Truth in the limit

Marcin Mostowski

Institute of Philosophy, Warsaw University
m.mostowski@uw.edu.pl

Abstract. We consider sl–semantics in which first order sentences are interpreted in potentially infinite domains. A potentially infinite domain is a growing sequence of finite models. We prove the completeness theorem for first order logic under this semantics. Additionally we characterize the logic of such domains as having a learnable, but not recursive, set of axioms.

The work is a part of author's research devoted to computationally motivated foundations of mathematics.

1   Introduction

We present here some results related to logic of potential infinity. The idea is slightly unconventional in mathematics of our days. Then we start with intuitions and some history.

The research reported here is motivated by searching computationally motivated foundations of mathematics. Inspirations for this search can be found in pre–computational era, particularly in works by Leopold Kronecker [10] and David Hilbert [9].

Kronecker postulates that natural numbers are based on counting procedure. So in every moment only finitely many of them are generated. Of course mathematics deals with what can happen further.

Hilbert – evidently influenced by Kronecker – recalled the Aristotelian notions of actual and potential infinity (see [1]). Actually infinite sets simply contain infinitely many members. Potentially infinite sets are finite, but they allow arbitrary finite enlargements. These enlargements can be repeated with no bounds. Any counting procedure determines such potentially infinite set of natural numbers.

Paradoxically one of the last works on foundations written in the spirit of potentially infinite mathematics was the Kurt Gödel work presenting the first version of the completeness theorem [4]. He had no tools for semantical considerations on models of arbitrary cardinality,[1] then he considered semantical notions only for finite models. The countable model, which he is constructing, is determined by finite approximations.

In more recent times the idea was recalled by Jan Mycielski [18] and [19]. In the first paper Mycielski discuss foundations of analysis defined on initial

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[1] It is know that the notion of truth was mathematized a few years later by Alfred Tarski in [21].

segments of natural numbers. His approach essentially agrees with presented in this paper. Nevertheless he proposes the general framework in the style of nonstandard analysis. As a result he leaves finite framework. It seems that Mycielski's motivations and intuitions are in agreement with those of our paper. Nevertheless technically his solutions go in essentially different direction.

Last but not least, let us mention another ancient idea, namely Euclidean plane geometry as presented in Elements [3]. His fifth postulate "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."([7], vol. 1, p. 151) seems to be meaningless when we think of straight lines as actually given.

This can be easily understood when we recall Aristotle's explanation from Physics: "Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish." (see [1]). It means that points on a straight line are added during our construction.

The fifth common notion of Euclid "The whole is greater than the part." ([7], vol. 1, p. 151) puts things even stronger. It is simply false when understood in a spirit of actual infinity.[2]

Summarizing, plain geometry of Euclid is determined by a sequence of constructions which on every stage are finite.

2 Potentially infinite domains

In this section we define sl–semantics or semantics of potential infinity. The idea of this semantics was formulated in [14], and it was investigated later in a few papers devoted to finite arithmetics: [17], [16], [13], [11], [12].

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[2] Quite competent modern commentator Heath tries to interpret The fifth common notion of Euclid as misunderstanding. He writes ([7], vol. 1, p. 232)

"The whole is greater than the part.

Proclus includes this "axiom" on the same ground as the preceding one. I think however there is force in the objection which Tannery takes to it, namely that it replaces a different expression in Eucl. I. 6, where it is stated that "the triangle DBC will be equal to the triangle ACB, *the less to the greater: which is absurd.*" The axiom appears to be an abstraction or generalisation substituted for an immediate inference from a geometrical figure, but it takes the form of a sort of definition of whole and part. The probabilities seem to be against its being genuine, notwithstanding Proclus' approval of it.

Clavius added the axiom that the whole is the equal to the sum of its parts."

Surely Heath knew very well the observation of Bernard Bolzano [2] that infinite sets are equicardinal with their proper subsets. However for finite sets it cannot happen. Therefore in potentially infinite domains the fifth common notion of Euclid is true.

In this paper we consider sl–semantics in general setting. However we start with recalling some ideas related to arithmetical models for two reasons. Firstly they adequately explain basic intuitions. Secondly, we need them as examples for the hardest cases.

2.1   FM–domains

Actually infinite domain of natural numbers is the set $N = \{0, 1, 2, \ldots\}$.

As an explication of potentially infinite domain of natural numbers we mean the family of finite approximations of the actually infinite domain. This is the following family:

$$\begin{aligned}
&\{0\}, \\
&\{0, 1\}, \\
&\{0, 1, 2\}, \\
&\{0, 1, 2, 3\}, \\
&\vdots
\end{aligned}$$

Therefore, having an arithmetical model $M = (N, R_1, \ldots, R_s)$, we define its potentially infinite version as the family $FM(M) = \{M_1, M_2, M_3, \ldots\}$, where $M_n = (N_n, R_1^{<n}, \ldots, R_s^{<n})$, $N_n = \{0, 1, \ldots, n-1\}$ and $R_i^{<n}$ is the restriction of R_i to the set $\{0, 1, \ldots, n-1\}$.

When the the basic model M will be just the standard model of addition and multiplication then the elements of $FM(M)$ will be called *arithmetical models*.

2.2 π–domains

Let σ be a vocabulary. The set of σ–sentences (closed formulae) is denoted by F_σ. We assume that all the considered vocabularies are purely relational, it means that there are no individual constants and function symbols. Moreover all vocabularies are finite.

Let K be a class of finite models for a given finite relational vocabulary σ. The sl–theory of K, $sl(K)$ is the set of all those sentences from F_σ which are true in almost all models from K, that is

$$sl(K) = \{\varphi \in F_\sigma : \exists k \forall M \in K (\mathrm{card}(M) > k \Rightarrow M \models \varphi)\}.$$

For a class of finite models K and a formula φ we define *truth in the limit* or *truth in all sufficiently large models* relation $\models_{sl}$ as follows

$$K \models_{sl} \varphi \text{ if and only if } \exists k \forall M \in K (\mathrm{card}(M) > k \Rightarrow M \models \varphi).$$

Thus we can define $sl(K)$ equivalently as

$$sl(K) = \{\varphi \in F_\sigma : K \models_{sl} \varphi\}.$$

Let us observe that if the class K contains only models of cardinality bounded by some natural number n then $sl(K) = F_\sigma$. So sl–theory of this class is in a sense inconsistent.

A class K of finite σ–models is unbounded if and only if for each n there is $M \in K$ such that $card(M) > n$. The class K will be called *unbounded σ–class* or simply *unbounded class* when a vocabulary will be clear from the context.

Unbounded classes containing for each cardinality at most one (up to isomorphisms) model approximate in a sense an infinite domain. Such classes will be called *potentially infinite domains*, or *pi–domains*, or *π–domains*. When a π–domain K describes an infinite structure in a stronger sense, that is $K = \{M_0, M_1, M_2, \ldots\}$ and for all n, $M_n \subseteq M_{n+1}$, then we say that K is a *proper π–domain*.

By $\mathbf{MOD}_\sigma$ we mean the class of all finite models of vocabulary σ. Because we consider only finite vocabularies then there is only countably many non isomorphic models in $\mathbf{MOD}_\sigma$. Moreover we can assume that all finite models are defined on the initial segments of natural numbers.[3] Therefore we can assume that $\mathbf{MOD}_\sigma$ is a reasonable, computationally manageable, countable set.

In what follows we will frequently use the set of purely logical sentences $A_{inf} = \{\xi_1, \xi_2, \xi_3, \ldots\}$, where ξ_n is the following:

$$\forall x_1 \ldots \forall x_n \exists y (x_1 \neq y \wedge \ldots \wedge x_n \neq y).$$

The sentence ξ_n says that *there are more than n objects.*

3 Logics of finite models

3.1 Logics L_{sl} and L_{fin}

When a vocabulary σ is fixed then sl–logic L_{sl} we define as $L_{sl} = sl(\mathbf{MOD}_\sigma)$. The logic of finite models L_{fin} is defined as $L_{fin} = th(\mathbf{MOD}_\sigma)$, where

$$th(K) = \{\varphi \in F_\sigma : M \models \varphi, \text{ for all } M \in K\}.$$

Of course we have $L_{fin} \subseteq L_{sl}$. Additionally the inclusion is proper because each statement ξ_n, for $n > 0$, belongs to $L_{sl} - L_{fin}$, where ξ_n is the following:

$$\forall x_1 \ldots \forall x_n \exists y (x_1 \neq y \wedge \ldots \wedge x_n \neq y).$$

Let us define the set A_{sl} as the union of L_{fin} and $A_{inf} = \{\xi_1, \xi_2, \ldots\}$, that is $A_{sl} = L_{fin} \cup A_{inf}$.

Theorem 1. *L_{sl} is the set of all first oder consequences of A_{sl}.*

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[3] Of course it does not mean that we consider only arithmetical models. We use only the fact that each finite model is isomorphic with a model having the set $\{0, 1, \ldots, n-1\}$ as the universe, for some n.

Proof. We know that $A_{sl} \subseteq L_{sl}$. Moreover, see [14], L_{sl} is closed on first oder consequences. Therefore it suffices to prove that for all $\varphi \in F_\sigma$, if $\varphi \in L_{sl}$ then $A_{sl} \vdash \varphi$.

Let us assume that $A_{sl} \nvdash \varphi$. It means that $\neg\varphi$ is consistent with A_{sl} and particularly the sentence $(\xi_n \wedge \neg\varphi)$ is consistent, for arbitrary n. Then $(\xi_n \wedge \neg\varphi)$ has a model. We claim that it has a finite model. Let us suppose that it has no finite model. Then $\neg(\xi_n \wedge \neg\varphi) \in L_{fin}$. It follows that $\xi_n \Rightarrow \varphi$ belongs to L_{fin}. Therefore φ is a first order consequence of A_{sl}, what is impossible by our assumption.

We have proved that the sentence $(\xi_n \wedge \neg\varphi)$ has a finite model, for each n. Therefore $\neg\varphi$ has arbitrary large finite models and φ cannot be true in almost all finite models, so $\varphi \notin L_{sl}$.

3.2 Trachtenbrot's theorem

Trachtenbrot proved that the logic L_{fin} is not recursively enumerable. Because we need here some of the refinements of his theorem then we give here a sketchy proof of it.

In what follows we need some properties of the Kleene predicate T and his function U. $T(e, c, n)$ says that *e is a Turing machine and c is its computation with the input n*. $U(c)$ gives the output of the computation c. The both T and U can be defined by arithmetical formulae with all quantifiers bounded by the condition "$< c$". So we get the following:

Lemma 1. *There are arithmetical formulae $\varphi_T(x, y, z)$ and $\varphi_U(x, y)$ such that for each $e, c, n, m \in N$ and each arithmetical finite model of cardinality greater than e, c, n, m we have*

$$T(e, c, n) \text{ if and only if } M \models \varphi_T(e, c, n)$$

and

$$U(c) = m \text{ if and only if } M \models \varphi_U(c, m).$$

Moreover in models too small – that is of cardinality $\leq c$ – the formulae are false.

In other words, the Kleene predicate T and the function U are definable in finite models.

It is known, see [15], that models isomorphic to finite arithmetical models can be separated by finite set of axioms in the class of finite models. Then we have the following.

Lemma 2. *Let us assume here that σ is the arithmetical vocabulary. There is a sentence φ_A such that the class $K = \{M \in \mathbf{MOD}_\sigma : M \models \varphi_A\}$ contains all arithmetical models and only models isomorphic to arithmetical models.*

Now we can prove the following.

Theorem 2 (Trachtenbrot's theorem). *The set L_{fin} is Π_1^0–complete.*

Proof. From definition it follows that the set L_{fin} is Π_1^0. Then we have to show that all Π_1^0–sets are recursively reducible to L_{fin}. We will show that Σ_1^0–complete set $H = \{(e,n) : \exists c T(e,c,n)\}$ is recursively reducible to the consistency in finite models – the problem which is equivalent to the complement of L_{fin}. The set H is so called "the halting problem".

We define $f : (e,n) \mapsto (\varphi_A \wedge \exists c \varphi_T(e,c,n))$. From the construction it follows that e hallts on the input n if and only if $f(e,n)$ has a finite model.

As a corollary we obtain the following.

Theorem 3. *The set A_{sl} is Π_1^0–complete.*

Proof. The set A_{sl} is the union of two disjoint sets: L_{fin} – which is Π_1^0–complete, and $\{\xi_1, \xi_2, \xi_3, \ldots\}$ – which is recursive. Therefore also A_{sl} is Π_1^0–complete.

3.3 Complexity of sl–logic

The main advantage of axiomatic method is the possibility of giving conclusive arguments for truth of our claims in a way that soundness of these arguments can be checked in a routine way. Therefore sets of theorems of practically useful theories have to be recursively enumerable or equivalently Σ_1^0. This is so because *φ is a theorem* is equivalent to:

$$\exists D \ (D \text{ is a proof for } \varphi),$$

and the relation "*is a proof for*" should be recursive.

The existence of complete proof procedure is the main advantage of first order logic in comparison with stronger logical systems. Therefore applicability of first order logic for theories based on sl–semantics seems to be a good argument for plausibility of sl–semantics. Unfortunately things are not so simple.

Theorem 4. *The set L_{sl} is Σ_2^0–complete.*

Proof. In the paper [17] it is proved that for the standard model of arithmetic M, the set $sl(FM(M))$ is Σ_2^0–complete. Therefore it suffices to reduce this set to L_{sl}. For any formula φ, we define $f(\varphi) = (\varphi_A \Rightarrow \varphi)$. Then we obtain

$$\varphi \in sl(FM(M)) \text{ if and only if } f(\varphi) \in L_{sl}.$$

4 The Completeness Theorem

The classical completeness theorem[4] says that for each $T \subseteq F_\sigma$ and $\varphi \in F_\sigma$ we have the equivalence

$$T \vdash \varphi \text{ if and only if } T \models \varphi,$$

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[4] The completeness theorem was proved for the first time in the earlier mentioned paper by Kurt Gödel [4]. However in this general form it was proved by Leon Henkin [8].

where on the left side we have the standard provability relation $\vdash$, and on the right side we have the standard semantical entailment $\models$. What will happen if we replace $\models$ by $\models_{sl}$?

In this section we are going to prove the standard completeness is also valid for sl–semantics.

Now we prove the crucial lemma needed for constructing suitable π–domains.

Lemma 3. *Let $T \subseteq F_\sigma$ contain sl–logic ($A_{sl} \subseteq T$) and $\varphi \in F_\sigma$.*

If φ is consistent with T ($T \nvdash \neg\varphi$) then for each finite subset $\{\psi_1, \ldots, \psi_n\} \subseteq T$ there are arbitrary large finite models M such that

$$M \models (\psi_1 \wedge \ldots \wedge \psi_n \wedge \varphi).$$

Proof. Let us assume that T satisfies the assumptions and $T \nvdash \neg\varphi$. Let us take any finite subset $\{\psi_1, \ldots, \psi_n\} \subseteq T$. Then the formula $(\psi_1 \wedge \ldots \wedge \psi_n \wedge \varphi)$ has a model. We have show that it has a finite model. Let us assume that it has only infinite models. Then the formula $\neg(\psi_1 \wedge \ldots \wedge \psi_n \wedge \varphi)$ belongs to L_{fin}, so also the formula $((\psi_1 \wedge \ldots \wedge \psi_n) \Rightarrow \neg\varphi)$ belongs to L_{fin}. However this means that $T \vdash \neg\varphi$, what contradicts to the assumptions. Therefore there is a finite model M such that

$$M \models (\psi_1 \wedge \ldots \wedge \psi_n \wedge \varphi).$$

It remains to prove that M can be arbitrary large. However we can always put into the considered finite set at the very begining the statement ξ_m, for arbitrary large m, so the size of M should be greater than m.

Now we are ready to prove of the main theorem of this section.

Theorem 5 (The Completeness Theorem). *For each $T \subseteq F_\sigma$ containing A_{sl} ($A_{sl} \subseteq T$) and $\varphi \in F_\sigma$ we have the equivalence*

$$T \vdash \varphi \text{ if and only if } T \models_{sl} \varphi.$$

Proof. Let $T \subseteq F_\sigma$ contain sl–logic ($L_{sl} \subseteq T$) and $\varphi \in F_\sigma$.

($\Rightarrow$) Let us assume that $T \vdash \varphi$. Then there is a finite subset $\{\psi_1, \ldots, \psi_n\} \subseteq T$ such that $((\psi_1 \wedge \ldots \wedge \psi_n) \Rightarrow \varphi)$ is a first order tautology.

Let K be π–domain (proper or not) such that $K \models_{sl} T$. Therefore for each $i = 0, 1, \ldots, n$ there is k_i such that ψ_i is true in all models from K of cardinality greater than k_i. We take $k = \max(k_0, k_1, \ldots, k_n)$. Then all the sentences $\psi_0, \psi_1, \ldots, \psi_n$ are true in all models from K of cardinality greater than k. Hence also φ is true in all models from K of cardinality greater than k. This proves that $T \models_{sl} \varphi$.

($\Leftarrow$) Now let us assume that $T \nvdash \varphi$. Therefore $\neg\varphi$ is consistent with T. Let $\psi_0, \psi_1, \psi_2, \ldots$ be enumeration of all sentences from T. We construct an infinite sequence of finite models $M_0, M_1, M_2, \ldots$ such that for all n, $\operatorname{card}(M_n) < \operatorname{card}(M_{n+1})$ and

$$M_n \models (\psi_0 \wedge \psi_1 \wedge \ldots \wedge \psi_n \wedge \neg\varphi).$$

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Let us assume that models $M_0, M_1, \ldots, M_{n-1}$ were defined. Then, by lemma 3, there is a model M_n such that $\text{card}(M_n) > \text{card}(M_{n-1})$ and

$$M_n \models (\psi_0 \wedge \psi_1 \wedge \ldots \wedge \psi_n \wedge \neg\varphi).$$

Finally we take $K = \{M_0, M_1, M_2, \ldots\}$. By the construction we have $K \models_{sl} T$ and $K \models_{sl} \neg\varphi$. Therefore $T \not\models_{sl} \varphi$.

Let us consider an example. In our language we have one unary predicate U and two binary predicates S and R.

$$T_1 = \{\forall x \forall y \forall y'(S(x,y) \wedge S(x,y') \Rightarrow y = y'),$$
$$\forall x \forall x' \forall y(S(x,y) \wedge S(x',y) \Rightarrow x = x'),$$
$$\forall x \exists y \; S(x,y),$$
$$\forall y \exists x \; S(x,y),$$
$$\forall x \forall y(S(x,y) \Rightarrow x \neq y),$$
$$\exists^{=1} x \; \neg U(x),$$

R restricted to U is a linear ordering with S as the successor$\}$,

$$T = T_1 \cup A_{sl}.$$

There is no proper π–domain K such that $K \models_{sl} T$.

5 Characteristic π–domains

In this section we prove a generalization of the theorem of Micha Krynicki, Jerzy Tomasik, and Konrad Zdanowski [12].

Theorem 6 (Existence of characteristic π–domains).
For each consistent $T \subseteq F_\sigma$ containing A_{sl} ($A_{sl} \subseteq T$) there is a π–domain K such that $sl(K)$ is the set of all first order consequences of T.

Proof. Let $T \subseteq F_\sigma$ be a consistent theory such that $A_{sl} \subseteq T$. Let us fix $\varphi_0, \varphi_1, \varphi_2, \ldots$ enumeration of all elements of T. Let $\psi_0, \psi_1, \psi_2, \ldots$ be enumeration of all sentences $\psi \in F_\sigma$ such that $T \not\vdash \psi$. Moreover we assume that all such ψ are repeated infinitely many times in this enumeration.

Now we construct – using lemma 3 – the sequence of models $M_0, M_1, M_2, \ldots$ such that for all n: $\text{card}(M_n) < \text{card}(M_{n+1})$ and

$$M_n \models (\varphi_0 \wedge \varphi_1 \wedge \ldots \wedge \varphi_n \wedge \neg\psi_n).$$

Finally we take $K = \{M_0, M_1, M_2, \ldots\}$.

Now let us assume that $T \vdash \varphi$. Then $sl(K) \vdash \varphi$ and $\varphi \in sl(K)$.

Now let us assume that $T \not\vdash \varphi$. Then – by the construction – there are infinitely many models in K not satisfying φ. Therefore $\varphi \notin sl(K)$.

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Let us observe that we cannot claim that the above would be a proper π–domain. The method applied in the proof of theorem 5 cannot be used here. Let us consider an example. Take K as the class of all finite fields. We claim that there is no proper π–domain K' such that $sl(K) = sl(K')$.

Let us suppose that there is a proper π–domain K' such that $sl(K) = sl(K')$. Let us observe that for $F, F' \in K'$, if $F \subseteq F'$ then F is a subfield of F' in usual sense and fields F, F' are of the same characteristic. Then all the fields in K' are of the same characteristic, say p. Therefore the sentence χ_p, saying that the characteristic is p, belongs to $sl(K')$, but not to $sl(K)$.

6 Some conclusions

We shown that the first order logic is correct and complete inference tool for sl–semantics. Unfortunately, interesting theories of potentially infinite domains usually are not axiomatizable in the standard sense. However this is true also for classical semantics which allows actually infinite models. We have no standard axiomatization of standard number theory either. Only some partial approximations are available, such as Peano Arithmetic.

We claim that sl–semantics is in much better situation than the classical one. The main problem is that we have the set of logical axioms A_{sl} being Π_1^0–complete. Then it cannot be replaced by any recursive set. Nevertheless it is Δ_2^0, so it is algorithmically learnable.[5] The classical number theory is in much worse position. The arithmetical truth – by the Tarski undefinability of truth theorem [21] – is beyond the arithmetical hierarchy.

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[5] In the sense independently invented by Mark Gold [5], [6] and Hilary Putnam [20].

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