathbb{Q} \cap (0, 1]$. 

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Polish $G$-spaces similar to logic $G$-spaces of continuous structures
Definition. A family $B$ of Borel graded subsets of the $G$-space $X$ is a nice basis w.r.to $\mathcal{R}$ if:

- $B$ is countable and generates the topol. finer than $\tau$;
- For all $\phi_1, \phi_2 \in B$, the functions $\min(\phi_1, \phi_2), \max(\phi_1, \phi_2), |\phi_1 - \phi_2|, \phi_1 - \phi_2, \phi_1 + \phi_2$ belong to $B$;
- For all $\phi \in B$ and rational $r \in [0, 1]$, $r\phi$ and $1 - \phi \in B$;
- For all $\phi \in B$ and $\rho \in \mathcal{R}$, $\phi^\rho, \phi^\Delta \rho \in B$;
- For any $\phi \in B$ there exists an open graded subgroup $H \in \mathcal{R}$ such that $\phi$ is invariant under the corresponding $H$-action.

A topology $t$ on $X$ is $\mathcal{R}$-nice for the $G$-space $\langle X, \tau \rangle$ if:

(a) $t$ is a Polish topology, $t$ is finer than $\tau$ and the $G$-action remains continuous with respect to $t$;
(b) there exists a nice basis $B$ so that $t$ is generated by all $\phi_{<q}$ with $\phi \in B$ and $q \in \mathbb{Q} \cap (0, 1]$.
The case of $U_L$

Let $U$ be **Urysohn** space of diameter 1: This is the unique Polish metric space which is universal and ultrahomogeneous, i.e. every isometry between finite subssts of $U$ extends to an isometry of $U$.

There is a ctble family $R$ consisting of cosets of clopen graded subgroups of $Iso(U)$ of the form

$$H_s : g \rightarrow d(g(s), s), \text{ where } s \subseteq S \text{ (ctble,dense)} ,$$

which generates the topology of $Iso(U)$.

Let $L$ be a continuous signature of continuity moduli $id$. Then the family of all continuous $L$-sentences

$$\phi(s) : M \rightarrow \phi^M(s), \text{ where } \bar{s} \in S,$$

forms an $R$-nice basis $B$ of the $G$-space $U_L$. 

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The case of $U_L$

Let $U$ be **Urysohn** space of diameter 1: This is the unique Polish metric space which is universal and ultrahomogeneous, i.e. every isometry between finite subsets of $U$ extends to an isometry of $U$.

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**Theorem.** Let \((G, \mathcal{R})\) be a Polish group with \(\mathcal{R}\) satisfying
(i) for every graded subgroup \(H \in \mathcal{R}\) if \(gH \in \mathcal{R}\), then \(H^g \in \mathcal{R}\);
(ii) \(\mathcal{R}\) is closed under \textbf{max} and multiplying by rationals.
Let \(\langle X, \tau \rangle\) be a \(G\)-space and \(\mathcal{U}\) be a countable family of Borel
graded subsets of \(X\) generating a topology finer than \(\tau\), so that
each \(\phi \in \mathcal{U}\) is invariant with respect to some graded subgroup
\(H \in \mathcal{R}\).
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of open graded subsets.
Lindström

$G$ is a Polish group with a graded basis $\mathcal{R}$ consisting of graded cosets, $\langle X, \tau \rangle$ is a Polish $G$-space, etc.

**Theorem**

Let $t$ be $\mathcal{R}$-nice. Let $X = Gx_0$ for some (any) $x_0 \in X$ and $X$ be a $G_\delta$-subset of $X$. Then both topologies $\tau$ and $t$ are equal on $X$. 

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*Let $X = Gx_0$ for some (any) $x_0 \in X$ and $X$ be a $G_\delta$-subset of $X$. Then both topologies $\tau$ and $t$ are equal on $X$.***
Let $\mathcal{B}$ be a nice basis defining $\mathcal{R}$-nice $\mathbf{t}$, $H$ be an open graded subgroup from $\mathcal{R}$, $X$ be an invariant $G_\delta$-subset of $X$ with respect to $\mathbf{t}$. 

(1) A family $\mathcal{F}$ of subsets of the form $\phi_{<r}$ with $H$-invariant $\phi \in \mathcal{B}$ is called an $H$-type in $X$, if it is maximal w.r. to the condition $X \cap \bigcap \mathcal{F} \neq \emptyset$.

(2) An $H$-type $\mathcal{F}$ is called principal if there is an $H$-invariant graded basic set $\phi \in \mathcal{B}$ and there is $r$ such that $\phi_{<r} \in \mathcal{F}$ and $\bigcap\{\overline{B} : B \in \mathcal{F}\} \cap X$ coincides with the closure of $\phi_{<r} \cap X$. Then we say that $\phi_{<r}$ defines $\mathcal{F}$. 
Let $\mathcal{B}$ be a nice basis defining $\mathcal{R}$-nice $t$, $H$ be an open graded subgroup from $\mathcal{R}$, $X$ be an invariant $G_\delta$-subset of $X$ with respect to $t$.

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Let $\mathcal{R}$ consist of clopen graded cosets. Let $\mathcal{B}$ be an $\mathcal{R}$-nice basis of a $G$-space $\langle X, \tau \rangle$ and $t$ be the corresponding nice topology,

**Theorem**

Assume that the action satisfies the approximation property for graded subgroups.

A piece $X$ of the canonical partition with respect to the topology $t$ is a $G$-orbit if and only if for any basic open graded subgroup $H \trianglelefteq G$ any $H$-type of $X$ is principal.
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Approximation property for graded subgroups

Definition

The \((G, \mathcal{R})\)-space \((X, \mathcal{U})\) has the **approximation property for graded subgroups** if for any \(\varepsilon > 0\)

for any graded subgroup \(H \in \mathcal{R}\), any \(c\) and \(c' \in X\) of the same \(G\)-orbit

if \(c, c'\) belong to the same subsets of the form \(\phi \leq t\) for \(H\)-invariant \(\phi \in \mathcal{U}\), then

\(c'\) can be approximated by the values \(g(c)\) with \(H(g) < \varepsilon\).

When \(G = Aut(M)\), where \(M\) is an approximately ultrahomogeneous separably categorical structure on \(\mathcal{Y}\), then this holds in the space of all \(L\)-expansions of \(M\).
Definition

The \((G, \mathcal{R})\)-space \((X, \mathcal{U})\) has the **approximation property for graded subgroups** if for any \(\varepsilon > 0\) for any graded subgroup \(H \in \mathcal{R}\), any \(c\) and \(c' \in X\) of the same \(G\)-orbit if \(c, c'\) belong to the same subsets of the form \(\phi \leq t\) for \(H\)-invariant \(\phi \in \mathcal{U}\), then \(c'\) can be approximated by the values \(g(c)\) with \(H(g) < \varepsilon\).

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Definition

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When \(G = Aut(M)\), where \(M\) is an approximately ultrahomogeneous separably categorical structure on \(Y\), then this holds in the space of all \(L\)-expansions of \(M\).
A relational continuous structure $M$ is **approximately ultrahomogeneous** if for any $n$-tuples $(a_1,..,a_n)$ and $(b_1,..,b_n)$ with the same quantifier-free type (i.e. with the same values of predicates for corresponding subtuples) and any $\varepsilon > 0$ there exists $g \in Aut(M)$ such that

$$\max\{d(g(a_j), b_j) : 1 \leq j \leq n\} \leq \varepsilon.$$ 

Any Polish group can be chosen as the automorphism group of a continuous metric structure which is approximately ultrahomogeneous.
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Any Polish group can be chosen as the automorphism group of a continuous metric structure which is approximately ultrahomogeneous.
Let \((\mathcal{Y}, d)\) be a Polish space.

Theorem

- There is a Borel subset \(SC \subset \mathcal{Y}_L\) of separably categorical continuous \(L\)-structures on \((\mathcal{Y}, d)\) so that any separably categorical continuous structure from \(\mathcal{Y}_L\) is isomorphic to a structure from \(SC\).

- There is a Borel subset \(SCU \subset \mathcal{Y}_L\) of separably categorical approximately ultahomogeneous continuous structures on \(\mathcal{Y}\) so that any sep.cat., appr. ultrhom. structure from \(\mathcal{Y}_L\) is isomorphic to a structure from \(SCU\).
Let \((Y, d)\) be a Polish space.

**Theorem**

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**Theorem**

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Polish $G$-spaces similar to logic $G$-spaces of continuous structures
Observation.
Let $M$ be a Polish approximately ultrahomogeneous continuous. Then $Aut(M)$ admits a compatible complete left-invariant metric if and only if there is no proper embedding of $M$ into itself.

The subset of $\mathcal{Y}_L$ consisting of structures $M$ so that $Aut(M)$ admits compatible complete left-invariant metric, is coanalytic in any Borel subset of $\mathcal{Y}_L$. It does not have any member in $\mathcal{SC}$. 
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