Full Satisfaction Classes in a General Setting

Albert Visser + Ali Enayat

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On Tarski(S, F)

Given a base theory B we wish to define certain canonical associated satisfaction theories here denoted B_{FS}, B_{IS}, and B_{FIS}, all of which are formulated in an expansion of the language L_B by adding a new binary predicate S(x, y).

Tarski(S, F) consists of the following axioms:

\begin{align*}
\text{tarski} \ 0 & : (S(x, F) \rightarrow \text{Form}(x)) \land (S(x, \alpha) \rightarrow F(x) \land \text{Asn}(\alpha, x)), \\
\text{tarski} \ 1, R & : (F(x) \land (x = \lfloor R(t_0, \ldots, t_{n-1}) \rfloor) \rightarrow (S(x, \alpha) \leftrightarrow R(\lfloor \alpha \rfloor t_0, \ldots, \lfloor \alpha \rfloor t_{n-1}))).
\end{align*}
Given a base theory $B$ we wish to define certain canonical associated satisfaction theories here denoted $B^{FS}$, $B^{IS}$, and $B^{FIS}$, all of which are formulated in an expansion of the language $\mathcal{L}_B$ by adding a new binary predicate $S(x, y)$.

Tarski(S, F) consists of the following axioms:

1. $\text{tarski}_0(S,F) := (F(x) \rightarrow \text{Form}(x)) \land (S(x,\alpha) \rightarrow F(x) \land \text{Asn}(\alpha, x))$.

2. $\text{tarski}_1, R(S,F) := (F(x) \land (x = \downarrow R(t_0, \cdots, t_{n-1})) \land \text{Asn}(\alpha, x)) \rightarrow (S(x,\alpha) \leftrightarrow R(\lceil \alpha \rceil t_0, \cdots, \lceil \alpha \rceil t_{n-1}))$. 
Given a base theory $B$ we wish to define certain canonical associated satisfaction theories here denoted $B^{FS}$, $B^{IS}$, and $B^{FIS}$, all of which are formulated in an expansion of the language $\mathcal{L}_B$ by adding a new binary predicate $S(x, y)$.

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- Tarski(S, F) consists of the following axioms:

  - $\text{tsk}_0(S, F) := (F(x) \rightarrow \text{Form}(x)) \land (S(x, \alpha) \rightarrow F(x) \land \text{Asn}(\alpha, x))$. 

- $\text{tsk}_1(S, F) := (F(x) \land (x = \lceil R(t_0, \ldots, t_{n-1}) \rceil)) \land \text{Asn}(\alpha, x) \rightarrow (S(x, \alpha) \leftrightarrow R(\lfloor \alpha \rfloor t_0, \ldots, \lfloor \alpha \rfloor t_{n-1}))$. 

On Tarski(S, F)

- Given a base theory $B$ we wish to define certain canonical associated satisfaction theories here denoted $B^{FS}$, $B^{IS}$, and $B^{FIS}$, all of which are formulated in an expansion of the language $\mathcal{L}_B$ by adding a new binary predicate $S(x, y)$.

- Tarski(S, F) consists of the following axioms:
  - $\text{tarski}_0(S, F) := (F(x) \to \text{Form}(x)) \land (S(x, \alpha) \to F(x) \land \text{Asn}(\alpha, x))$.
  - $\text{tarski}_{1,R}(S, F) := (F(x) \land (x = \lceil R(t_0, \cdots, t_{n-1}) \rceil) \land \text{Asn}(\alpha, x)) \to (S(x, \alpha) \leftrightarrow R([\alpha]_{t_0} \cdots, [\alpha]_{t_{n-1}}))$. 
On Tarski$(S, F)$, continued

\[
t_{2}(S, F) := (F(x) \land (x = \neg y)) \land \text{Asn}(\alpha, x) \rightarrow (S(x, \alpha) \leftrightarrow \neg S(y, \alpha)).
\]

\[
t_{3}(S, F) := (F(x) \land (x = \neg (y_1 \lor y_2))) \land \text{Asn}(\alpha, x) \rightarrow (S(x, \alpha) \leftrightarrow (S(y_1, \alpha \restriction \text{FV}(y_1)) \lor S(y_2, \alpha \restriction \text{FV}(y_2)))).
\]

\[
t_{4}(S, F) := (F(x) \land (x = \neg (\exists v_i y))) \land \text{Asn}(\alpha, x) \rightarrow (S(x, \alpha) \leftrightarrow \exists \alpha' (\alpha \sim v_i \alpha' \land S(y, \alpha')))\]
On Tarski\((S, F)\), continued

\[
\text{tarski}_2(S, F) := \left( F(x) \land (x = \neg y) \land \text{Asn}(\alpha, x) \right) \rightarrow \\
(S(x, \alpha) \leftrightarrow \neg S(y, \alpha)).
\]
On Tarski$(S, F)$, continued

\[
\begin{align*}
\text{tarski}_2(S, F) & := \left( F(x) \land (x = \lnot y) \land \text{Asn}(\alpha, x) \right) \to \\
& \quad (S(x, \alpha) \leftrightarrow \neg S(y, \alpha)). \\
\text{tarski}_3(S, F) & := \\
& \quad \left( F(x) \land (x = y_1 \lor y_2) \land \text{Asn}(\alpha, x) \right) \to \\
& \quad \left( S(x, \alpha) \leftrightarrow \left( S(y_1, \alpha \restriction \text{FV}(y_1)) \lor S(y_2, \alpha \restriction \text{FV}(y_2)) \right) \right). 
\end{align*}
\]
On Tarski(S, F), continued

- $\text{tarski}_2(S, F) := (F(x) \land (x = \neg y^\perp) \land \text{Asn}(\alpha, x) \land (S(x, \alpha) \iff \neg S(y, \alpha))).$

- $\text{tarski}_3(S, F) :=$
  
  
  
  \[
  \left( F(x) \land (x = y_1 \lor y_2^\perp) \land \text{Asn}(\alpha, x) \land (S(x, \alpha) \iff \left( S(y_1, \alpha \upharpoonright \text{FV}(y_1)) \lor S(y_2, \alpha \upharpoonright \text{FV}(y_2)) \right) \right).
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- $\text{tarski}_4(S, F) :=$
  
  
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  \]
**B^FS, B^IS, and B^FIS**

B^FS := B^∪ Tarski (S, Form).

B^IS := B^∪ \{Tarski (S, Form^n) : n ∈ ω\} ∪ Ind(S), where Form^n is the collection of formulas L^B with quantifier alternation depth at most n and Ind(S) is the full scheme of induction on N in the language L^B(S) := L^B ∪ {S}.

B^FIS := B^FS ∪ Ind(S) = B^FS ∪ B^IS.
B^{FS}, B^{IS}, and B^{FIS}

- $B^{FS} := B \cup \text{Tarski}(S, \text{Form})$.
\( B^{FS} := B \cup \text{Tarski (}S, \text{Form)} \).

\( B^{IS} := B \cup \{\text{Tarski (}S, \text{Form}_n) : n \in \omega\} \cup \text{Ind}(S)\), where \( \text{Form}_n \) is the collection of formulas \( \mathcal{L}_B \) with quantifier alternation depth at most \( n \) and \( \text{Ind}(S) \) is the full scheme of induction on \( \mathbb{N} \) in the language \( \mathcal{L}_B(S) := \mathcal{L}_B \cup \{S\} \).
\[ B^{FS} := B \cup \text{Tarski} (S, \text{Form}). \]

\[ B^{IS} := B \cup \{ \text{Tarski} (S, \text{Form}_n) : n \in \omega \} \cup \text{Ind}(S), \text{ where Form}_n \]

is the collection of formulas \( \mathcal{L}_B \) with quantifier alternation depth at most \( n \) and \( \text{Ind}(S) \) is the full scheme of induction on \( \mathbb{N} \) in the language \( \mathcal{L}_B(S) := \mathcal{L}_B \cup \{S\} \).

\[ B^{FIS} := B^{FS} \cup \text{Ind}(S) = B^{FS} \cup B^{IS}. \]
Satisfaction Classes

Suppose $M| = B$, $F \subseteq \text{Form}_M$, where $F$ is closed under direct subformulas, and let $S$ be a binary relation on $M$. $S$ is an $F$-satisfaction class if $(M, S, F) | = \text{Tarski}(S, F)$.

If $F = \text{Form}_M \cap \omega$, then we say that $F$ is the set of standard $L_B$-formulas of $M$. In this case there is a unique $F$-satisfaction class on $M$, which we refer to as the Tarskian satisfaction class on $M$.

$S$ is a full satisfaction class on $M$ if $S$ is an $F$-satisfaction class for $F := \text{Form}_M$. This is equivalent to $(M, S) | = B_{FS}$.

$S$ is a satisfaction class on $M$, if either (i) $M$ is $\omega$-standard and $S$ is the usual Tarskian satisfaction relation $\text{Sat}_M$ on $M$; or $M$ is not $\omega$-standard and there is some nonstandard integer $c$ of $M$ such that the expansion $(M, S, F \leq c) | = \text{Tarski}(S, F)$.

A satisfaction class $S$ on $M$ is said to be an inductive satisfaction class on $M$ if $(M, S) | = B_{IS}$.
Suppose $\mathcal{M} \models B$, $F \subseteq \text{Form}^\mathcal{M}$, where $F$ is closed under direct subformulas, and let $S$ be a binary relation on $\mathcal{M}$. A satisfaction class $S$ on $\mathcal{M}$ is said to be an inductive satisfaction class on $\mathcal{M}$ if $(\mathcal{M}, S) \models B IS$. If $F = \text{Form}^\mathcal{M} \cap \omega$, then we say that $F$ is the set of standard $L_B$-formulas of $\mathcal{M}$. In this case there is a unique $F$-satisfaction class on $\mathcal{M}$, which we refer to as the Tarskian satisfaction class on $\mathcal{M}$. 

A satisfaction class $S$ on $\mathcal{M}$ is a full satisfaction class on $\mathcal{M}$ if $S$ is an $F$-satisfaction class for $F := \text{Form}^\mathcal{M}$. This is equivalent to $(\mathcal{M}, S) \models B FS$. S is a satisfaction class on $\mathcal{M}$, if either (i) $\mathcal{M}$ is $\omega$-standard and $S$ is the usual Tarskian satisfaction relation Sat$_\mathcal{M}$ on $\mathcal{M}$; or $\mathcal{M}$ is not $\omega$-standard and there is some nonstandard integer $c$ of $\mathcal{M}$ such that the expansion $(\mathcal{M}, S, F \leq c) \models Tarski(S, F)$. 

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Full Satisfaction Classes in a General Setting
Satisfaction Classes

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Satisfaction Classes

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- If $F = \text{Form}^\mathcal{M} \cap \omega$, then we say that $F$ is the set of standard $\mathcal{L}_B$-formulas of $\mathcal{M}$. In this case there is a unique $F$-satisfaction class on $\mathcal{M}$, which we refer to as the Tarskian satisfaction class on $\mathcal{M}$. 
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A satisfaction class $S$ on $\mathcal{M}$ is said to be an inductive satisfaction class on $\mathcal{M}$ if $(\mathcal{M}, S) \models B^{IS}$. 
A base theory $B$ is strongly reflexive if $B$ is bi-interpretable with the theory $T_B$ formulated in the language of set theory $\{\in\}$ such that $T_B$ satisfies the following two properties; note that property (a) implies that $B$ is an inductive base theory.

(a) $T_B \vdash \text{KP} + \text{Ind} + \text{Infinity}$, where KP is Kripke-Platek set theory.

(b) For each sentence $\phi$ in the language of set theory, $T_B$ proves the implication $\phi \rightarrow \exists x \phi(x)$, where $x$ does not occur in $\phi$ and $\phi(x)$ is the formula obtained by relativizing all of the quantifiers of $\phi$ to $x$. 
A base theory $B$ is *strongly reflexive* if $B$ is bi-interpretable with the theory $T_B$ formulated in the language of set theory $\{\in\}$ such that $T$ satisfies the following two properties; note that property (a) implies that $B$ is an *inductive* base theory.

**(a)** $T_B \vdash KP + \text{Ind} + \text{Infinity}$, where KP is Kripke-Platek set theory.

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$$\varphi \rightarrow \exists x \varphi(x),$$

where $x$ does not occur in $\varphi$ and $\varphi(x)$ is the formula obtained by relativizing all of the quantifiers of $\varphi$ to $x$. 
Examples. Any extension (in the same language) of the following theories is strongly reflexive. Zermelo-Fraenkel set theory $\text{ZF}$. Second Order Arithmetic $\text{Z}_2$ augmented with the full scheme $\Pi_1^\infty$-$\text{DC}$ of dependent choice. Kelley-Morse theory of classes $\text{KM}$ augmented with the full scheme $\Pi_1^\infty$-$\text{DC}$ of dependent choice.

Theorem. Every model $M$ of a strongly reflexive theory base theory $B$ is elementarily equivalent to a model $N$ that carries a full inductive satisfaction class $S$. 
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Theorem. Every model \(\mathcal{M}\) of a strongly reflexive theory base theory \(B\) is elementarily equivalent to a model \(\mathcal{N}\) that carries a full inductive satisfaction class \(S\).
Corollary.
Every countable recursively saturated model of a strongly reflexive base theory carries an inductive full satisfaction class.

Corollary (Conservativity Results).
(a) $\mathcal{B}_{FS} + \text{Ind}(S)$ is conservative over $\mathcal{B}$ for every strongly reflexive base theory $\mathcal{B}$.
(b) $\text{ZF}_{FS} + \text{Sep}(S)$ is a conservative extension of ZF.
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Corollary (Conservativity Results).
(a) $B^{FS} + \text{Ind}(S)$ is conservative over $B$ for every strongly reflexive base theory $B$.
(b) $ZF^{FS} + \text{Sep}(S)$ is a conservative extension of $ZF$. 
Core Lemma. Let $N_0 \models B$ and suppose $S_0$ is an $F_0$-satisfaction class, where $F_0 \subseteq F_1 := \text{Form}_{N_0}$. Then there is an elementary extension $N_1$ of $N_0$ that carries an $F_1$-satisfaction class $S_1 \supseteq S_0$.

Proof: Let $L^+_{B}(N_0)$ be the language obtained by enriching $L_{B}$ with constant symbols for each member of $N_0$, and new unary predicates $U_c$ for each $c \in \text{Form}_{N_0}$. 
Core Lemma. Let $\mathcal{N}_0 \models B$ and suppose $S_0$ is an $F_0$-satisfaction class, where $F_0 \subseteq F_1 := \text{Form}^{\mathcal{N}_0}$. Then there is an elementary extension $\mathcal{N}_1$ of $\mathcal{N}_0$ that carries an $F_1$-satisfaction class $S_1 \supseteq S_0$. 
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Proof: Let $\mathcal{L}_B^+(\mathcal{N}_0)$ be the language obtained by enriching $\mathcal{L}_B$ with constant symbols for each member of $\mathcal{N}_0$, and new unary predicates $U_c$ for each $c \in \text{Form}^{\mathcal{N}_0}$. 
If $R \in \mathbb{L}$ and $N|_c = \downarrow R(t_0, \ldots, t_{n-1})$, then

$$\theta_c := \forall \alpha (U_c(\alpha) \iff \alpha \in A_c \land R([\alpha] t_0, \ldots, [\alpha] t_{n-1}))$$

If $N|_c = \downarrow \neg d$ then

$$\theta_c := \forall \alpha (U_c(\alpha) \iff \alpha \in A_c \land \neg U_d(\alpha))$$

If $N|_c = \downarrow d_1 \lor d_2$ then

$$\theta_c := \forall \alpha (U_c(\alpha) \iff \alpha \in A_c \land (U_{d_1}(\alpha|_{\text{FV}(d_1)}) \lor U_{d_2}(\alpha|_{\text{FV}(d_2)})))$$

If $N|_c = \downarrow \exists v a b$ then

$$\theta_c := \forall \alpha (U_c(\alpha) \iff \exists \alpha' (\alpha \sim v a \alpha' \land U_b(\alpha')))$$

Let $\Gamma := \{ U_c(\alpha) : c \in F_0 \land (c, \alpha) \in S_0 \}$ and define

$$\text{Th}^+(N_0) := \text{Th}(N_0, c) \cup \Theta \cup \Gamma.$$
The Core Construction (2)

- If $R \in \mathcal{L}_B$ and $\mathcal{N}_0 \models c = \neg R(t_0, \cdots, t_{n-1})$, then
  \[
  \theta_c := \forall \alpha \left( \bigcup_c (\alpha) \leftrightarrow \alpha \in A_c \land R([\alpha]_{t_0}, \cdots, [\alpha]_{t_{n-1}}) \right).
  \]
\begin{itemize}
  \item If $R \in \mathcal{L}_B$ and $\mathcal{N}_0 \models c = \neg R(t_0, \ldots, t_{n-1})$, then
    \[ \theta_c := \forall \alpha \left( U_c(\alpha) \leftrightarrow \alpha \in A_c \land R([\alpha]_{t_0}, \ldots, [\alpha]_{t_{n-1}}) \right). \]
  \item If $\mathcal{N} \models c = \neg d$ then
    \[ \theta_c := \forall \alpha \left( U_c(\alpha) \leftrightarrow \alpha \in A_c \land \neg U_d(\alpha) \right). \]
\end{itemize}
If \( R \in \mathcal{L}_B \) and \( \mathcal{N}_0 \models c = \neg R(t_0, \ldots, t_{n-1}) \), then
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\]

If \( \mathcal{N} \models c = d_1 \lor d_2 \), then
\[
\theta_c := \\
\forall \alpha \left( U_c(\alpha) \leftrightarrow \alpha \in A_c \land (U_{d_1}(\alpha \upharpoonright \text{FV}(d_1)) \lor U_{d_2}(\alpha \upharpoonright \text{FV}(d_2))) \right).
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  \[ \theta_c := \forall \alpha \left( U_c(\alpha) \leftrightarrow \alpha \in A_c \land \left( U_{d_1}(\alpha \upharpoonright \text{FV}(d_1)) \lor U_{d_2}(\alpha \upharpoonright \text{FV}(d_2)) \right) \right). \]

- If $\mathcal{N} \models c = \neg \neg \exists v_a b \neg$, then
  \[ \theta_c := \forall \alpha \left( U_c(\alpha) \leftrightarrow \exists \alpha' \left( \alpha \sim p v_a \alpha' \land U_b(\alpha') \right) \right). \]
If \( R \in \mathcal{L}_B \) and \( N_0 \models c = \neg R(t_0, \ldots, t_{n-1}) \), then
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\theta_c := \forall \alpha \left( U_c(\alpha) \leftrightarrow \exists \alpha' (\alpha \sim_{v_a} \alpha' \land U_b(\alpha')) \right).
\]

Let \( \Gamma := \{ U_c(\alpha) : c \in F_0 \text{ and } (c, \alpha) \in S_0 \} \) and define
\[
\text{Th}^+(N_0) := \text{Th}(N_0, c)_{c \in N_0} \cup \Theta \cup \Gamma.
\]
We now proceed to show that $\text{Th}^+ (\mathbb{N}_0)$ is consistent by demonstrating that each finite subset of $\text{Th}^+ (\mathbb{N}_0)$ is interpretable in $(\mathbb{N}_0, S_0)$. To this end, suppose $T_0$ is a finite subset of $\text{Th}^+ (\mathbb{N}_0)$ and let $C$ consist of the collection of constants $c$ that appear in at least one of the sentences in $T_0 \cap \Theta$.

If $C = \emptyset$, $T_0$ is readily seen to be consistent, so we shall assume that $C \neq \emptyset$ for the rest of the argument. Our goal is to construct subsets $\{U_c : c \in C\}$ of $\mathbb{N}_0$ such that the following two conditions hold when $U_c$ is interpreted by $U_c$:

1. $(\mathbb{N}_0, U_c) | _c \in C | _c = \{ \theta_c : c \in C \}$
2. $\alpha \in U_c$ whenever $c \in C \cap F_0$ and $(c, \alpha) \in S_0$.

We shall construct $\{U_c : c \in C\}$ in stages, beginning with the simplest formulas in $C$, and working our way up using Tarski rules for more complex ones.
We now proceed to show that $\text{Th}^+(\mathcal{N}_0)$ is consistent by demonstrating that each finite subset of $\text{Th}^+(\mathcal{N}_0)$ is interpretable in $(\mathcal{N}_0, S_0)$. To this end, suppose $T_0$ is a finite subset of $\text{Th}^+(\mathcal{N}_0)$ and let $C$ consist of the collection of constants $c$ that appear in at least one of the sentences in $T_0 \cap \Theta$. If $C = \emptyset$, $T_0$ is readily seen to be consistent, so we shall assume that $C \neq \emptyset$ for the rest of the argument. Our goal is to construct subsets $\{U_c : c \in C\}$ of $\mathcal{N}_0$ such that the following two conditions hold when $U_c$ is interpreted by $U_c$:

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(1) $(\mathcal{N}_0, U_c)_{c \in C} \models \{\theta_c : c \in C\}$ and

(2) $\alpha \in U_c$ whenever $c \in C \cap F_0$ and $(c, \alpha) \in S_0$. 

We shall construct $\{U_c : c \in C\}$ in stages, beginning with the simplest formulas in $C$, and working our way up using Tarski rules for more complex ones.
The Core Construction (3)

We now proceed to show that $\text{Th}^+(\mathcal{N}_0)$ is consistent by demonstrating that each finite subset of $\text{Th}^+(\mathcal{N}_0)$ is interpretable in $(\mathcal{N}_0, S_0)$.

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We shall construct $\{U_c : c \in C\}$ in stages, beginning with the simplest formulas in $C$, and working our way up using Tarski rules for more complex ones.
Let $c \triangleleft d$ express "$c$ is a direct subformula of $d$".

Define $\triangleleft^*$ on $C$ by:

$c \triangleleft^* d$ iff $(c \triangleleft d) \land \theta_d \in T_0 \cap \Theta$. 

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Full Satisfaction Classes in a General Setting
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Define $\triangleleft^*$ on $C$ by:

$$c \triangleleft^* d \iff (c \triangleleft d)^{N_0} \text{ and } \theta_d \in T_0 \cap \Theta.$$
The finiteness of $C$ implies that $(C, \lhd \ast)$ is well-founded, which in turn helps us define a useful measure of complexity for $c \in C$ using the following recursive definition:

$$\text{rank}_{C}(x) := \sup \{ \text{rank}_{C}(y) + 1 : x \in C \text{ and } (y \lhd \ast x) \geq 0 \}.$$ 

Note that $\text{rank}_{G}(c) = 0$ precisely when there is no $x \in C$ such that $(x \lhd \ast c) \geq 0$.

Next, let $C_{i} := \{ x \in C : \text{rank}_{C}(x) \leq i \}$. Observe that since $C$ is finite, $C_{0} \neq \emptyset$, and $c \in C_{0}$ iff $c \in C$ and $C$ does not contain the code of any subformula of the formula coded by $c$. Moreover, if $c \in C_{i+1}$, then the codes of every immediate subformula of the formula coded by $c$ are in $C_{i}$. This observation ensures that the following recursive clauses yield a well-defined $U_{c}$ for each $c \in C$. 
The finiteness of $C$ implies that $(C, \triangleleft^*)$ is well-founded, which in turn helps us define a useful measure of complexity for $c \in C$ using the following recursive definition:
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Albert Visser + Ali Enayat

Full Satisfaction Classes in a General Setting
If \( c \in C_0 \) then \( U_c := \{ \alpha : (c, \alpha) \in S_0 \} \), if \( c \in F_0 \); \( U_c := \emptyset \), if \( c \notin F_0 \).

If \( c \in C_i+1 \setminus C_i \) and \( \downarrow c = \neg d \), then \( U_c := \{ \alpha \in A_c : \alpha \not\in U_d \} \).

If \( c \in C_i+1 \setminus C_i \) and \( c = \downarrow a \lor b \), then \( U_c := \{ \alpha \in A_c : \alpha \upharpoonright \text{FV}(a) \in U_a \lor \alpha \upharpoonright \text{FV}(b) \in U_b \} \).

If \( c \in C_i+1 \setminus C_i \) and \( c = \downarrow \exists v \, a \, b \), then \( U_c := \{ \alpha \in A_c : \exists \alpha' \in N(\alpha \sim v \, a \, \alpha') \land \alpha \in U_b \} \).
If $c \in C_0$ then $U_c := \begin{cases} \{ \alpha : (c, \alpha) \in S_0 \}, & \text{if } c \in F_0; \\ \emptyset, & \text{if } c \notin F_0. \end{cases}$
The Core Construction (6)

If $c \in C_0$ then $U_c := \begin{cases} \{ \alpha : (c, \alpha) \in S_0 \}, & \text{if } c \in F_0; \\ U_c := \emptyset, & \text{if } c \notin F_0. \end{cases}$

If $c \in C_{i+1} \setminus C_i$ and $\neg c = \neg d \upharpoonright$, then

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- If $c \in C_{i+1} \setminus C_i$ and $c = \neg a \lor b^\neg$, then

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Core Theorem. Let \( M_0 \) be a model of \( B \) of any cardinality. There is an elementary extension \( M \) of \( M_0 \) that admits a full satisfaction class.

Proof: Let \( F_0 \) be the set of atomic \( N \)-formulas and let \( S_0 \) be the obvious satisfaction predicate for \( F_0 \). Then by the Lemma there is an elementary extension \( M_1 \) of \( M_0 \) that carries a Form \( M_0 \)-satisfaction class. Thanks to Lemma 3.1, this argument can be carried out countably many times to yield two sequences \( \langle M_i : i \in \omega \rangle \) and \( \langle S_i : i \in \omega \rangle \) that satisfy the following two properties:

1. \( M_i \preceq M_{i+1} \);
2. \( S_{i+1} \) is a Form \( M_i \)-satisfaction class on \( M_{i+1} \) for each \( i \in \omega \).

\[ M := \bigcup_{i \in \omega} M_i, \quad S := \bigcup_{i \in \omega} S_i. \]
Core Theorem. Let $\mathcal{M}_0$ be a model of $\text{B}$ of any cardinality. There is an elementary extension $\mathcal{M}$ of $\mathcal{M}_0$ that admits a full satisfaction class.

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Full Satisfaction Classes in a General Setting
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$\mathcal{M} := \bigcup_{i \in \omega} \mathcal{M}_i$, and $S := \bigcup_{i \in \omega} S_i$. 
Corollary. $B_{FS}$ is a conservative extension of $B$ for every base theory $B$.

Theorem. Let $<L$ be a $B$-definable linear order on $\mathbb{N}$ in the sense of $B$. Every model of $B$ has an elementary extension to a model that expands to $B_{FS}L$. Consequently, $B_{FS}L$ is conservative over $B$ for every base theory $B$. 
**Corollary.** $B^{FS}$ is a conservative extension of $B$ for every base theory $B$. 
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Definition.
(a) Suppose $M$ is a model of some base theory, and $N$ is a structure in a finite language $L$. $N$ is strongly interpretable in $M$ if $M$ can interpret an isomorphic copy $N_0$ of $N$; and moreover there is an $M$-definable $F$-satisfaction class $S$ on $N_0$, where $F$ is the collection of all $L$-formulas in the sense of $M$.

(b) $B$ strongly interprets $B_{FS0}$, i.e., every model $M|\models B$ strongly interprets a structure $(N, S)|\models B_{FS0}$ in a uniform manner.

Theorem.
Suppose $B$ is an inductive base theory such that $B \vdash \text{Con}(B_0)$, where $B_0$ is some r.e. base theory. Then $B$ strongly interprets $B_{FS0}$.
**Definition.**

(a) Suppose $\mathcal{M}$ is a model of some base theory, and $\mathcal{N}$ is a structure in a finite language $\mathcal{L}$. $\mathcal{N}$ is strongly interpretable in $\mathcal{M}$ if $\mathcal{M}$ can interpret an isomorphic copy $\mathcal{N}_0$ of $\mathcal{N}$; and moreover there is an $\mathcal{M}$-definable $F$-satisfaction class $S$ on $\mathcal{N}_0$, where $F$ is the collection of all $\mathcal{L}$-formulas in the sense of $\mathcal{M}$.
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Theorem. Suppose $B$ is an inductive base theory such that $B \vdash \text{Con}(B_0)$, where $B_0$ is some r.e. base theory. Then $B$ strongly interprets $B_0^{FS}$. 
Corollary.

1. \( B \vdash \text{Con}(B_{FS0}) \) for every finitely axiomatized base theory \( B_0 \subseteq B \).

2. \( B_{FS} \) and \( B_{IS} \) are not finitely axiomatizable for inductive base theories \( B \).
Corollary. If $B$ is an inductive theory, then:

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Corollary. If $B$ is an inductive theory, then:

1. $B \vdash \text{Con}(B_0^{\text{FS}})$ for every finitely axiomatized base theory $B_0 \subseteq B$.

2. $B^{\text{IS}}$ and $B^{\text{FS}}$ are not finitely axiomatizable for inductive base theories $B$. 
The Arithmetization of the Core Construction (3)

Theorem. The following statement (∗) is provable within WKL₀:

(∗) Every consistent base theory B has a model M that carries a full satisfaction class S and which has the property that the Tarskian satisfaction relation of (M, S) is coded by some X ⊆ ω.

Theorem. PRA ⊢ "B FS is conservative over B" for every r.e. base theory B.
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Theorem. PRA \vdash "B^{FS} is conservative over B" for every r.e. base theory B.
Pathological Satisfaction Classes

Definition. For any standard formula $\sigma$ of $L_B$, and for each $a \in \mathbb{N}^M$, where $M$ is some prescribed model of $B$, the 'formula' $\sigma_a$ is defined by internal recursion in $M_0$ via $\sigma_0 := \sigma$; and $\sigma_{n+1} := \sigma_n \lor \sigma_n$.

Theorem. Let $\sigma := \exists v_0 (v_0 = v_0)$ (or $\sigma =$ any other logically valid sentence), and $M_0$ be a model of $B$ of any cardinality. Then $M_0$ has an elementary extension $M$ that carries a full satisfaction class $S$ such that $\{ a \in \mathbb{N}^M : \sigma_a$ is $S$-valid $\} = \omega$. 
Definition. For any standard formula $\sigma$ of $\mathcal{L}_B$, and for each $a \in \mathbb{N}^\mathcal{M}$, where $\mathcal{M}$ is some prescribed model of $B$, the ‘formula’ $\sigma_a$ is defined by internal recursion in $\mathcal{M}_0$ via $\sigma_0 := \sigma$; and $\sigma_{n+1} := \sigma_n \lor \sigma_n$. 
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Theorem. Let $\sigma := \exists v_0 (v_0 = v_0)$ (or $\sigma = \text{any other logically valid sentence}$), and $\mathcal{M}_0$ be a model of $B$ of any cardinality. Then $\mathcal{M}_0$ has an elementary extension $\mathcal{M}$ that carries a full satisfaction class $S$ such that

$$\{ a \in \mathbb{N}^\mathcal{M} : \sigma_a \text{ is } S\text{-valid} \} = \omega.$$
Theorem. Let $M_0 | B$, where $B$ is a base theory. There is an elementary extension $M$ of $M_0$ that carries full satisfaction classes $S_1, S_2,$ and $S_3$ such that:

1. $S_1$ is schematically correct;
2. $S_2$ is both existentially and disjunctively correct; and
3. $S_3$ is both extensional and alphabetically correct.
Theorem. Let $M_0 \models B$, where $B$ is a base theory. There is an elementary extension $M$ of $M_0$ that carries full satisfaction classes $S_1, S_2, \text{ and } S_3$ such that:

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Desirable Satisfaction Classes (1)

**Theorem.** Let $M_0 \models B$, where $B$ is a base theory. There is an elementary extension $M$ of $M_0$ that carries full satisfaction classes $S_1, S_2, \text{ and } S_3$ such that:

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2. $S_2$ is both existentially and disjunctively correct; and
3. $S_3$ is both extensional and alphabetically correct.
Moreover, if $B$ is an inductive base theory, then $M$ carries a full satisfaction class $S_4$ such that:

$$(4) \quad S_4 \text{ is } \Sigma^B_\infty \text{-correct},$$

and there is a family $\{S_5,s\} : s \in \mathbb{N}_M$ of full satisfaction classes on $M$ such that for each $s \in \mathbb{N}_M$ there is a cut $I$ of $\mathbb{N}_M$ with $I|_s = \text{PA}$ with $s \in I$ such that:

$$(5) \quad S_5,s \text{ is } I \text{-deductively correct}.$$
Moreover, if $B$ is an inductive base theory, then $\mathcal{M}$ carries a full satisfaction class $S_4$ such that:

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$\text{(5}_s) : S_{5,s}$ is $I$-deductively correct.
Let ACA be the strengthening of ACA_0 with the full scheme of induction. It has been long known that ACA and PA_{FIS} are ‘proof-theoretically equivalent’. The result below provides a more precise relationship between the two theories.

Theorem. There is a sentence σ in the language of ACA_0 such that PA_{FIS} and ACA + σ are bi-interpretable.

Theorem. B_{IS} and B_{FS} are both interpretable in B for every inductive recursively axiomatizable base theory B.
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Interpretability Issues (2)

(a) The theories \{PA, PA_{IS}, PA_{FS}\} are mutually interpretable.

(b) Each of the theories \{PA, PA_{IS}, PA_{FS}\} is interpretable in ACA₀, but none of them interprets ACA₀.

(c) No pair of the theories \{PA, PA_{FS}, PA_{IS}, ACA₀\} are bi-interpretable.
Theorem (Interpretability among \( \text{PA}, \text{PA}^{\text{IS}}, \text{PA}^{\text{FS}}, \) and \( \text{ACA}_0 \)).
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Interpretability Issues (3)

Theorem. If $B$ is a consistent finitely axiomatizable base theory, then neither $B$ nor $B'$ is interpretable in $B$. 
Theorem. If \( B \) is a consistent finitely axiomatizable base theory, then neither \( B^{IS} \) nor \( B^{FS} \) is interpretable in \( B \).